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New strong divisibility sequences*

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Abstract

We provide new families of divisibility and strong divisibility sequences based on some factorization properties of Chebyshev polynomials.

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1 Introduction

A sequence of any integer numbers $\{a_n\}$ is said to be a *divisibility sequence* if

$$a_m \mid a_n, \quad \text{whenever } m \mid n,$$

and is called a *strong divisibility sequence* if

$$\gcd(a_m, a_n) = a_{\gcd(m,n)}.$$

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The strong divisibility sequences and its weaker version have been studied for more than one century. Actually, the Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \dots$$

are perhaps the best known non-trivial strong divisibility sequence. For earlier questions, open problems, and general characterizations, the reader is referred to [4, 10, 11, 12, 21, 22].

As a particular case of the *general conditional recurrence sequences* defined in [16], recently it was proposed in [20] the study of the *conditional recurrence sequences* $\{f_n\}$ satisfying the recurrence relations of integers

$$f_n = \begin{cases} a_1 f_{n-1} + b_2 f_{n-2}, & \text{if } n \text{ is odd,} \\ a_2 f_{n-1} + b_1 f_{n-2}, & \text{if } n \text{ is even.} \end{cases}$$

for $n \geqslant 2$, with $f_0 = 1$ and $f_1 = a_1$, aiming to generate new strong divisibility sequences. Indeed, the authors were able to obtain sufficient conditions for which certain subsequences of $\{f_n\}$ are strong divisible.

Theorem 1.1 ([20])**.** *Let* $\tilde{f}_n = f_{2n-1}$. *If* $a_1 = 1$ *and* $\gcd(a_1a_2 + b_1 + b_2, b_1b_2) = 1$*, then*

$$\gcd(\tilde{f}_m, \tilde{f}_n) = \tilde{f}_{\gcd(m,n)}.$$

Corollary 1.2 ([20])**.** *Let* $\tilde{f}_n = f_{2n-1}$. *If* $\gcd(a_1a_2 + b_1 + b_2, b_1b_2) = 1$*, then* $\{\tilde{f}_n\}$ *is a strong divisibility sequence.*

Theorem 1.3 ([20])**.** *Let* $\tilde{f}_n = f_{2n-1}$. *Thus* $\tilde{f}_m \mid \tilde{f}_n$*, whenever* $m \mid n$.

For example, setting $a_1 = 3, a_2 = 1 = b_1$, and $b_2 = 2$, we get

n	1	2	3	4	5	6	7	8	9
f_n	3	4	18	22	102	124	576	700	3252

This means that the first terms of the subsequence of odd indices of $\{f_n\}$ are

n	1	2	3	4	5	6
$\tilde{f}_n$	3	18	102	576	3252	18360

While $\{\tilde{f}_n\}$ is a divisibility sequence, it is clear that is not strong.

Another interesting result obtained in [20] is the following:

Theorem 1.4. *Let* $\tilde{f}_1 = 1$ *and* $\tilde{f}_n = f_{n-1}$*, for* $n > 1$. *If* $a_1 = 1$*,* $b_1 = b_2$*, and* $\gcd(a_2, b_1) = 1$*, then* $\{\tilde{f}_n\}$ *is a strong divisibility sequence.*

For the weaker divisibility, the following general result was obtained:

Corollary 1.5. *Under the conditions of Theorem 1.4,* $\{\tilde{f}_n\}$ *is a divisibility sequence.*

Our aim here is to extend the above results to a more general setting, namely for the sequences of integers defined by the recurrence relations

$$f_n = \begin{cases} a_1 f_{n-1} + b_k f_{n-2}, & \text{if } n \equiv 1 \pmod{k}, \\ a_2 f_{n-1} + b_1 f_{n-2}, & \text{if } n \equiv 2 \pmod{k}, \\ a_3 f_{n-1} + b_2 f_{n-2}, & \text{if } n \equiv 3 \pmod{k}, \\ \cdots & \cdots \\ a_{k-1} f_{n-1} + b_{k-2} f_{n-2}, & \text{if } n \equiv k-1 \pmod{k}, \\ a_k f_{n-1} + b_{k-1} f_{n-2}, & \text{if } n \equiv 0 \pmod{k}, \end{cases} \tag{1.1}$$

for $n \geqslant 2$, with $f_0 = 1$ and $f_1 = a_1$. The previous results will be recovered by making $k = 2$. Consequently, we answer to the open problem proposed in [20].

In this paper, we will relate (1.1) with the so-called *periodic continuants* [6, 18] (for recent applications, the reader is referred to [1, 2, 3]). This relation is established by using Chebyshev polynomials of the second kind. Then, from $\{f_n\}$ we can, under certain conditions, generate new strong divisibility sequences. At the same time, we can recover the connection between the sequences defined by recurrence relations with two terms and the determinants of tridiagonal matrices. This is effectively in the spirit of some ideas we can find in [15], proposed by Édouard Lucas back to 1878.

2 The determinant of a tridiagonal k-Toeplitz matrix

The matrices of the form

$$A_n = \begin{pmatrix} a_1 & b_1 & & & & & & & & \\ c_1 & \ddots & \ddots & & & & & & & \\ & \ddots & a_k & b_k & & & & & & \\ & & c_k & a_1 & b_1 & & & & & \\ & & & c_1 & \ddots & \ddots & & & & \\ & & & & \ddots & a_k & b_k & & & \\ & & & & & c_k & a_1 & b_1 & & \\ & & & & & & c_1 & \ddots & \ddots & \\ & & & & & & & \ddots & & \end{pmatrix}_{n\times n},$$

i.e., tridiagonal matrices $A_n = (a_{ij})$ with entries satisfying

$$a_{i+k,j+k} = a_{ij}, \quad \text{for } i, j = 1, 2, \ldots, n-k,$$

are known as *tridiagonal k-Toeplitz*. The determinant of such matrix is known as a *periodic continuant* [18].

For $k = 1$, we get a tridiagonal Toeplitz matrix and its determinant was known in [18] as a *continuant*. The characteristic polynomial of such a matrix was found by V. Lovass-Nagy and P. Rózsa [13, 14], in 1963. Notwithstanding, the particular case when $k = 2$ and the two subdiagonals are constant equal to 1, had been considered in 1947 in D. E. Rutherford's seminal paper [19], followed soon after by J. F. Elliott with his Master's thesis [5, Section IV.4]. In 1966, Rózsa held a seminar at the University of Hamburg on tridiagonal k-Toeplitz matrices motivated mainly by problems of lattice dynamics, of ladder networks, and of structural analysis. In that year, L. Elsner and R. M. Redheffer [6] studied A_n for special cases of k and, two years later, P. Rózsa in [18] originally proved a general formula for the determinant of A_n. Independently, the spectrum of a tridiagonal 2-Toeplitz matrix was also studied by M. J. C. Gover in 1994 [9]. In [7], it is considered the case when $k = 3$ and, later on, the characteristic polynomial of A_n was stated, for any k, when analyzing the invertibility conditions for A_n based on orthogonal polynomials theory (cf. [8]).

We recall now Rózsa's solution. Let $\Delta_{i_1,i_2,\ldots,i_p}$ be the principal minor of A_n indexed by the rows and columns $i_1, i_2, \ldots, i_p$. The determinant of A_n is given in [18] as

$$\det A_n = (\sqrt{b_1c_1\cdots b_kc_k})^q \Bigg(\Delta_{1,\ldots,r}U_q(x) + \frac{\sqrt{b_kc_kb_1c_1\cdots b_rc_r}}{\sqrt{b_{r+1}c_{r+1}\cdots b_{k-1}c_{k-1}}}\Delta_{r+2,\ldots,k-1}\,U_{q-1}(x)\Bigg)$$

with $n = qk + r$ and

$$x = \frac{\Delta_{1,\ldots,k} - b_kc_k\Delta_{2,\ldots,k-1}}{2\sqrt{b_1c_1\cdots b_kc_k}},$$

assuming that $\Delta_{1,\ldots,r} = 1$ and $\Delta_{2,\ldots,r} = 0$, for $r = 0$, and with $\{U_n(x)\}_{n\geqslant 0}$ standing for the Chebyshev polynomials of the second kind. These polynomials satisfy the three-term recurrence relation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad \text{for all } n = 1, 2, \ldots, \tag{2.1}$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. We recall that the main explicit formula for the Chebyshev polynomials of the second kind could be

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi), \tag{2.2}$$

for all $n = 0, 1, 2\ldots$. While (2.2) is more common to find in the orthogonal polynomials theory, there are other explicit representations and relations for $U_n(x)$. Among them, the most frequent are

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}},$$

an immediate consequence of de Moivre's formula, and

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{n-k}{k}(2x)^{n-2k}.$$

Taking into account the definition of A_n, we can redefine (1.1) in terms of the determinant of A_n, namely,

$$f_n = \begin{vmatrix} a_1 & b_1 & & & & & & & & & \\ -1 & \ddots & \ddots & & & & & & & & \\ & \ddots & a_k & b_k & & & & & & & \\ & & -1 & a_1 & b_1 & & & & & & \\ & & & -1 & \ddots & \ddots & & & & & \\ & & & & \ddots & a_k & b_k & & & & \\ & & & & & -1 & a_1 & b_1 & & & \\ & & & & & & -1 & \ddots & \ddots & & \\ & & & & & & & \ddots & & & \end{vmatrix}_{n\times n} . \tag{2.3}$$

That means that the determinant (2.3) is

$$f_n = (i^k\sqrt{b_1\cdots b_k})^q \left(\Delta_{1,\ldots,r}\, U_q(x) + \frac{i^{r+1}\sqrt{b_k b_1\cdots b_r}}{i^{k-r-1}\sqrt{b_{r+1}\cdots b_{k-1}}}\,\Delta_{r+2,\ldots,k-1}\, U_{q-1}(x)\right),$$

where

$$x = \frac{\Delta_{1,\ldots,k} + b_k\Delta_{2,\ldots,k-1}}{i^k 2\sqrt{b_1\cdots b_k}}.$$

In particular, if $r = k-1$, then

$$f_n = (i^k\sqrt{b_1\cdots b_k})^q \Delta_{1,\ldots,k-1}\, U_q(x), \tag{2.4}$$

which we will explore to generate new strong divisibility sequences in the next sections.

Before that, we recall a general result relating distinct minors, which can be found for example in [18].

Lemma 2.1. *For any positive integer n and $i < j$,*

$$\Delta_{1,\ldots,j-1}\Delta_{i+1,\ldots,n} - (-1)^{j-i} b_i\cdots b_{j-1}\Delta_{1,\ldots,i-1}\Delta_{j+1,\ldots,n} = \Delta_{1,\ldots,n}\Delta_{i+1,\ldots,j-1}.$$

In fact, Lemma 2.2 in [20] is a particular case of Lemma 2.1.

3 New divisibility sequences

In [20], the authors asked for conditional (strong) divisibility sequences for $r > 2$, i.e., satisfying (1.1). We start with the weaker condition.

Let us recall several factorization properties for Chebyshev polynomials disclosed in [17].

Theorem 3.1 ([17]). *Let $m \geqslant n$ be two positive integers. Then $U_m(x)$ is a multiple of $U_n(x)$ if and only if $m = (\ell+1)n + \ell$, for some nonnegative integer ℓ. More precisely, if ℓ is even, then*

$$U_m(x) = U_n(x)\left(2\sum_{k=0}^{\frac{\ell}{2}} T_{m-(2k+1)n-2k}(x) - 1\right),$$

and if ℓ is odd, then

$$U_m(x) = 2U_n(x) \sum_{k=0}^{\frac{\ell-1}{2}} T_{m-(2k+1)n-2k}(x).$$

In Theorem 3.1, $\{T_n(x)\}_{n\geqslant 0}$ stands for the Chebyshev polynomial of the first kind. These polynomials satisfy the same recurrence (2.1), here with initial conditions $T_0(x) = 1$ and $T_1(x) = x$. An explicit formula for such polynomials is $T_n(x) = \cos n\theta$, with $x = \cos\theta$.

The next two results, naturally connected to those in Section 1, can be found in [17].

Theorem 3.2. *Let m and n be two nonnegative integers and $d = \gcd(m, n)$. Then*

$$\gcd(U_{m-1}(x), U_{n-1}(x)) = U_{d-1}(x).$$

Corollary 3.3. *If m and n are coprime, then $\gcd(U_{m-1}(x), U_{n-1}(x)) = 1$.*

The general sequences that we consider are

$$f_n = (\pm\sqrt{b})^{n-1} U_{n-1}\left(\frac{a}{\pm 2\sqrt{b}}\right),$$

where a, b are nonzero integers (possibly with $b < 0$), for $n \geqslant 1$. In particular, $f_0 = 0$, $f_1 = 1$ and $f_2 = a$.

It is worth mentioning that the symbol $\pm$ can be ignored, that is to say:

$$f_n = (\pm\sqrt{b})^{n-1} U_{n-1}\left(\frac{a}{\pm 2\sqrt{b}}\right) = (\sqrt{b})^{n-1} U_{n-1}\left(\frac{a}{2\sqrt{b}}\right), \tag{3.1}$$

since the Chebyshev polynomials of the second kind $U_n(x)$ have the same parities as n.

We may now state our first main result.

Theorem 3.4. *For any integers a and b, $\{f_n\}$ as defined in (3.1) is a divisibility sequence.*

Proof. Assume that $n \mid m$, say $m = sn$, where $s \geqslant 1$. For simplicity, set $x = \frac{1}{2\sqrt{b}}$. So

$$f_n = \frac{U_{n-1}(ax)}{(2x)^{n-1}} \quad \text{and} \quad f_m = \frac{U_{sn-1}(ax)}{(2x)^{sn-1}},$$

which implies that

$$\frac{f_m}{f_n} = \frac{U_{sn-1}(ax)}{(2x)^{(s-1)n} U_{n-1}(ax)}.$$

Set $\ell = s - 1$, we have $sn - 1 = (\ell + 1)(n - 1) + \ell$. From Theorem 3.1, $U_{sn-1}(x)$ is a multiple of $U_{n-1}(x)$. More precisely, when s is even,

$$U_{sn-1}(x) = 2U_{n-1}(x) \sum_{t=0}^{\frac{s-2}{2}} T_{(s-2t-1)n}(x),$$

and when s is odd,

$$U_{sn-1}(x) = U_{n-1}(x)\left(2\sum_{t=0}^{\frac{s-1}{2}} T_{(s-2t-1)n}(x) - 1\right).$$

Therefore

$$\frac{f_m}{f_n} = \frac{U_{sn-1}(ax)}{(2x)^{(s-1)n}U_{n-1}(ax)} = \frac{2}{(2x)^{(s-1)n}}\sum_{t=0}^{\frac{s-2}{2}} T_{(s-2t-1)n}(ax)$$

when s is even, and

$$\frac{f_m}{f_n} = \frac{U_{sn-1}(ax)}{(2x)^{(s-1)n}U_{n-1}(ax)} = \frac{2}{(2x)^{(s-1)n}}\sum_{t=0}^{\frac{s-1}{2}} T_{(s-2t-1)n}(ax) - \frac{1}{(2x)^{(s-1)n}}$$

when s is odd.

We will prove $\frac{U_{sn-1}(ax)}{(2x)^{(s-1)n}U_{n-1}(ax)}$ is an integer whether s is even or odd, by involving with the following two claims.

Claim 1. *$2T_{(s-2t-1)n}\left(\frac{a}{2}\right)$ is an integer, for any $0 \leqslant t \leqslant \lfloor \frac{s-1}{2} \rfloor$.*

This claim follows immediately from the recurrence relation about $T_n(x)$ as shown in (2.1).

Claim 2. *$(\sqrt{b})^{(s-1)n}T_{(s-2t-1)n}\left(\frac{1}{\sqrt{b}}\right)$ is an integer, for any $0 \leqslant t \leqslant \lfloor \frac{s-1}{2} \rfloor$.*

Observe that among all the terms in $T_{(s-2t-1)n}\left(\frac{1}{\sqrt{b}}\right)$, the maximum degree of denominator is $(\sqrt{b})^{(s-1)n}$, which means that all the denominators of $T_{(s-2t-1)n}\left(\frac{1}{\sqrt{b}}\right)$ would be canceled by $(\sqrt{b})^{(s-1)n}$. It leads to this claim.

Combining the above claims, it leads to

$$\frac{2}{(2x)^{(s-1)n}}\sum_{t=0}^{\lfloor \frac{s-1}{2} \rfloor} T_{(s-2t-1)n}(ax) = 2(\sqrt{b})^{(s-1)n}\sum_{t=0}^{\lfloor \frac{s-1}{2} \rfloor} T_{(s-2t-1)n}\left(\frac{a}{2\sqrt{b}}\right)$$

is an integer. When s is even, $f_n \mid f_m$ follows now. When s is odd, together with the fact that $\frac{1}{(2x)^{(s-1)n}} = (\sqrt{b})^{(s-1)n}$ is an integer, $f_n \mid f_m$ also holds. □

4 Strong divisibility sequences

The sequence $\{f_n\}$ defined in (3.1) can have negative terms. Therefore, in our strongly divisibility definition, we are assuming that $\gcd(a_m, a_n) = |a_{\gcd(m,n)}|$. Since we are interested in positive conditional recurrence sequences (1.1), all the terms of $\{f_n\}$ will be considered as positive or, equivalently, $a > 0$ and $a^2 - 4b \geqslant 0$. Notice that the zeros of the Chebyshev polynomials of the second kind are in the interval $(-1, 1)$ and, from its definition, $\lim_{x\to+\infty} U_n(x) = +\infty$.

In order to provide our characterization to the strong divisibility property of $\{f_n\}$, let us state several straightforward relations involving f_n, as defined in (3.1). From (2.1), we have

$$U_n\left(\frac{a}{2\sqrt{b}}\right) = \frac{a}{\sqrt{b}}U_{n-1}\left(\frac{a}{2\sqrt{b}}\right) - U_{n-2}\left(\frac{a}{2\sqrt{b}}\right)$$

and

$$f_n = af_{n-1} - bf_{n-2}. \tag{4.1}$$

A more general identity can be obtained from (2.1), namely

$$U_{s+t}(x) = U_s(x)U_t(x) - U_{s-1}(x)U_{t-1}(x),$$

and then,

$$f_{s+t} = f_{s+1}f_t - bf_sf_{t-1}. \tag{4.2}$$

The next result is an extension of some other results we can find in the literature, as for example related to the Fibonacci numbers.

Lemma 4.1. *If* $\gcd(a,b) = 1$, *then* $\gcd(f_n, f_{n+1}) = 1$ *for any* $n \geqslant 1$.

Proof. We claim that $\gcd(f_n, b) = 1$, for any $n \geqslant 1$, which can be proved by induction. From $f_1 = 1$ and $f_2 = a$, this claim holds when $n = 1, 2$. Assume that $\gcd(f_{n-1}, b) = 1$ and $\gcd(f_{n-2}, b) = 1$. Suppose to the contrary that $\gcd(f_n, b) = s$, where $s > 1$. From (4.1), $s \mid af_{n-1}$. Notice that $\gcd(s, a) = 1$, otherwise it is a contradiction to the hypothesis that $\gcd(a, b) = 1$. So $s \mid f_{n-1}$. However, this is another contradiction to the inductive hypothesis stating $\gcd(f_{n-1}, b) = 1$.

Now we are ready to show that $\gcd(f_n, f_{n+1}) = 1$. Again, from $f_1 = 1$ and $f_2 = a$, we know that $\gcd(f_n, f_{n+1}) = 1$ is true when $n = 1$. Suppose to the contrary that $\gcd(f_{n-2}, f_{n-1}) = 1$, for some $n \geqslant 3$, but $\gcd(f_{n-1}, f_n) = t$ with $t > 1$. From (4.1), $t \mid bf_{n-2}$. Note that $\gcd(t, f_{n-2}) = 1$, otherwise we get a contradiction with $\gcd(f_{n-2}, f_{n-1}) = 1$. Thus, $t \mid b$ means that t is a common divisor of b and f_n, a contradiction to the above claim that $\gcd(f_n, b) = 1$.

The proof is now completed. □

We are now able to prove the main result of this section.

Theorem 4.2. *The sequence* $\{f_n\}$ *defined in* (3.1) *is strongly divisible if and only if* $\gcd(a, b) = 1$.

Proof. The necessity part is easy. Assume that $\{f_n\}$ is a strong divisibility sequence. Suppose to the contrary that $\gcd(a, b) \neq 1$. From (4.1), we may obtain the first few values: $f_1 = 1$, $f_2 = a$, $f_3 = a^2 - b$, $f_4 = a^3 - 2ab$. Clearly, $\gcd(f_3, f_4) \neq 1 = f_1$ follows from $\gcd(a, b) \neq 1$, which is a contradiction to the strong divisibility property of $\{f_n\}$.

Now we prove the part of sufficiency. Suppose that $\gcd(a, b) = 1$. Set $g = \gcd(n, m)$ and $d = \gcd(f_n, f_m)$. We would like to show that $\gcd(f_n, f_m) = |f_{\gcd(n,m)}|$, i.e., $d = |f_g|$, which comes from $f_g \mid d$ and $d \mid f_g$.

On one hand, from $g \mid n$ and $g \mid m$, we get $f_g \mid f_n$ and $f_g \mid f_m$, since $\{f_n\}$ is a divisibility sequence from Theorem 3.4. Thus, $f_g \mid d$.

On the other hand, we still need to show that $d \mid f_g$. Since, $g = \gcd(n, m)$, we may assume that there exist positive integers s, k such that $sn = g + km$. From (4.2), we have

$$f_{sn} = f_{g+km} = f_gf_{km+1} - bf_{g-1}f_{km}.$$

From $d \mid f_n$, and $f_n \mid f_{sn}$ (since $\{f_n\}$ is a divisibility sequence), we get $d \mid f_{sn}$. Similarly, we have $d \mid f_{km}$. Therefore, $d \mid f_g f_{km+1}$. Notice that $\gcd(d, f_{km+1}) = 1$, otherwise, together with $d \mid f_{km}$, it leads to $\gcd(f_{km}, f_{km+1}) \neq 1$, which is a contraction to Lemma 4.1. Now, it follows that $d \mid f_g$.

Combining $f_g \mid d$ and $d \mid f_g$, we obtain $d = |f_g|$, which reveals the strong divisibility property of $\{f_n\}$. □

5 Examples

In this final section, from the above results, we provide several examples of new (conditional) strong divisibility sequences.

Setting $k = 3$, $r = 2$, we have

$$x = \frac{\Delta_{1,\ldots,k} + b_k \Delta_{2,\ldots,k-1}}{i^k 2\sqrt{b_1 \cdots b_k}}.$$

In particular, if $r = k - 1$, then

$$f_n = (-i\sqrt{b_1 b_2 b_3})^q (a_1 a_2 + b_1)\, U_q \left(\frac{a_1 a_2 a_3 + a_3 b_1 + a_1 b_2 + a_2 b_3}{-i2\sqrt{b_1 b_2 b_3}} \right).$$

So, if we consider the sequence defined by

$$f_n = \begin{cases} f_{n-1} + 3f_{n-2}, & \text{if} n \equiv 1 \pmod 3, \\ 2f_{n-1} + f_{n-2}, & \text{if} n \equiv 2 \pmod 3, \\ 4f_{n-1} + 2f_{n-2}, & \text{if } n \equiv 0 \pmod 3, \end{cases}$$

we have

$$f_n = (-i\sqrt{6})^q\, 3\, U_q \left(\frac{20}{-i2\sqrt{6}} \right).$$

Now set

$$g_{q+1} = (-i\sqrt{6})^q\, U_q \left(\frac{20}{-i2\sqrt{6}} \right),$$

for $q \geqslant 0$. The first terms are:

n	g_n
1	1
2	20
3	406
4	8240
5	167236
6	3394160
7	68886616
8	1398097280
9	28375265296
10	575893889600

Now, we can check, for example, that $g_3 \mid g_6$ or $g_5 \mid g_{10}$. However,

$$\gcd(g_8, g_{10}) = 320.$$

Instead, we take the recurrence relation

$$f_n = \begin{cases} 2f_{n-1} + 3f_{n-2}, & \text{if } n \equiv 1 \pmod 3, \\ f_{n-1} + f_{n-2}, & \text{if } n \equiv 2 \pmod 3, \\ 4f_{n-1} + 2f_{n-2}, & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

Setting

$$g_{q+1} = (-i\sqrt{6})^q\, U_q\left(\frac{19}{-i2\sqrt{6}}\right),$$

for $q \geqslant 0$, the first terms are:

n	g_n
1	1
2	19
3	367
4	7087
5	136855
6	2642767
7	51033703
8	985496959
9	19030644439
10	367495226095

Now we can check, for example, that $g_4 \mid g_8$ or $g_5 \mid g_{10}$. Moreover,

$$\gcd(g_8, g_{10}) = g_2 \quad \text{or} \quad \gcd(g_6, g_9) = g_3,$$

and, of course,

$$\gcd(g_4, g_9) = g_1.$$

Let us consider now two more elaborated examples, for $k = 4$. We start with the following one

$$f_n = \begin{cases} 2f_{n-1} + 4f_{n-2}, & \text{if } n \equiv 1 \pmod 4, \\ f_{n-1} + 3f_{n-2}, & \text{if } n \equiv 2 \pmod 4, \\ 2f_{n-1} + f_{n-2}, & \text{if } n \equiv 3 \pmod 4, \\ 3f_{n-1} + f_{n-2}, & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Setting

$$g_{q+1} = (\sqrt{12})^q\, U_q\left(\frac{53}{2\sqrt{12}}\right),$$

for $q \geqslant 0$, the first terms are:

n	g_n
1	1
2	53
3	2797
4	147605
5	7789501
6	411072293
7	21693357517
8	1144815080885
9	60414878996701
10	3188250805854533

Straightforward verification shows, for example, that $g_4 \mid g_8$ or $g_5 \mid g_{10}$. Furthermore,

$$\gcd(g_8, g_{10}) = g_2 \quad \text{or} \quad \gcd(g_6, g_9) = g_3,$$

and, of course,

$$\gcd(g_4, g_9) = g_1.$$

Finally, we study

$$f_n = \begin{cases} 2f_{n-1} + 4f_{n-2}, & \text{if } n \equiv 1 \pmod 4, \\ f_{n-1} + 2f_{n-2}, & \text{if } n \equiv 2 \pmod 4, \\ 2f_{n-1} + f_{n-2}, & \text{if } n \equiv 3 \pmod 4, \\ 3f_{n-1} + f_{n-2}, & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Setting

$$g_{q+1} = (\sqrt{8})^q \, U_q\left(\frac{46}{2\sqrt{8}}\right),$$

for $q \geqslant 0$, the first terms are:

n	g_n
1	1
2	46
3	2108
4	96600
5	4426736
6	202857056
7	9296010688
8	425993635200
9	19521339133696
10	894573651068416

Now we can check, for instance, that $g_4 \mid g_8$ or $g_5 \mid g_{10}$. However,

$$\gcd(g_8, g_{10}) = 2944.$$

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