

Well-totally-dominated graphs*

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Abstract

A subset of vertices in a graph is called a total dominating set if every vertex of the graph is adjacent to at least one vertex of this set. A total dominating set is called minimal if it does not properly contain another total dominating set. In this paper, we study graphs whose all minimal total dominating sets have the same size, referred to as well-totally-dominated (WTD) graphs. We first show that WTD graphs with bounded total domination number can be recognized in polynomial time. Then we focus on WTD graphs with total domination number two. In this case, we characterize triangle-free WTD graphs and WTD graphs with packing number two, and we show that there are only finitely many planar WTD graphs with minimum degree at least three. Lastly, we show that if the minimum degree is at least three then the girth of a WTD graph is at most 12. We conclude with several open questions.

Keywords: Total domination, well-totally-dominated graphs, minimal total dominating sets.

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1 Introduction

Total domination in graphs has been extensively studied in the literature (see [15]) and has numerous applications. For instance, consider a computer network where a core group of file servers has the ability to communicate directly with every computer outside the core group. Moreover, each file server is directly linked to at least one other backup file server where duplicate information is stored. This core group of servers corresponds to a total dominating set in the graph representing the computer network. Another application area is a specific committee selection mechanism such that every non-member of the committee knows at least one member of the committee and every member of the committee knows at least one other member of the committee to avoid feelings of isolation and thus enhance cooperation (see [14]).

Let G be a graph with no isolated vertices. A subset S of $V(G)$ is called a *total dominating set* (TDS) of G if every vertex in G is adjacent to at least one element in S . A total dominating set is *minimal* if it contains no other TDS of G . The minimum size of a total dominating set of a graph G is called the *total domination number* and denoted by $\gamma_t(G)$, while the maximum size of a minimal total dominating set is called the *upper total domination number* and denoted by $\Gamma_t(G)$. G is called *well-totally-dominated* (WTD) if every minimal TDS of G is of the same size, that is, $\gamma_t(G) = \Gamma_t(G)$. WTD graphs with $\gamma_t = k$ are denoted by $\text{WTD}(k)$.

Given a graph, computing its total domination number and its upper total domination number are NP-hard in general [18, 6] and already NP-hard even in specific graph classes such as bipartite graphs, comparability graphs and claw-free graphs [15]. One way to deal with such a problem is to consider “trivial” instances where these two parameters have the same value. Examples of graph classes defined in this way in the literature include well-covered graphs (whose all maximal independent sets have the same size), well-dominated graphs (whose all minimal dominating sets have the same size), and equimatchable graphs (whose all maximal matchings have the same size). Structural properties of each one of these graph classes have been studied extensively in the literature. In this paper, we take the same approach for the total dominating sets. Works on total domination in the literature mostly focused on the relation of the total domination number with other graph parameters and characterized graphs with total domination number being equal to an upper bound (e.g. [4, 3]). Inequalities relating the total domination number to other domination parameters and characterization of graphs that tightly attain these bounds have also been studied (see [16, 1]).

Clearly, if the total domination number and the upper total domination number are polynomial time solvable for a given class of graphs, then the recognition of WTD graphs belonging to this class of graphs is polynomial. However, the complexity of recognizing WTD graphs in general is unknown. In such a situation, a classical approach consists in studying the structure of WTD graphs in restricted graph classes and providing structural characterizations along with efficient recognition algorithms whenever possible.

WTD graphs were initially introduced in [12], where WTD cycles and paths are characterized and several constructions of WTD trees are given. They also proved that a WTD graph with minimum degree at least two has girth at most 14. The work in [7] focused on the composition and decomposition of WTD trees and proved that any WTD tree can be constructed from a family of three small trees. To the best of our knowledge, [12] and [7] are the only work on WTD graphs. A graph class resembling WTD graphs is *well-dominated graphs*, which are graphs whose minimal dominating sets have the same size. It

is known that well-dominated graphs form a proper subset of well-covered graphs [8]. We note that well-covered graphs are graphs whose maximal independent sets have the same size and there is a rich literature about them (see [19, 13]). Well-dominated graphs were introduced by Finbow et.al. [8], who provided a characterization of bipartite well-dominated graphs and well-dominated graphs with girth at least 5. Characterizations of these graphs within other graph classes were also obtained [20, 10, 9, 17]. Although their definitions resemble each other, there is not a containment relationship between WTD graphs and well-dominated graphs. For instance, a cycle on six vertices is WTD but not well-dominated, whereas the graph T_{10} described in [17] is well-dominated but not WTD.

It follows from the previous studies on WTD graphs that we do not know much about their structure. In this paper, we investigate the study of WTD graphs from a structural point of view. We first study WTD graphs with bounded total domination number. We prove in Section 2 that the recognition of WTD graphs with total domination number k is solvable in polynomial time for every positive integer k . We then focus on WTD graphs with total domination number 2, referred to as WTD(2) graphs in Section 3. We characterize triangle-free WTD(2) graphs and WTD(2) graphs with packing number 2 (or equivalently of diameter 3). We also show that there is a finite number of planar WTD(2) graphs with minimum degree at least 3. Subsequently, we study the girth of WTD graphs in Section 4. In particular, building on a result in [12], we prove that WTD graphs with minimum degree at least three have girth at most 12. Finally, we discuss several open research directions.

2 WTD Graphs with Bounded Total Domination Number

Recall that the complexity of recognizing WTD graphs is unknown. In this section, we show that for any positive integer k , WTD(k) graphs can be recognized in polynomial time. To this end, we will use an equivalent description of WTD(k) graphs using transversal hypergraphs. Let us first introduce necessary definitions. A *hypergraph* H is a pair $H = (X, E)$ where X is a set of elements called *vertices*, and E is a set of nonempty subsets of X called *hyperedges*. Therefore, a hypergraph might have a vertex which belongs to none of the hyperedges, but cannot have multiple hyperedges. A *transversal* (or *hitting set*) of a hypergraph $H = (X, E)$ is a set $T \subseteq X$ that has nonempty intersection with every hyperedge of H . A transversal of a collection of sets is a transversal of the hypergraph whose hyperedges are the given collection. A transversal T is called *minimal* if no proper subset of T is a transversal. The *transversal hypergraph* of $H = (X, E)$ is the hypergraph $H^* = (X, F)$ whose hyperedge set F consists of all minimal transversals of H .

Let G be a graph with no isolated vertex. Let H_G be the hypergraph whose vertex set is $V(G)$ and hyperedges are open neighborhoods of the vertices of G . Let also $MTDS(G)$ denote the set of all minimal total dominating sets of G .

Lemma 2.1. *$MTDS(G)$ consists of hyperedges of the transversal hypergraph of H_G .*

Proof. Let T be a hyperedge of H_G^* , that is a minimal transversal of the set of open neighborhoods of G . This means that T contains a neighbor of every vertex in G , thus it is a total dominating set. By minimality of the transversal T , it is also a minimal total dominating set of G . Conversely, let S be a minimal total dominating set of G . Then, every vertex in G is adjacent to at least one vertex in S . That is, S has a nonempty intersection with every open neighborhood in G . Therefore, S is a transversal of the hypergraph H_G and minimality of S implies that it is a minimal transversal. Thus S is a hyperedge of H_G^* . \square

Proposition 2.2. *Let G be a graph. Then, for any minimal transversal T of $MTDS(G)$, there exists a vertex v in G such that $N(v) = T$.*

Proof. Let $MTDS(G) = \{A_1, \dots, A_m\}$. Since T has nonempty intersection with each A_i , $V(G) \setminus T$ contains none of the minimal total dominating sets A_1, \dots, A_m . Therefore, $V(G) \setminus T$ is not a TDS of G , and hence there exists at least one vertex $v \in V(G)$ such that $N(v) \cap (V(G) \setminus T) = \emptyset$. Thus, we see that $N(v) \subseteq T$. Suppose that $N(v) \neq T$. Then $T \setminus N(v) \neq \emptyset$ and let $u \in T \setminus N(v)$. Since T is a minimal transversal, $T \setminus \{u\}$ is disjoint with at least one of A_1, \dots, A_m , say A_1 . As $u \in T \setminus N(v)$, we have $N(v) \subseteq T \setminus \{u\}$, and hence $N(v) \cap A_1 = \emptyset$. That is, v is not dominated by A_1 , which is a contradiction. Therefore, $N(v) = T$. \square

A hypergraph H is said to be *Sperner* if no hyperedge of H contains another hyperedge. The following result shows that any finite collection of finite sets which forms a Sperner hypergraph corresponds to the set of all minimal total dominating sets of a graph.

Proposition 2.3. *Let H be a Sperner hypergraph. Then there exists a graph G such that $E(H) = MTDS(G)$.*

Proof. Let $E(H) = \{A_1, \dots, A_m\}$ and $A = \cup_{i=1}^m A_i$. Consider a graph with vertex set A and draw edges between its vertices such that each vertex is adjacent to at least one vertex in A_i for all $i = 1, \dots, m$ (for example, draw all possible edges). Then, in accordance with Proposition 2.2, for each minimal transversal T of H , add a vertex v_T to the graph such that $N(v_T) = T$. Let G be the resulting graph.

We first show that each A_i is a TDS of G . By construction, every vertex of A is adjacent to at least one vertex in A_i . Moreover, for every minimal transversal T of A_1, \dots, A_m we have $T \cap A_i \neq \emptyset$, and hence, each v_T is dominated by A_i . Therefore, A_i is a TDS for $i = 1, \dots, m$.

We next show that every TDS of G contains at least one of A_1, \dots, A_m . Let S be a TDS of G and suppose that $A_i \not\subseteq S$ for $i = 1, \dots, m$. Then, $V(G) \setminus S$ is a transversal of A_1, \dots, A_m , and hence, there exists a minimal transversal T of A_1, \dots, A_m such that $T \subseteq V(G) \setminus S$. On the other hand, we have $N(v_T) = T$ and thus, we get $N(v_T) \cap S = \emptyset$, which contradicts with S being a TDS of G .

Consequently, a set other than A_1, \dots, A_m can not be a minimal TDS of G . We finally show that each A_i is a minimal TDS of G . Suppose that A_i is not minimal for some i . Then, $A_i \setminus \{x\}$ is still a TDS of G for some $x \in A_i$, and therefore, $A_j \subseteq A_i \setminus \{x\}$ for some j , which implies $A_j \subseteq A_i$, which is a contradiction to H being Sperner. Therefore, minimal TDSs of G are exactly A_1, \dots, A_m . \square

Remark 2.4. One can extend G to another graph whose minimal TDSs are A_1, \dots, A_m as follows: Let G' be a graph disjoint from G . Draw edges between the vertices of G' and A in such a way that every vertex of G' is adjacent to at least one vertex of A_i for $i = 1, \dots, m$. By following the same arguments, it is easy to check that minimal TDSs of the resulting graph are A_1, \dots, A_m .

Notice that any finite collection consisting of distinct sets of size k corresponds to a Sperner hypergraph and therefore, Proposition 2.3 implies the following result.

Corollary 2.5. *For every integer $k \geq 2$, $WTD(k)$ is an infinite graph family.*

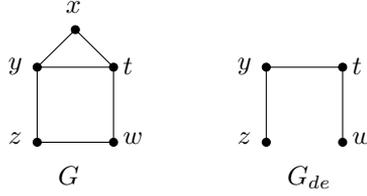


Figure 1: A WTD(2) graph G and the graph G_{de} obtained by the dominating edges of G .

The HYPERGRAPH TRANSVERSAL problem is the decision problem that takes as input two Sperner hypergraphs H and H' and asks whether H' is the transversal hypergraph H^* of H .

Theorem 2.6 ([5], [2]). *For every positive integer k , the HYPERGRAPH TRANSVERSAL problem is solvable in polynomial time if all hyperedges of one of the two hypergraphs H and H' are of size at most k .*

Theorem 2.6 has the following consequence:

Corollary 2.7. ([11]) *For every positive integer k , the following problem is solvable in polynomial time: Given a Sperner hypergraph H , determine whether all minimal transversals of H are of size k .*

The complexity of recognition of WTD graphs with bounded total domination number can now be derived from Corollary 2.7.

Theorem 2.8. *For every positive integer k , the problem of recognizing WTD(k) graphs can be solved in polynomial time.*

Proof. Let G be a graph with no isolated vertices. Consider the hypergraph $H_G = (V, \mathcal{E})$, where \mathcal{E} contains the inclusion-minimal elements of $\{N(v) : v \in V\}$. Observe that H_G is Sperner and that the minimal transversals of H_G are exactly the minimal total dominating sets of G by Lemma 2.1. It follows that G is WTD if and only if all minimal transversals of H_G are of size k . By Corollary 2.7, this condition can be tested in polynomial time. \square

3 WTD Graphs with Total Domination Number Two

In this section, we study WTD graphs whose total domination number is 2. We give complete characterizations of WTD(2) graphs with packing number 2 and triangle-free WTD(2) graphs. We also show that planar WTD(2) graphs with minimum degree at least 3 have at most 16 vertices.

Let G be a WTD(2) graph. Note that every minimal TDS of G is a pair consisting of endpoints of an edge of G . Consequently, every WTD(2) graph is connected. We will call an edge of G whose endpoints is a TDS of G a *dominating edge* of G . Let G_{de} be the graph with vertex set $\cup_{S \in MTDS(G)} S$ (i.e., vertices of G serve as an endpoint of a dominating edge) and edge set which consists of dominating edges of G . In other words, G_{de} is the edge-induced subgraph of G obtained by the dominating edges. See Figure 1 for an example.

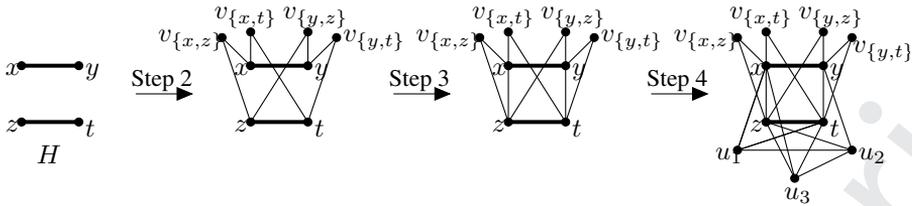


Figure 2: A graph in W_2 obtained by the given process. Bold edges represent the dominating edges.

Remark 3.1. Notice that the graph G_{de} and the subgraph of G induced by $V(G_{de})$ are not necessarily the same. In general, G_{de} is a subgraph of G but not necessarily an induced subgraph of G with respect to a set of vertices.

A set S is a *vertex cover* of a graph G if every edge of G has an endpoint from S . Let $MVC(G)$ denote the set of all minimal vertex covers of the graph G .

Proposition 3.2. Let G be a $WTD(2)$ graph. For every minimal vertex cover S of G_{de} there exists a vertex v_S in G such that $N(v_S) = S$.

Proof. Note that every minimal vertex cover S of G_{de} is a minimal transversal of $MTDS(G)$. Therefore, by Proposition 2.2 there exists a vertex in G whose neighborhood is exactly S . □

3.1 Characterization of $WTD(2)$ Graphs with Packing Number 2

A set $S \subseteq V(G)$ is called a *packing* of G if $N[u] \cap N[v] = \emptyset$ for every distinct $u, v \in S$. The *packing number* $\rho(G)$ is the maximum size of a packing of G . It is well-known that for any graph G we have $\rho(G) \leq \gamma(G) \leq \gamma_t(G)$. Therefore, if $\gamma_t(G) = 2$, then $\rho(G)$ is either 1 or 2. In this subsection, we provide a characterization of $WTD(2)$ graphs G with $\rho(G) = 2$. In particular, this characterization allows us to construct any $WTD(2)$ graph with $\rho(G) = 2$.

Let W_2 be the set of graphs obtained as follows:

- Step 1:** Choose a bipartite graph H with no isolated vertices.
- Step 2:** For every $S \in MVC(H)$, choose a new vertex v_S and draw edges from v_S to every vertex in S .
- Step 3:** For each edge uv in H and every $w \in V(H) \setminus \{u, v\}$, add the edges wu and/or wv if needed to make sure w is adjacent to at least one of u and v .
- Step 4:** Choose a new graph H' (might be the empty graph) which is disjoint from the current graph. Then for each edge uv in H and every $w \in V(H')$, draw at least one of the edges wu and wv .

A graph in W_2 is given in Figure 2.

Lemma 3.3. If a graph G is in W_2 , then G is a $WTD(2)$ graph with $\rho(G) = 2$.

Proof. Let $G \in W_2$ and $H = (U, V, E)$ be the bipartite graph in the first step of the construction of G . We first show that the packing number of G is 2. As H has no isolated

vertices, both U and V are minimal vertex covers of H . Thus, the vertices v_U and v_V have disjoint closed neighborhoods since $N(v_U) = U$ and $N(v_V) = V$ and hence, we get $\rho(G) \geq 2$. Clearly, by construction, every edge of H is a dominating edge of G . Therefore, we get $\gamma_t(G) = 2$. Since $\rho(G) \leq \gamma_t(G)$, we obtain $\rho(G) \leq 2$ and hence, $\rho(G) = 2$.

Now let T be a minimal TDS of G other than the edges of H . Then T contains at most one endpoint of an edge of H because otherwise T contains a TDS, which contradicts with T being minimal. Therefore, $V(H) \setminus T$ is a vertex cover of H and hence, it contains a minimal vertex cover S of H . By construction there exists a vertex v_S with $N(v_S) = S$. As $S \subseteq V(H) \setminus T$, we obtain $N(v_S) \cap T = \emptyset$, which contradicts with T being a TDS of G . Consequently, edges of H are the only minimal TDSs of G and hence, G is a WTD(2) graph and $G_{de} = H$. \square

Lemma 3.4. *Let G be a WTD(2) graph with $\rho(G) = 2$. Then, G is in W_2 .*

Proof. Let $\{x, y\}$ be a packing with minimum $|N[x]| + |N[y]|$. Note that every dominating edge of G has one endpoint from $N(x)$ and one from $N(y)$ and hence, G_{de} is a bipartite graph, say with parts X and Y where $X \subseteq N(x)$ and $Y \subseteq N(y)$.

We next show that $X = N(x)$ and $Y = N(y)$. By symmetry, it suffices to prove $X = N(x)$. Notice that G_{de} has no isolated vertices and therefore, X is a minimal vertex cover of G_{de} . By Proposition 3.2 there exists a vertex v_X satisfying $N(v_X) = X$. Suppose that $X \neq N(x)$. Then, we get $X \subset N(x)$. Clearly $v_X \neq y$. Moreover, $v_X \notin N(y)$ since $y \notin X = N(v_X)$. Thus, we get $N[v_X] \cap N[y] = \emptyset$ and hence $\{v_X, y\}$ is a packing of G . However, we obtain $|N[v_X]| + |N[y]| < |N[x]| + |N[y]|$ since $X \subset N(x)$, which yields a contradiction with the definition of the packing $\{x, y\}$. Consequently, we get $X = N(x)$ and hence, we may take $v_X = x$. Similarly, we have $Y = N(y)$ and we may assume $v_Y = y$.

Now let S be a minimal vertex cover of G_{de} . By Proposition 3.2 there exists a vertex v_S satisfying $N(v_S) = S$. If $S = X$ or $S = Y$, we can take v_S to be x or y , respectively, and in both cases, we have $v_S \notin V(G_{de})$. Otherwise, suppose that $v_S \in V(G_{de}) = X \cup Y$. Without loss of generality, let $v_S \in X$. Then, as $X = N(x)$, we get $x \in N(v_S) = S \subseteq N(x) \cup N(y)$, which is a contradiction. Therefore, v_S is not a vertex of G_{de} , that is, $v_S \in V(G) \setminus V(G_{de})$.

Finally, we see that one can obtain the graph G by following the procedure in the definition of W_2 with the initial bipartite graph $H = G_{de}$. \square

Combining the results in Lemma 3.3 and Lemma 3.4 gives the following structural characterization of WTD(2) graphs with $\rho(G) = 2$. Moreover, by definition of the class W_2 , this provides us with a procedure to construct any WTD(2) graph with $\rho(G) = 2$.

Theorem 3.5. *A graph G is WTD(2) with $\rho(G) = 2$ if and only if $G \in W_2$.*

Given a graph G , the *diameter* of G , denoted by $diam(G)$ is the maximum length of a shortest path between any pair of vertices of G . Let G be a graph such that $\gamma_t(G) = 2$. Then, it is easy to see that $diam(G) \leq 3$. Moreover, whenever $\gamma_t(G) = 2$, we have $diam(G) = 3$ if and only if $\rho(G) = 2$ and therefore, in all the statements in Lemma 3.3, Lemma 3.4 and Theorem 3.5, the condition $\rho(G) = 2$ can be replaced with $diam(G) = 3$.

Corollary 3.6. *A graph G is WTD(2) with $diam(G) = 3$ if and only if $G \in W_2$.*

One may attempt to modify the description of W_2 graphs in order to describe all WTD(2) graphs with $\rho(G) = 1$. In the first step of the process of building a graph in W_2 , if one starts with a non-bipartite graph H with no isolated vertices, then the resulting graph is still WTD(2) but has packing number 1. However, not every WTD(2) graph G with $\rho(G) = 1$ can be obtained in this way. For example, consider the graph presented in Figure 1. To obtain this graph G , in Step 1 one should definitely choose H to be the graph with vertex set $\{z, y, t, w\}$ and edge set $\{zy, yt, tw\}$ which is indeed G_{de} . However, in Step 2 if one chooses a new vertex v_S for $S = \{y, w\}$ (which is a minimal vertex cover of G_{de}), then the graph G can not be obtained. So, the complete characterization of WTD(2) graphs with $\rho(G) = 1$ is left as an open question.

3.2 Triangle-free WTD(2) Graphs

In this subsection, we provide characterization of triangle-free WTD(2) graphs.

Lemma 3.7. *If G is a triangle-free graph with $\gamma_t(G) = 2$, then G is a bipartite graph and we have*

$$\rho(G) = \begin{cases} 1, & \text{if } G \text{ is complete bipartite} \\ 2, & \text{otherwise} \end{cases}$$

Proof. Let uv be a dominating edge of G . Then we have $N(u) \cup N(v) = V(G)$. As G is triangle-free, none of two adjacent vertices have a common neighbor. Therefore, we have $N(u) \cap N(v) = \emptyset$ and also see that both $N(u)$ and $N(v)$ are independent sets. We consequently obtain that G is a bipartite graph with parts $N(u)$ and $N(v)$. Since $\rho(G) \leq \gamma_t(G) = 2$, we have $\rho(G) \in \{1, 2\}$. Moreover, it is clear that $\rho(G) = 1$ if and only if each vertex in $N(u)$ is adjacent to all the vertices in $N(v)$, i.e., G is a complete bipartite graph. \square

For a bipartite graph with parts X and Y , define $X_u = \{x \in X : N(x) = Y\}$ and $Y_u = \{y \in Y : N(y) = X\}$. In other words, X_u (resp. Y_u) is the set of vertices in X (resp. Y) which are adjacent to every vertex in Y (resp. X). The following result characterizes all triangle-free WTD(2) graphs.

Theorem 3.8. *The following three statements are equivalent:*

- (i) G is a triangle-free WTD(2) graph.
- (ii) G is a bipartite WTD(2) graph.
- (iii) G is complete bipartite graph or G is a bipartite graph with parts X and Y such that there exist vertices $a \in X \setminus X_u$ and $b \in Y \setminus Y_u$ satisfying $N(a) = Y_u \neq \emptyset$ and $N(b) = X_u \neq \emptyset$.

Proof. By Lemma 3.7 we see that (i) implies (ii). On the other hand, the implication (iii) \rightarrow (i) can be easily verified and hence, the proof finishes if we show that (ii) implies (iii). Now let G be a bipartite WTD(2) graph, say with parts X and Y . Clearly we will only consider the case when G is not a complete bipartite graph. By definition of X_u and Y_u , note that every dominating edge of G has one endpoint in $X_u \neq \emptyset$ and one endpoint in $Y_u \neq \emptyset$. Moreover, any edge xy where $x \in X_u$ and $y \in Y_u$ is a dominating edge of G . Therefore, G_{de} is the subgraph of G induced by $X_u \cup Y_u$ and it is complete bipartite. Thus, G_{de} has only two minimal vertex covers, namely X_u and Y_u . Then, definition of a graph in W_2 and Theorem 3.5 imply the existence of the vertices $a \in X \setminus X_u$ and $b \in Y \setminus Y_u$ with $N(a) = Y_u$ and $N(b) = X_u$. \square

Although a polynomial time recognition algorithm for WTD(2) graphs follows from Theorem 2.8, the characterization in Theorem 3.8 provides us with a simple linear time recognition algorithm.

Corollary 3.9. *Triangle-free WTD(2) graphs can be recognized in linear time.*

Proof. Given a graph G , one can check whether it is a connected bipartite graph and if so, find its unique bipartition (X, Y) in linear time (in the number of vertices and edges of G). Then, sets X_u and Y_u can be identified simply by assigning every vertex $x \in X$ such that $d(x) = |Y|$ into X_u , and $y \in Y$ such that $d(y) = |X|$ into Y_u . According to Theorem 3.8, G is triangle-free WTD(2) if and only if either $X_u = X$ and $Y_u = Y$ (thus, G is complete bipartite), or the removal of X_u and Y_u leaves at least one isolated vertex in each one of X and Y . Clearly, all these checks take only linear time. \square

3.3 Planar WTD(2) Graphs

In this subsection, we study planar WTD(2) graphs whose minimum degree is at least three and show that such graphs can have at most sixteen vertices. Throughout this section, we frequently use the fact that a graph obtained by an edge contraction of a planar graph is also planar. Recall also that a planar graph contains no K_5 or $K_{3,3}$.

Observation 3.10. *Let G be a WTD(2) graph. The vertex obtained by edge contraction of a dominating edge is a universal vertex in the new graph.*

Let $\nu(G)$ denote the matching number of a graph G .

Lemma 3.11. *Let G be a planar WTD(2) graph. If $\nu(G_{de}) \geq 3$, then $|V(G)| \leq 8$.*

Proof. Suppose that $\nu(G_{de}) \geq 3$ and G has at least 9 vertices. Then, G has three independent dominating edges, say u_1v_1, u_2v_2 and u_3v_3 , and three vertices other than $u_1, u_2, u_3, v_1, v_2, v_3$, say w_1, w_2 and w_3 . Now contract the edges u_1v_1, u_2v_2 and u_3v_3 . In the resulting graph, new three vertices and w_1, w_2, w_3 contain a $K_{3,3}$, which contradicts with the planarity. \square

Lemma 3.12. *If G is a WTD(2) graph with $\delta(G) \geq 3$, then $\nu(G_{de}) \geq 2$.*

Proof. Let G be a WTD(2) graph with $\delta(G) \geq 3$. It suffices to show that G has two independent dominating edges. Let xy be a dominating edge of G . Since the minimum degree is at least three, each vertex of G has at least one neighbor in $V(G) \setminus \{x, y\}$. Therefore, $V(G) \setminus \{x, y\}$ is a TDS of G and hence, it contains a dominating edge ab since G is WTD(2). As the dominating edges xy and ab share no vertex, we get $\nu(G_{de}) \geq 2$. \square

Combining the results in Lemmas 3.11 and 3.12 gives the following result.

Proposition 3.13. *If G is a planar WTD(2) graph with $\delta(G) \geq 3$, then $\nu(G_{de}) = 2$ or $|V(G)| \leq 8$.*

We next study planar WTD(2) graphs whose minimum degree is at least 3 and matching number is 2.

Proposition 3.14. *If G is a planar WTD(2) graph with $\delta(G) \geq 3$ and $\nu(G_{de}) = 2$, then $|V(G)| \leq 16$.*

Proof. Let ab and xy be two independent dominating edges of G and $H = G - \{a, b, x, y\}$. Let H_1, \dots, H_m be the connected components of H and order of H_i be h_i for $i = 1, \dots, m$. Note that it suffices to show that $h_1 + \dots + h_m \leq 12$.

We first prove that each H_i is a path or a singleton. Note that it suffices to show that maximum degree of H is at most 2 and H contains no cycle. Suppose that a vertex v of H has three neighbors, say v_1, v_2, v_3 , in H . Then contraction of the edges ab and xy gives rise to a $K_{3,3}$ with parts $\{ab, xy, v\}$ and $\{v_1, v_2, v_3\}$, which is a contradiction. Therefore, every vertex in H has at most two neighbors in H . Suppose that H has a cycle, say v_1, v_2, \dots, v_k . Contract the edge $v_k v_{k-1}$ and denote the new point by v_{k-1} . Then contract the edge $v_{k-1} v_{k-2}$ and denote the new point by v_{k-2} and so on. Follow this process until we get a triangle v_1, v_2, v_3 . Then contracting the edges ab and xy yields a K_5 with vertices ab, xy, v_1, v_2, v_3 , which is a contradiction. Thus, H has no cycle and hence, H is a disjoint union of paths and singletons.

We next show that for every vertex $u \in H$ we have $|N(u) \cap \{a, b, x, y\}| \geq 3$ or $|(N(u) \cup N(v)) \cap \{a, b, x, y\}| \geq 3$ for some neighbor $v \in V(H)$ of u . Since both ab and xy are dominating edges, the intersection $N(u) \cap \{a, b, x, y\}$ has at least two elements: one from $\{a, b\}$ and one from $\{x, y\}$. Consider the case when $|N(u) \cap \{a, b, x, y\}| = 2$. Without loss of generality, let $N(u) \cap \{a, b, x, y\} = \{a, x\}$. Since the minimum degree of G is at least 3, there is no vertex $v \in G$ such that $N(v) = \{a, x\}$. Hence, by Proposition 3.2 the set $\{a, x\}$ is not a vertex cover of G_{de} . Then, there exists an edge wv of G_{de} such that $\{w, v\} \cap \{a, x\} = \emptyset$. Thus, as $\nu(G_{de}) = 2$ and $ab, xy \in G_{de}$, we have $wv = by$ or $w \in \{b, y\}$ and $v \in V(H)$. Recall that wv is a dominating edge in G and hence, u is adjacent to w or v . Therefore, the case $wv = by$ is impossible and we see that v is adjacent to u . Consequently, we get $|(N(u) \cup N(v)) \cap \{a, b, x, y\}| \geq 3$ since $w \in \{b, y\}$ is a neighbor of v . Note that this result implies that if $\{u\}$ is a component of H , then u has at least three neighbors among a, b, x, y ; otherwise, contraction of the edge uv gives rise to a vertex adjacent to at least three of a, b, x, y .

We then apply the following process for each $i = 1, \dots, m$: If $h_i \leq 3$, contract the edges of H_i and obtain a singleton. If $h_i \geq 4$, let H_i be the path v_1, v_2, \dots, v_k where $k = h_i$. First, contract $v_1 v_2$ and $v_{k-1} v_k$. Then contract the paths $v_3 v_4 v_5, v_6 v_7 v_8, \dots$ and so on. Note that for every i we obtain at least $2 + \lfloor (h_i - 4)/3 \rfloor = \lfloor (h_i + 2)/3 \rfloor$ vertices adjacent to at least three of a, b, x, y . Therefore, each such vertex is adjacent to both a and b or adjacent to both x and y . Assume that the number of vertices having at least three neighbors among a, b, x, y in the resulting graph is more than 4. Then, by pigeonhole principle, there will be three distinct vertices u_1, u_2 and u_3 each of which is adjacent, without loss of generality, to both a and b . Then, contraction of the edge xy gives a $K_{3,3}$ with parts $\{a, b, xy\}$ and $\{u_1, u_2, u_3\}$, contradicting with the planarity of G . Thus, there are at most 4 vertices having at least three neighbors among a, b, x, y once the contraction process is terminated, that is, $\sum_{i=1}^m \lfloor (h_i + 2)/3 \rfloor \leq 4$. Since h_i is an integer, the inequality $h_i/3 \leq \lfloor (h_i + 2)/3 \rfloor$ holds, implying that $\sum_{i=1}^m h_i/3 \leq 4$ which yields $\sum_{i=1}^m h_i \leq 12$ as desired. \square

Propositions 3.13 and 3.14 imply that, unlike the general case stated in Corollary 2.5, there is a finite number of planar WTD(2) graphs with $\delta(G) \geq 3$.

Theorem 3.15. *If G is a planar WTD(2) graph with $\delta(G) \geq 3$, then $|V(G)| \leq 16$.*

In contrast, there is no upper bound on the number of vertices for planar WTD(2) graphs

with minimum degree 1 or 2. For example, consider a star with arbitrarily many leaves and a graph with arbitrarily many triangles sharing a common edge, respectively.

4 Girth of WTD Graphs

In this section, we provide a relation between the minimum degree and the girth for WTD graphs. We show that if the minimum degree is more than two in a WTD graph, then the graph contains a cycle of length at most twelve. It is shown in [12] that if G is a WTD graph with $\delta(G) \geq 2$, then the girth of G , $g(G)$, is at most 14.

Theorem 4.1 (Theorem 4.1 in [12]). *Suppose G is a connected graph with no leaves such that G has girth at least fifteen. Then $\gamma_t(G) < \Gamma_t(G)$.*

By following the idea in the proof of Theorem 4.1 in [12], one can find other relations between $\delta(G)$ and $g(G)$ of a WTD graph G . Before presenting such extensions, we need the following useful lemma, which is also given in [12]:

Lemma 4.2. *Let G be a WTD graph, u_1v_1, \dots, u_mv_m be a subset of the edges of G and $A = \cup_{i=1}^m \{u_i, v_i\}$. If the subgraph of G induced by A is disjoint union of m complete graphs of order 2 and $G - N[A]$ has no isolated vertices, then $G - N[A]$ is also WTD.*

Proof. Let S be a minimal TDS of $G - N[A]$. We claim that $S \cup A$ is a minimal TDS of G . It is easy to see that it is a TDS of G . Suppose that $S \cup A$ contains another TDS of G , say T . Then $T \cap S$ is a TDS of $G - N[A]$ and hence, since S is minimal we get $T \cap S = S$. Therefore, we obtain $T = S \cup A'$ where $A' \subseteq A$. If $A \setminus A'$ is nonempty, then without loss of generality we assume that $u_1 \in A \setminus A'$. But then, v_1 is not dominated by T , which is a contradiction. Therefore, we have $A' = A$, which implies that $T = S \cup A$, that is, $S \cup A$ is minimal.

As every minimal TDS of G has the same size, $|S| + 2m$ is independent of S and hence, $G - N[A]$ is a WTD graph as well. \square

Theorem 4.3. *If G is a WTD graph with $\delta(G) \geq 3$, then $g(G) \leq 12$.*

Proof. Assume that G is a WTD graph with $\delta(G) \geq 3$ and $g(G) \geq 13$. Let $P = v_1, v_2, v_3, v_4, v_5$ be a path in G . For any vertex v in G , let $d_P(v) = \min_{1 \leq i \leq 5} \text{dist}(v, v_i)$. Define N_k to be the set of vertices v with $d_P(v) = k$ for $k = 1, 2, \dots$.

First note that every vertex in N_k has a neighbor in N_{k-1} for every $k \geq 2$. Moreover, for $k = 1, 2, 3$, N_k is an independent set since otherwise we obtain a cycle of length at most 11. We will now show that for $k = 1, \dots, 4$, any vertex in N_k has at least one neighbor in N_{k+1} . Suppose that there exist $k \leq 4$ and $v \in N_k$ such that v is adjacent to no vertex in N_{k+1} . By definition, it is clear that v has no neighbor in N_l for any $l \geq k + 2$. Therefore, all the neighbors of v are in $\cup_{1 \leq i \leq k} N_i$. Thus, as v has at least three neighbors, there exist three paths from v to P such that one of them has length k and two of them have length at most $k + 1$. By a simple case analysis, considering the vertices of these paths on P gives that there exist a cycle of length at most $2k + 3 \leq 11$, which is a contradiction.

Now, let $N_2 = \{w_1, \dots, w_m\}$. For every $i = 1, \dots, m$, choose a neighbor of w_i in N_3 , say u_i . Let $A = \cup_{i=1}^m \{w_i, u_i\}$. For any $i \neq j$, w_i is not adjacent to u_j because otherwise we obtain a cycle of length at most 10. Therefore, the induced subgraph of G induced by A is a disjoint union of m complete graphs of order 2.

Next, consider the graph $H = G - N[A]$. Note that $N[A]$ consists of N_1, N_2, N_3 and some vertices in N_4 . Therefore, P is a connected component of H . As any vertex in N_4 has a neighbor in N_5 , no vertex $v \in N_4 \cap V(H)$ is isolated in H . Clearly, no vertex in N_k with $k \geq 6$ is isolated in H since it has a neighbor in N_{k-1} . Suppose to the contrary that a vertex v in N_5 is isolated in H . Then v has no neighbor in N_5 and N_6 , and thus, all its neighbors are in N_4 . Therefore, since there exist three paths from v to P , this yields a cycle of length at most 12, which is a contradiction. Consequently, H has no isolated vertices and we can apply Lemma 4.2 and conclude that H is a WTD graph. However, P is a component of H and hence, it should be WTD as well. Nevertheless, a path of length 4 is not a WTD graph (both $\{v_1, v_2, v_4, v_5\}$ and $\{v_2, v_3, v_4\}$ are minimal TDSs of P), which is a contradiction. \square

5 Conclusion

In this work, we studied graphs whose all minimal total dominating sets have the same size. We say these graphs are well-totally-dominated. We proved that well-totally-dominated graphs with bounded total domination number can be recognized in polynomial time. We then analyzed well-totally-dominated graphs with total domination number two for the special cases of triangle-free graphs and planar graphs. Finally, we focused on the girth of well-totally-dominated graphs. In particular, we proved that a well-totally-dominated graph with minimum degree at least three has girth at most 12. We now conclude with several future research directions.

Although we proved in this paper that the problem of recognizing well-totally-dominated graphs with bounded total domination number can be solved in polynomial time, the complexity of the general case is an open research problem. Hence, we pose the following question:

Problem-1: What is the computational complexity of recognizing well-totally-dominated graphs?

We have already characterized WTD(2) graphs with packing number $\rho(G) = 2$ in Theorem 3.5. Since WTD(2) graphs have $\rho(G) \leq 2$, in order to complete the characterization of all WTD(2) graphs, it remains to answer the following question:

Problem-2: What are WTD(2) graphs with $\rho(G) = 1$?

Along the same line, one may consider to generalize our result in Theorem 3.5. It is well known that $\rho(G) \leq \gamma_t(G) \leq \Gamma_t(G)$; hence graphs with $\rho(G) = \Gamma_t(G)$ form a subclass of WTD graphs. This suggests our next open problem:

Problem-3: What are WTD(k) graphs with $\rho(G) = k$?

Lastly, we have shown in Theorem 3.15 that planar WTD(2) graphs with $\delta(G) \geq 3$ have at most 16 vertices. Our intuition is that 16 is not a tight bound. Thus, we pose the following question:

Problem-4: Is the upper bound of 16 for the number of vertices of a planar WTD(2)

graph with $\delta(G) \geq 3$ tight? Can we determine all (finitely many) planar WTD(2) graphs?

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