

Trivalent dihedrants and bi-dihedrants*

Mi-Mi Zhang 

School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, 050024, China

Jin-Xin Zhou [†] 

Department of Mathematics, Beijing Jiaotong University, Beijing, 100044, China

Received 3 July 2020, accepted 22 February 2021, published online 21 October 2021

Abstract

A Cayley (resp. bi-Cayley) graph on a dihedral group is called a *dihedrant* (resp. *bi-dihedrant*). In 2000, a classification of trivalent arc-transitive dihedrants was given by Marušič and Pisanski, and several years later, trivalent non-arc-transitive dihedrants of order $4p$ or $8p$ (p a prime) were classified by Feng et al. As a generalization of these results, our first result presents a classification of trivalent non-arc-transitive dihedrants. Using this, a complete classification of trivalent vertex-transitive non-Cayley bi-dihedrants is given, thus completing the study of trivalent bi-dihedrants initiated in our previous paper [Discrete Math. 340 (2017) 1757–1772]. As a by-product, we generalize a theorem in [The Electronic Journal of Combinatorics 19 (2012) #P53].

Keywords: Cayley graph, non-Cayley, bi-Cayley, dihedral group, dihedrant, bi-dihedrant.

Math. Subj. Class. (2020): 05C25, 20B25

1 Introduction

In this paper we describe an investigation of trivalent Cayley graphs on dihedral groups as well as vertex-transitive trivalent bi-Cayley graphs over dihedral groups. To be brief, we shall say that a Cayley (resp. bi-Cayley) graph on a dihedral group a *dihedrant* (resp. *bi-dihedrant*).

Cayley graphs are usually defined in the following way. Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\text{Cay}(G, S)$ on G with respect to S is a graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. For any $g \in G$, $R(g)$ is the

*This work was supported by the National Natural Science Foundation of China (11671030, 12101181) and the Natural Science Foundation of Hebei Province (A2019205180)

[†]Corresponding author.

E-mail addresses: mmzhang@hebtu.edu.cn (Mi-Mi Zhang), jxzhou@bjtu.edu.cn (Jin-Xin Zhou)

permutation of G defined by $R(g) : x \mapsto xg$ for $x \in G$. Set $R(G) := \{R(g) \mid g \in G\}$. It is well-known that $R(G)$ is a subgroup of $\text{Aut}(\text{Cay}(G, S))$. We say that the Cayley graph $\text{Cay}(G, S)$ is *normal* if $R(G)$ is normal in $\text{Aut}(\text{Cay}(G, S))$ (see [19]).

In 2000, Marušič and Pisanski [13] initiated the study of automorphisms of dihedrants, and they gave a classification of trivalent arc-transitive dihedrants. Following this work, highly symmetrical dihedrants have been extensively studied, and one of the remarkable achievements is the complete classification of 2-arc-transitive dihedrants (see [7, 12]). In contrast, however, relatively little is known about the automorphisms of non-arc-transitive dihedrants. In [1], the authors claimed that every trivalent non-arc-transitive dihedrant is normal. However, this is not true. There exist non-arc-transitive and non-normal dihedrants. Actually, in [22, 26], the automorphism groups of trivalent dihedrants of order $4p$ and $8p$ are determined for each prime p , and the result reveals that every non-arc-transitive trivalent dihedrant of order $4p$ or $8p$ is either a normal Cayley graph, or isomorphic to the so-called cross ladder graph. For an integer $m \geq 2$, the *cross ladder graph*, denoted by CL_{4m} , is a trivalent graph of order $4m$ with vertex set $V_0 \cup V_1 \cup \dots \cup V_{2m-2} \cup V_{2m-1}$, where $V_i = \{x_i^0, x_i^1\}$, and edge set $\{\{x_{2i}^r, x_{2i+1}^r\}, \{x_{2i+1}^s, x_{2i+2}^s\} \mid i \in \mathbb{Z}_m, r, s \in \mathbb{Z}_2\}$ (see Fig. 1 for CL_{4m}). It is worth mentioning that the cross ladder graph plays an important role in the

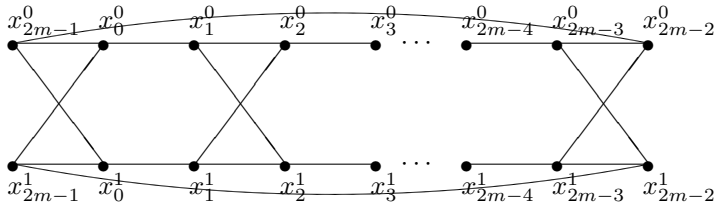


Figure 1: The cross ladder graph CL_{4m}

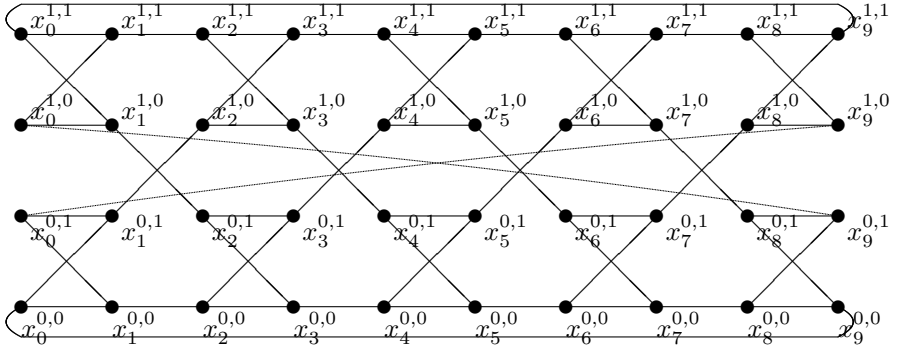
study of automorphisms of trivalent graphs (see, for example, [5, 21, 26]). Motivated by the above mentioned facts, we shall focus on trivalent non-arc-transitive dihedrants. Our first theorem generalizes the results in [22, 26] to all trivalent dihedrants.

Theorem 1.1. *Let $\Sigma = \text{Cay}(H, S)$ be a connected trivalent Cayley graph, where $H = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle (n \geq 3)$. If Σ is non-arc-transitive and non-normal, then n is even and $\Sigma \cong \text{CL}_{4 \cdot \frac{n}{2}}$ and $S^\alpha = \{b, ba, ba^{\frac{n}{2}}\}$ for some $\alpha \in \text{Aut}(H)$.*

Recall that for an integer $m \geq 2$, the cross ladder graph CL_{4m} has vertex set $V_0 \cup V_1 \cup \dots \cup V_{2m-2} \cup V_{2m-1}$, where $V_i = \{x_i^0, x_i^1\}$. The *multi-cross ladder graph*, denoted by $\text{MCL}_{4m,2}$, is the graph obtained from CL_{4m} by blowing up each vertex x_i^r of CL_{4m} into two vertices $x_i^{r,0}$ and $x_i^{r,1}$. The edge set is $\{\{x_{2i}^{r,s}, x_{2i+1}^{r,t}\}, \{x_{2i+1}^{r,s}, x_{2i+2}^{s,r}\} \mid i \in \mathbb{Z}_m, r, s, t \in \mathbb{Z}_2\}$ (see Fig. 2 for $\text{MCL}_{20,2}$).

Note that the multi-cross ladder graph $\text{MCL}_{4m,2}$ is just the graph given in [23, Definition 7]. From [6, Proposition 3.3] we know that every $\text{MCL}_{4m,2}$ is vertex-transitive. However, not all multi-cross ladder graphs are Cayley graphs. Actually, in [23, Theorem 9], it is proved that $\text{MCL}_{4p,2}$ is a vertex-transitive non-Cayley graph for each prime $p > 7$. Our second theorem generalizes this result to all multi-cross ladder graphs.

Theorem 1.2. *The multi-cross ladder graph $\text{MCL}_{4m,2}$ is a Cayley graph if and only if either m is even, or m is odd and $3 \mid m$.*


 Figure 2: The multi-cross ladder graph $MCL_{20,2}$

Both of the above two theorems are crucial in attacking the problem of classification of trivalent vertex-transitive non-Cayley bi-dihedrants. Before proceeding, we give some background to this topic, and set some notation.

Let R, L and S be subsets of a group H such that $R = R^{-1}$, $L = L^{-1}$ and $R \cup L$ does not contain the identity element of H . The *bi-Cayley graph* $\text{BiCay}(H, R, L, S)$ over H relative to R, L, S is a graph having vertex set the union of the *right part* $H_0 = \{h_0 \mid h \in H\}$ and the *left part* $H_1 = \{h_1 \mid h \in H\}$, and edge set the union of the *right edges* $\{\{h_0, g_0\} \mid gh^{-1} \in R\}$, the *left edges* $\{\{h_1, g_1\} \mid gh^{-1} \in L\}$ and the *spokes* $\{\{h_0, g_1\} \mid gh^{-1} \in S\}$. If $|R| = |L| = s$, then $\text{BiCay}(H, R, L, S)$ is said to be an *s-type bi-Cayley graph*.

In [20] we initiated a program to investigate the automorphism groups of the trivalent vertex-transitive bi-dihedrants. This was partially motivated by the following facts. As one of the most important finite graphs, the Petersen graph is a bi-circulant, but it is not a Cayley graph. Note that a *bi-circulant* is a bi-Cayley graph over a cyclic group. The Petersen graph is the initial member of a family of graphs $P(n, t)$, known now as the *generalized Petersen graphs* (see [17]), which can be also constructed as bi-circulants. Let $n \geq 3$, $1 \leq t < n/2$ and set $H = \langle a \rangle \cong \mathbb{Z}_n$. The generalized Petersen graph $P(n, t)$ is isomorphic to the bi-circulant $\text{BiCay}(H, \{a, a^{-1}\}, \{a^t, a^{-t}\}, \{1\})$. The complete classification of vertex-transitive generalized Petersen graphs has been worked out in [8, 14]. Latter, this was generalized by Marušič et al. in [13, 15] where all trivalent vertex-transitive bi-circulants were classified, and more recently, all trivalent vertex-transitive bi-Cayley graphs over abelian groups were classified in [24]. The characterization of trivalent vertex-transitive bi-dihedrants is the next natural step.

Another motivation for us to consider trivalent vertex-transitive bi-dihedrants comes from the excellent work in a highly cited article [16], where the authors give a census of trivalent vertex-transitive graphs of order up to 1280. This is very important in the study of trivalent vertex-transitive graphs. Actually, by checking this census of graphs of order up to 1000, we find out that there are 981 non-Cayley graphs, and among these graphs, 233 graphs are non-Cayley bi-dihedrants. This may suggest bi-dihedrants form an important class of trivalent vertex-transitive non-Cayley graphs.

In [20], we gave a classification of trivalent arc-transitive bi-dihedrants, and we also proved that every trivalent vertex-transitive 0- or 1-type bi-dihedrant is a Cayley graph, and gave a classification of trivalent vertex-transitive non-Cayley bi-dihedrants of order $4n$ with

n odd. The goal of this paper is to complete the classification of trivalent vertex-transitive non-Cayley bi-dihedrants.

Before stating the main result, we need the following concepts. For a bi-Cayley graph $\Gamma = \text{BiCay}(H, R, L, S)$ over a group H , we can assume that the identity 1 of H is in S (see Proposition 2.3 (2)). The triple (R, L, S) of three subsets R, L, S of a group H is called *bi-Cayley triple* if $R = R^{-1}, L = L^{-1}$, and $1 \in S$. Two bi-Cayley triples (R, L, S) and (R', L', S') of a group H are said to be *equivalent*, denoted by $(R, L, S) \equiv (R', L', S')$, if either $(R', L', S') = (R, L, S)^\alpha$ or $(R', L', S') = (L, R, S^{-1})^\alpha$ for some automorphism α of H . The bi-Cayley graphs corresponding to two equivalent bi-Cayley triples of the same group are isomorphic (see Proposition 2.3 (3)-(4)).

Theorem 1.3. *Let $\Gamma = \text{BiCay}(R, L, S)$ be a trivalent vertex-transitive bi-dihedrant where $H = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$ is a dihedral group. Then either Γ is a Cayley graph or one of the following occurs:*

- (1) $(R, L, S) \equiv (\{b, ba\}, \{a, a^{-1}\}, \{1\})$, where $n = 5$.
- (2) $(R, L, S) \equiv (\{b, ba^{\ell+1}\}, \{ba, ba^{\ell^2+\ell+1}\}, \{1\})$, where $n \geq 5, \ell^3 + \ell^2 + \ell + 1 \equiv 0 \pmod{n}, \ell^2 \not\equiv 1 \pmod{n}$.
- (3) $(R, L, S) \equiv (\{ba^{-\ell}, ba^\ell\}, \{a, a^{-1}\}, \{1\})$, where $n = 2m$ and $\ell^2 \equiv -1 \pmod{m}$.
Furthermore, Γ is also a bi-Cayley graph over an abelian group $\mathbb{Z}_n \times \mathbb{Z}_2$.
- (4) $(R, L, S) \equiv (\{b, ba\}, \{b, ba^{2m}\}, \{1\})$, where $n = 2(2m + 1), m \not\equiv 1 \pmod{3}$, and the corresponding graph is isomorphic the multi-cross ladder graph $\text{MCL}_{4m,2}$.
- (5) $(R, L, S) \equiv (\{b, ba\}, \{ba^{24\ell}, ba^{12\ell-1}\}, \{1\})$, where $n = 48\ell$ and $\ell \geq 1$.

Moreover, all of the graphs arising from (1)-(4) are vertex-transitive non-Cayley.

2 Preliminaries

All groups considered in this paper are finite, and all graphs are finite, connected, simple and undirected. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [3, 18].

2.1 Definitions and notations

For a positive integer, let \mathbb{Z}_n be the cyclic group of order n and \mathbb{Z}_n^* be the multiplicative group of \mathbb{Z}_n consisting of numbers coprime to n . For two groups M and $N, N \rtimes M$ denotes a semidirect product of N by M . For a subgroup H of a group G , denote $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H of G . Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G . We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular.

For a finite, simple and undirected graph Γ , we use $V(\Gamma), E(\Gamma), A(\Gamma), \text{Aut}(\Gamma)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. For any subset B of $V(\Gamma)$, the subgraph of Γ induced by B will be denoted by $\Gamma[B]$. For any $v \in V(\Gamma)$ and a positive integer i no more than the diameter of Γ , denote by $\Gamma_i(v)$ be the set of vertices at distance i from v . Clearly, $\Gamma_1(v)$ is just the neighborhood of v . We shall often abuse the notation by using $\Gamma(v)$ to replace $\Gamma_1(v)$.

A graph Γ is said to be *vertex-transitive*, and *arc-transitive* (or *symmetric*) if $\text{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$ and $A(\Gamma)$, respectively. Let Γ be a connected vertex-transitive

graph, and let $G \leq \text{Aut}(\Gamma)$ be vertex-transitive on Γ . For a G -invariant partition \mathcal{B} of $V(\Gamma)$, the *quotient graph* $\Gamma_{\mathcal{B}}$ is defined as the graph with vertex set \mathcal{B} such that, for any two different vertices $B, C \in \mathcal{B}$, B is adjacent to C if and only if there exist $u \in B$ and $v \in C$ which are adjacent in Γ . Let N be a normal subgroup of G . Then the set \mathcal{B} of orbits of N in $V(\Gamma)$ is a G -invariant partition of $V(\Gamma)$. In this case, the symbol $\Gamma_{\mathcal{B}}$ will be replaced by Γ_N . The original graph Γ is said to be a N -cover of Γ_N if Γ and Γ_N have the same valency.

2.2 Cayley graphs

Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph on G with respect to S . Then Γ is vertex-transitive due to $R(G) \leq \text{Aut}(\Gamma)$. In general, we have the following proposition.

Proposition 2.1 ([2, Lemma 16.3]). *A vertex-transitive graph Γ is isomorphic to a Cayley graph on a group G if and only if its automorphism group has a subgroup isomorphic to G , acting regularly on the vertex set of Γ .*

In 1981, Godsil [9] proved that the normalizer of $R(G)$ in $\text{Aut}(\text{Cay}(G, S))$ is $R(G) \times \text{Aut}(G, S)$, where $\text{Aut}(G, S)$ is the group of automorphisms of G fixing the set S set-wise. This result has been successfully used in characterizing various families of Cayley graphs $\text{Cay}(G, S)$ such that $R(G) = \text{Aut}(\text{Cay}(G, S))$ (see, for example, [9, 10]). Recall that a Cayley graph $\text{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\text{Aut}(\text{Cay}(G, S))$ (see [19]).

Proposition 2.2 ([19, Proposition 1.5]). *The Cayley graph $\Gamma = \text{Cay}(G, S)$ is normal if and only if $A_1 = \text{Aut}(G, S)$, where A_1 is the stabilizer of the identity 1 of G in $\text{Aut}(\Gamma)$.*

2.3 Basic properties of bi-Cayley graphs

In this subsection, we let Γ be a connected bi-Cayley graph $\text{BiCay}(H, R, L, S)$ over a group H . It is easy to prove some basic properties of such a Γ , as in [24, Lemma 3.1].

Proposition 2.3. *The following hold.*

- (1) H is generated by $R \cup L \cup S$.
- (2) Up to graph isomorphism, S can be chosen to contain the identity of H .
- (3) For any automorphism α of H , $\text{BiCay}(H, R, L, S) \cong \text{BiCay}(H, R^\alpha, L^\alpha, S^\alpha)$.
- (4) $\text{BiCay}(H, R, L, S) \cong \text{BiCay}(H, L, R, S^{-1})$.

Next, we collect several results about the automorphisms of bi-Cayley graph $\Gamma = \text{BiCay}(H, R, L, S)$. For each $g \in H$, define a permutation as follows:

$$\mathcal{R}(g) : h_i \mapsto (hg)_i, \quad \forall i \in \mathbb{Z}_2, h \in H. \quad (2.1)$$

Set $\mathcal{R}(H) = \{\mathcal{R}(g) \mid g \in H\}$. Then $\mathcal{R}(H)$ is a semiregular subgroup of $\text{Aut}(\Gamma)$ with H_0 and H_1 as its two orbits.

For an automorphism α of H and $x, y, g \in H$, define two permutations of $V(\Gamma) = H_0 \cup H_1$ as follows:

$$\begin{aligned} \delta_{\alpha, x, y} : h_0 &\mapsto (xh^\alpha)_1, h_1 \mapsto (yh^\alpha)_0, \forall h \in H, \\ \sigma_{\alpha, g} : h_0 &\mapsto (h^\alpha)_0, h_1 \mapsto (gh^\alpha)_1, \forall h \in H. \end{aligned} \quad (2.2)$$

Set

$$\begin{aligned}
 I &= \{\delta_{\alpha,x,y} \mid \alpha \in \text{Aut}(H) \text{ s.t. } R^\alpha = x^{-1}Lx, L^\alpha = y^{-1}Ry, S^\alpha = y^{-1}S^{-1}x\}, \\
 F &= \{\sigma_{\alpha,g} \mid \alpha \in \text{Aut}(H) \text{ s.t. } R^\alpha = R, L^\alpha = g^{-1}Lg, S^\alpha = g^{-1}S\}.
 \end{aligned}
 \tag{2.3}$$

Proposition 2.4 ([25, Theorem 1.1]). *Let $\Gamma = \text{BiCay}(H, R, L, S)$ be a connected bi-Cayley graph over the group H . Then $N_{\text{Aut}(\Gamma)}(\mathcal{R}(H)) = \mathcal{R}(H) \rtimes F$ if $I = \emptyset$ and $N_{\text{Aut}(\Gamma)}(\mathcal{R}(H)) = \mathcal{R}(H)\langle F, \delta_{\alpha,x,y} \rangle$ if $I \neq \emptyset$ and $\delta_{\alpha,x,y} \in I$. Furthermore, for any $\delta_{\alpha,x,y} \in I$, we have the following:*

- (1) $\langle \mathcal{R}(H), \delta_{\alpha,x,y} \rangle$ acts transitively on $V(\Gamma)$;
- (2) if α has order 2 and $x = y = 1$, then Γ is isomorphic to the Cayley graph $\text{Cay}(\bar{H}, R \cup \alpha S)$, where $\bar{H} = H \rtimes \langle \alpha \rangle$.

3 Cross ladder graphs

The goal of this section is to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that $\Sigma = \text{Cay}(H, S)$ is a connected trivalent Cayley graph which is neither normal nor arc-transitive, where $H = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle (n \geq 3)$. Then S is a generating subset of H and $|S| = 3$. So S must contain an involution of H outside $\langle a \rangle$. As $\text{Aut}(H)$ is transitive on the coset $b\langle a \rangle$, we may assume that $S = \{b, x, y\}$ for $x, y \in H \setminus \langle b \rangle$.

Suppose first that x is not an involution. Then we must have $y = x^{-1}$. Since S generates H , one has $\langle a \rangle = \langle x \rangle$, and so $bx = x^{-1}$. Then there exists an automorphism of H sending b, x to b, a respectively. So we may assume that $S = \{b, a, a^{-1}\}$. Now it is easy to check that Σ is isomorphic to the generalized Petersen graph $P(n, 1)$. Since Σ is not arc-transitive, by [8, 14], we have $|\text{Aut}(\Sigma)| = 2|H|$, and so Σ would be a normal Cayley graph of H , a contradiction.

Therefore, both x and y must be involutions. Suppose that $x \in \langle a \rangle$. Then n is even and $x = a^{n/2}$. Again since S generates H , one has $y = ba^j$, where $1 \leq j \leq n - 1$ and either $(j, n) = 1$ or $(j, n) = 2$ and $\frac{n}{2}$ is odd. Note that the subgroup of $\text{Aut}(H)$ fixing b is transitive on the set of generators of $\langle a \rangle$ and that $\langle a^{n/2} \rangle$ is the center of H . There exists $\alpha \in \text{Aut}(H)$ such that

$$S^\alpha = \{b, ba, a^{\frac{n}{2}}\} \text{ or } \{b, ba^2, a^{\frac{n}{2}}\}.$$

Without loss of generality, we may assume that $S = \{b, ba, a^{\frac{n}{2}}\}$ or $\{b, ba^2, a^{\frac{n}{2}}\}$. If $S = \{b, ba^2, a^{\frac{n}{2}}\}$, we shall prove that $\Sigma \cong P(n, 1)$. Note that the generalized Petersen graph $P(n, 1)$ has vertex set $\{u_i, v_i \mid i \in \mathbb{Z}_n\}$ and edge set $\{\{u_i, u_{i+1}\}, \{v_i, v_{i+1}\}, \{u_i, v_i\} \mid i \in \mathbb{Z}_n\}$. Define a map from $V(\Sigma)$ to $V(P(n, 1))$ as follows:

$$\begin{aligned}
 \varphi : \quad & a^{2i} \mapsto u_{2i}, & a^{2i+\frac{n}{2}} \mapsto v_{2i}, \\
 & ba^{2i} \mapsto u_{2i-1}, & ba^{2i+\frac{n}{2}} \mapsto v_{2i-1},
 \end{aligned}$$

where $0 \leq i \leq \frac{n}{2} - 1$. It is easy to see that φ is an isomorphism from Σ to $P(n, 1)$. Since Σ is not arc-transitive, by [8, 14], we have $|\text{Aut}(\Sigma)| = 2|H|$, and so Σ would be a normal Cayley graph of H , a contradiction. If $S = \{b, ba, a^{\frac{n}{2}}\}$, then Σ has a connected subgraph $\Sigma_1 = \text{Cay}(H, \{b, ba\})$ which is a cycle of length $2n$, and Σ is just the graph obtained from

Σ_1 by adding a 1-factor such that each vertex g of Σ_1 is adjacent to its antipodal vertex $a^{\frac{n}{2}}g$. Then $R(H) \rtimes \mathbb{Z}_2 \cong \text{Aut}(\Sigma_1) \leq \text{Aut}(\Sigma)$, and then since Σ is assumed to be not arc-transitive, $\text{Aut}(\Sigma)$ will fix the 1-factor $\{\{g, a^{\frac{n}{2}}g\} \mid g \in H\}$ setwise. This implies that $\text{Aut}(\Sigma) \leq \text{Aut}(\Sigma_1)$ and so $\text{Aut}(\Sigma) = \text{Aut}(\Sigma_1)$. Consequently, we have Σ is a normal Cayley graph of H , a contradiction.

Similarly, we have $y \notin \langle a \rangle$. Then we may assume that $x = ba^i$ and $y = ba^j$ for some $1 \leq i, j \leq n-1$ and $i \neq j$. Then $S = \{b, ba^i, ba^j\} \subseteq b\langle a \rangle$. This implies that Σ is a bipartite graph with $\langle a \rangle$ and $b\langle a \rangle$ as its two partition sets. Since Σ is not arc-transitive, $\text{Aut}(\Sigma)_1$ is intransitive on the neighbourhood S of 1, and since Σ is not a normal Cayley graph of H , there exists a unique element, say $s \in S$, such that $\text{Aut}(\Sigma)_1 = \text{Aut}(\Sigma)_s$. Considering the fact that $\text{Aut}(H)$ is transitive on $b\langle a \rangle$, without loss of generality, we may assume that $\text{Aut}(\Sigma)_1 = \text{Aut}(\Sigma)_b$ and $\text{Aut}(\Sigma)_1$ swaps ba^i and ba^j . Then for any $h \in H$, we have

$$\text{Aut}(\Sigma)_h = (\text{Aut}(\Sigma)_1)^{R(h)} = (\text{Aut}(\Sigma)_b)^{R(h)} = \text{Aut}(\Sigma)_{bh}.$$

Direct computation shows that

$$\begin{aligned} \Sigma_2(1) &= \{a^{-i}, a^{-j}, a^i, a^{i-j}, a^j, a^{j-i}\}, \\ \Sigma_3(1) &= \{ba^{-i}, ba^{j-i}, ba^{-j}, ba^{i-j}, ba^{2i}, ba^{j+i}, ba^{2i-j}, ba^{2j}, ba^{2j-i}\}. \end{aligned}$$

Let $\text{Aut}(\Sigma)_1^*$ be the kernel of $\text{Aut}(\Sigma)_1$ acting on S . Take an $\alpha \in \text{Aut}(\Sigma)_1^*$. Then α fixes every element in S . As $\text{Aut}(\Sigma)_h = \text{Aut}(\Sigma)_{bh}$ for any $h \in H$, α will fix $b(ba^i) = a^i$ and $b(ba^j) = a^j$. Note that $\Sigma(ba^i) \setminus \{1, a^i\} = \{a^{i-j}\}$ and $\Sigma(ba^j) \setminus \{1, a^j\} = \{a^{j-i}\}$. Then α also fixes a^{i-j} and a^{j-i} , and then α also fixes ba^{i-j} and ba^{j-i} .

If $|\Sigma_2(1)| = 6$, then it is easy to check that a^{-i} is the unique common neighbor of b and ba^{j-i} . So α also fixes a^{-i} . Now one can see that α fixes every vertex in $\Sigma_2(1)$. If $|\Sigma_2(1)| < 6$ and either $|\Sigma_1(b) \cap \Sigma_1(ba^i)| > 1$ or $|\Sigma_1(b) \cap \Sigma_1(ba^j)| > 1$, then α also fixes every vertex in $\Sigma_2(1)$. In the above two cases, by the connectedness and vertex-transitivity of Σ , α would fix all vertices of Σ , implying that $\alpha = 1$. Hence, $\text{Aut}(\Sigma)_1^* = 1$ and $\text{Aut}(\Sigma)_1 \cong \mathbb{Z}_2$. This forces that Σ is a normal Cayley graph of H , a contradiction.

Thus, we have $|\Sigma_2(1)| < 6$ and $|\Sigma_1(b) \cap \Sigma_1(ba^i)| = |\Sigma_1(b) \cap \Sigma_1(ba^j)| = 1$. This implies that $\Sigma_1(ba^i) \cap \Sigma_1(ba^j) = \{1, a^{i-j}\} = \{1, a^{j-i}\}$, and so $a^{i-j} = a^{j-i}$. It follows that a^{i-j} is an involution, and hence n is even and $a^{i-j} = a^{n/2}$. So $S = \{b, ba^i, ba^{i+n/2}\}$. As S generates H , one has $\langle a^i, a^{n/2} \rangle = \langle a \rangle$. So either $(i, n) = 1$ or $(i, n) = 2$ and $\frac{n}{2}$ is odd. Note that the subgroup of $\text{Aut}(H)$ fixing b is transitive on the set of generators of $\langle a \rangle$ and that $\langle a^{n/2} \rangle$ is the center of H . There exists $\alpha \in \text{Aut}(H)$ such that

$$S^\alpha = \{b, ba, ba^{1+\frac{n}{2}}\} \text{ or } \{b, ba^2, ba^{2+\frac{n}{2}}\}.$$

Let β_ϵ be the automorphism of H induced by the map $a \mapsto a^{-1}, b \mapsto ba^\epsilon$, where $\epsilon \in \mathbb{Z}_2$. Then

$$\{b, ba, ba^{1+\frac{n}{2}}\}^{\beta_1} = \{b, ba, ba^{\frac{n}{2}}\}, \text{ and } \{b, ba^2, ba^{2+\frac{n}{2}}\}^{\beta_2} = \{b, ba^2, ba^{\frac{n}{2}}\}.$$

If $\frac{n}{2}$ is odd, then the map $\eta : a \mapsto a^{2+\frac{n}{2}}, b \mapsto ba^{\frac{n}{2}}$ induces an automorphism of H , and $\{b, ba, ba^{\frac{n}{2}}\}^\eta = \{b, ba^2, ba^{\frac{n}{2}}\}$. So there always exists $\gamma \in \text{Aut}(H)$ such that $S^\gamma = \{b, ba, ba^{\frac{n}{2}}\}$, completing the proof of the first part of our theorem.

Finally, we shall prove $\Sigma \cong \text{CL}_4.\frac{n}{2}$. Without loss of generality, assume that $S = \{b, ba, ba^{\frac{n}{2}}\}$. Recall that $V(\text{CL}_4.\frac{n}{2}) = \{x_i^r \mid i \in \mathbb{Z}_{2n}, r \in \mathbb{Z}_2\}$ and $E(\text{CL}_4.\frac{n}{2}) =$

$\{\{x_i^r, x_{i+1}^r\}, \{x_{2i}^r, x_{2i+1}^r\}, | i \in \mathbb{Z}_{2n}, r \in \mathbb{Z}_2\}$. Let ϕ be a map from $V(\Sigma)$ to $V(\text{CL}_{4, \frac{n}{2}})$ as following:

$$\phi : \begin{matrix} a^i \mapsto x_{2i}^0, & a^{i+\frac{n}{2}} \mapsto x_{2i}^1, \\ ba^j \mapsto x_{2j-1}^0, & ba^{j+\frac{n}{2}} \mapsto x_{2j-1}^1, \end{matrix}$$

where $0 \leq i \leq \frac{n}{2} - 1$ and $1 \leq j \leq \frac{n}{2}$. It is easy to check that ϕ is an isomorphism from Σ and $X(\text{CL}_{4, \frac{n}{2}})$, as desired. □

4 Multi-cross ladder graphs

The goal of this section is to prove Theorem 1.2. We first show that each $\text{MCL}_{4m,2}$ is a bi-Cayley graph.

Lemma 4.1. *The multi-cross ladder graph $\text{MCL}_{4m,2}$ is isomorphic to the bi-Cayley graph $\text{BiCay}(H, \{c, ca\}, \{ca, ca^2b\}, \{1\})$, where*

$$H = \langle a, b, c \mid a^m = b^2 = c^2 = 1, a^b = a, a^c = a^{-1}, b^c = b \rangle.$$

Proof. For convenience, let Γ be the bi-Cayley graph given in our lemma, and let $X = \text{MCL}_{4m,2}$. Let ϕ be a map from $V(X)$ to $V(\Gamma)$ defined by the following rule:

$$\phi : \begin{matrix} x_{2t}^{1,1} \mapsto (a^t)_0, & x_{2t+1}^{1,1} \mapsto (ca^{t+1})_0, & x_{2t}^{1,0} \mapsto (ca^{t+1})_1, & x_{2t+1}^{1,0} \mapsto (a^t)_1, \\ x_{2t}^{0,1} \mapsto (ca^{t+1}b)_1, & x_{2t+1}^{0,1} \mapsto (a^tb)_1, & x_{2t}^{0,0} \mapsto (a^tb)_0, & x_{2t+1}^{0,0} \mapsto (ca^{t+1}b)_0, \end{matrix}$$

where $t \in \mathbb{Z}_m$.

It is easy to see that ϕ is an adjacency preserving isomorphism from X to Γ . □

Remark 1 Let m be odd, let $e = ab$ and $f = ca$. Then the group given in Lemma 4.1 has the following presentation:

$$H = \langle e, f \mid e^{2m} = f^2 = 1, e^f = e^{-1} \rangle.$$

Clearly, in this case, H is a dihedral group. Furthermore, the corresponding bi-Cayley graph given in Lemma 4.1 will be

$$\text{BiCay}(H, \{f, fe\}, \{f, fe^{m-1}\}, \{1\}).$$

Proof of Theorem 1.2. By Lemma 4.1, we may let $\Gamma = \text{MCL}_{4m,2}$ be just the bi-Cayley graph $\text{BiCay}(H, R, L, S)$, where

$$\begin{matrix} H = \langle a, b, c \mid a^m = b^2 = c^2 = 1, a^b = a, a^c = a^{-1}, b^c = b \rangle, \\ R = \{c, ca\}, L = \{ca, ca^2b\}, S = \{1\}. \end{matrix}$$

We first prove the sufficiency. Assume first that m is even. Then the map

$$a \mapsto ab, b \mapsto b, c \mapsto cb$$

induces an automorphism, say α of H of order 2. Furthermore, $R^\alpha = \{c, ca\}^\alpha = caLca$, $L^\alpha = \{ca, ca^2b\}^\alpha = caRca$ and $S^\alpha = \{1\}^\alpha = ca\{1\}ca = S^{-1}$. By Proposition 2.4, $\delta_{\alpha,ca,ca} \in \text{Aut}(\Gamma)$ and $\mathcal{R}(H) \rtimes \langle \delta_{\alpha,ca,ca} \rangle$ acts regularly on $V(\Gamma)$. Consequently, by Proposition 2.1, Γ is a Cayley graph.

Assume now that m is odd and $3 \mid m$. In this case, we shall use the bi-Cayley presentation for Γ as in Remark 5.1, that is,

$$\Gamma = \text{BiCay}(H, \{f, fe\}, \{f, fe^{m-1}\}, \{1\}),$$

where

$$H = \langle e, f \mid e^{2m} = f^2 = 1, ef = e^{-1} \rangle.$$

Let β be a permutation of $V(\Gamma)$ defined as following:

$$\begin{aligned} \beta : \quad & (f^i e^{3t+1})_i \leftrightarrow (f^i e^{m+3t+1})_i, & (f^{i+1} e^{3t+1})_i \leftrightarrow (f^i e^{m+3t+1})_{i+1}, \\ & (f^{i+1} e^{3t+2})_i \leftrightarrow (f^{i+1} e^{m+3t+2})_i, & (f^i e^{3t+2})_i \leftrightarrow (f^{i+1} e^{m+3t+2})_{i+1}, \\ & (e^{3t})_i \leftrightarrow (f e^{3t})_{i+1}, & (e^{m+3t})_i \leftrightarrow (f e^{m+3t})_{i+1}, \end{aligned}$$

where $t \in \mathbb{Z}_{\frac{m}{3}}$ and $i \in \mathbb{Z}_2$. It is easy to check that β is an automorphism of Γ of order 2. Furthermore, $\mathcal{R}(e)$, $\mathcal{R}(f)$ and β satisfy the following relations:

$$\begin{aligned} \mathcal{R}(e)^{2m} = \mathcal{R}(f)^2 = \beta^2 = 1, \quad \mathcal{R}(f)^{-1} \mathcal{R}(e) \mathcal{R}(f) = \mathcal{R}(e)^{-1}, \quad \mathcal{R}(f)^{-1} \beta \mathcal{R}(f) = \beta, \\ \mathcal{R}(e)^6 \beta = \beta \mathcal{R}(e)^6, \quad \mathcal{R}(e)^2 \beta = \beta \mathcal{R}(e)^4 \beta \mathcal{R}(e)^{-2}. \end{aligned}$$

Let $G = \langle \mathcal{R}(e^2), \mathcal{R}(f), \beta \rangle$ and $P = \langle \mathcal{R}(e^2), \beta \rangle$. Then $\mathcal{R}(f) \notin P$ and $G = P \langle \mathcal{R}(f) \rangle$. Since $\mathcal{R}(e)^6 \beta = \beta \mathcal{R}(e)^6$, we have $\mathcal{R}(e^6) \in Z(P)$. Since $\mathcal{R}(e)^2 \beta = \beta \mathcal{R}(e)^4 \beta \mathcal{R}(e)^{-2}$, it follows that

$$(\mathcal{R}(e)^2 \beta)^3 = \mathcal{R}(e)^2 \beta [\beta \mathcal{R}(e)^4 \beta \mathcal{R}(e)^{-2}] \mathcal{R}(e)^2 \beta = \mathcal{R}(e^6).$$

Let $N = \langle \mathcal{R}(e^6) \rangle$. Clearly, N is a normal subgroup of G . Furthermore,

$$P/N = \langle \mathcal{R}(e^2)N, \beta N \mid \mathcal{R}(e^2)^3 N = \beta^2 N = (\mathcal{R}(e^2)\beta)^3 N = N \rangle \cong A_4.$$

Therefore, $|P| = 4m$ and $|G| \leq 8m$.

Let

$$\begin{aligned} \Delta_{00} = \{x_0 \mid x \in \langle e^2, f \rangle\}, \quad \Delta_{10} = \{(ex)_0 \mid x \in \langle e^2, f \rangle\}, \\ \Delta_{01} = \{x_1 \mid x \in \langle e^2, f \rangle\}, \quad \Delta_{11} = \{(ex)_1 \mid x \in \langle e^2, f \rangle\}. \end{aligned}$$

Then Δ_{ij} 's ($i, j \in \mathbb{Z}_2$) are four orbits of $\langle \mathcal{R}(e^2), \mathcal{R}(f) \rangle$. Moreover,

$$1_0^{\beta \mathcal{R}(f)} = 1_1 \in \Delta_{01}, \quad e_0^\beta = (e^{m+1})_0 \in \Delta_{00}, \quad e_1^\beta = (f e^{m+1})_0 \in \Delta_{00}.$$

This implies that G is transitive on $V(\Gamma)$. Hence, $|G| = 8m$ and so G is regular on $V(\Gamma)$, and by Proposition 2.1, Γ is a Cayley graph.

To prove the necessity, it suffices to prove that if m is odd and $3 \nmid m$, then Γ is a non-Cayley graph. In this case, we shall use the original definition of $\Gamma = \text{MCL}_{4m,2}$. Suppose that m is odd and $3 \nmid m$. We already know from [6, Proposition 3.3] that Γ is vertex-transitive. Let $A = \text{Aut}(\Gamma)$. For $m = 5$ or 7 , using Magma [4], Γ is a non-Cayley graph. In what follows, we assume that $m \geq 11$.

For each $j \in \mathbb{Z}_m$, $C_j^0 = (x_{2j}^{0,0}, x_{2j+1}^{0,0}, x_{2j}^{0,1}, x_{2j+1}^{0,1})$ and $C_j^1 = (x_{2j}^{1,1}, x_{2j+1}^{1,1}, x_{2j}^{1,0}, x_{2j+1}^{1,0})$ are two 4-cycles. Set $\mathcal{F} = \{C_j^i \mid i \in \mathbb{Z}_2, j \in \mathbb{Z}_m\}$. From the construction of $\Gamma = \text{MCL}_{4m,2}$, it is easy to see that in $\Gamma = \text{MCL}_{4m,2}$ passing each vertex there is exactly one 4-cycle, which belongs to \mathcal{F} . Clearly, any two distinct 4-cycles in \mathcal{F} are vertex-disjoint. This implies that $\Delta = \{V(C_j^i) \mid i \in \mathbb{Z}_2, j \in \mathbb{Z}_m\}$ is an A -invariant partition of $V(\Gamma)$. Consider

the quotient graph Γ_Δ , and let T be the kernel of A acting on Δ . Then $\Gamma_\Delta \cong C_m[2K_1]$, the lexicographic product of a cycle of length m and an empty graph of order 2. Hence $A/T \leq \text{Aut}(C_m[2K_1]) \cong \mathbb{Z}_2^m \rtimes D_{2m}$. Note that between any two adjacent vertices of Γ_Δ there is exactly one edge of $\Gamma = \text{MCL}_{4m,2}$. Then T fixes each vertex of Γ and hence $T = 1$. So we may view A as a subgroup of $\text{Aut}(\Gamma_\Delta) \cong \text{Aut}(C_m[2K_1]) \cong \mathbb{Z}_2^m \rtimes D_{2m}$.

For convenience, we will simply use the C_j^i 's to represent the vertices of Γ_Δ . Then Γ_Δ has vertex set

$$\{C_j^0, C_j^1 \mid j \in \mathbb{Z}_m\}$$

and edge set

$$\{\{C_j^0, C_{j+1}^0\}, \{C_j^1, C_{j+1}^1\}, \{C_j^0, C_{j+1}^1\}, \{C_j^1, C_{j+1}^0\} \mid j \in \mathbb{Z}_m\}.$$

Let $\mathcal{B} = \{\{C_j^0, C_j^1\} \mid j \in \mathbb{Z}_m\}$. Then \mathcal{B} is an $\text{Aut}(\Gamma_\Delta)$ -invariant partition of $V(\Gamma_\Delta)$. Let K be the kernel of $\text{Aut}(\Gamma_\Delta)$ acting on \mathcal{B} . Then $K = \langle k_0 \rangle \times \langle k_2 \rangle \times \dots \times \langle k_{m-1} \rangle$, where we use k_i to denote the transposition $(C_j^0 C_j^1)$ for $j \in \mathbb{Z}_m$. Clearly, K is the maximal normal 2-subgroup of $\text{Aut}(\Gamma_\Delta)$.

Suppose to the contrary that $\Gamma = \text{MCL}_{4m,2}$ is a Cayley graph. By Proposition 2.1, A has a subgroup, say G acting regularly on $V(\Gamma)$. Then G has order $8m$, and

$$G/(G \cap K) \cong GK/K \leq \text{Aut}(\Gamma_\Delta)/K \lesssim D_{2m}.$$

Since m odd, it follows that $|G \cap K| = 4$ or 8 , and so $G \cap K \cong \mathbb{Z}_2^2$ or \mathbb{Z}_2^3 .

If $G \cap K \cong \mathbb{Z}_2^2$, then $|GK/K| = 2m$ and $GK/K = \text{Aut}(\Gamma_\Delta)/K \cong D_{2m}$. So $GK = \text{Aut}(\Gamma_\Delta) \cong \mathbb{Z}_2^m \rtimes D_{2m}$. Let M be a Hall $2'$ -subgroup of G . Then $M \cong \mathbb{Z}_m$ and M is also a Hall $2'$ -subgroup of $\text{Aut}(\Gamma_\Delta)$. Clearly, $\text{Aut}(\Gamma_\Delta)$ is solvable, so all Hall $2'$ -subgroups of $\text{Aut}(\Gamma_\Delta)$ are conjugate. Without loss of generality, we may let $M = \langle \alpha \rangle$, where α is the following permutation on $V(\Gamma_\Delta)$:

$$\alpha = (C_0^0 C_1^0 \dots C_{m-1}^0)(C_0^1 C_1^1 \dots C_{m-1}^1).$$

Then $K \rtimes \langle \alpha \rangle$ acts transitively on $V(\Gamma_\Delta)$. Clearly, $C_K(\alpha)$ is contained in the center of $K \rtimes \langle \alpha \rangle$. So $C_K(\alpha)$ is semiregular on $V(\Gamma_\Delta)$. This implies that

$$C_K(\alpha) = \langle k_0 k_1 \dots k_{m-1} \rangle \cong \mathbb{Z}_2.$$

On the other hand, let $L = (G \cap K)M$. Clearly, $G \cap K \trianglelefteq G$, so L is a subgroup of G of order $4m$. For any odd prime factor p of m , let P be a Sylow p -subgroup of M . Then P is also a Sylow p -subgroup of L , and since M is cyclic, one has $M \leq N_L(P)$. By Sylow theorem, we have $|L : N_L(P)| = kp + 1 \mid 4$ for some integer k . Since $3 \nmid m$, one has $L = N_L(P)$. It follows that $M \trianglelefteq L$ and so $L = M \times (G \cap K)$. This implies that $G \cap K \leq C_K(M) = C_K(\alpha) \cong \mathbb{Z}_2$, a contradiction.

If $G \cap K \cong \mathbb{Z}_2^3$, then $|GK/K| = m$. Furthermore, $GK/K \cong \mathbb{Z}_m$ and GK/K acts on \mathcal{B} regularly. Since G is transitive on $V(\Gamma)$, there exists $g \in G$ such that $(x_0^{1,1})^g = x_1^{1,1}$, where $x_0^{1,1}, x_1^{1,1} \in C_0^1$. As $V(\Gamma_\Delta) = \{C_j^i \mid i \in \mathbb{Z}_2, j \in \mathbb{Z}_m\}$, g fixes the 4-cycle $C_0^1 = (x_0^{1,1}, x_1^{1,1}, x_0^{1,0}, x_1^{1,0})$. Since $\mathcal{B} = \{\{C_j^0, C_j^1\} \mid j \in \mathbb{Z}_m\}$ is also A -invariant, g fixes $\{C_0^0, C_0^1\}$ setwise. Since GK/K acts on \mathcal{B} regularly, g fixes $\{C_j^0, C_j^1\}$ setwise for every $j \in \mathbb{Z}_m$. Observe that $\{x_0^{1,1}, x_{2m-1}^{1,1}\}$ and $\{x_1^{1,1}, x_2^{1,1}\}$ are the unique edges of Γ between C_0^1 and C_{m-1}^1 , C_0^1 and C_2^1 , respectively. This implies that g will map C_{m-1}^1 to C_2^1 , contradicting that g fixes $\{C_j^0, C_j^1\}$ setwise for every $j \in \mathbb{Z}_m$. \square

5 A family of trivalent VNC bi-dihedrants

The goal of this section is to prove the following lemma which gives a new family of trivalent vertex-transitive non-Cayley bi-dihedrants. To be brief, a vertex-transitive non-Cayley graph is sometimes simply called a *VNC graph*.

Lemma 5.1. *Let $H = \langle a, b \mid a^n = b^2 = 1, a^b = a^{-1} \rangle$ be a dihedral group, where $n = 48\ell$ and $\ell \geq 1$. Then $\Gamma = \text{BiCay}(H, \{b, ba\}, \{ba^{24\ell}, ba^{12\ell-1}\}, \{1\})$ is a VNC dihedrant.*

Proof. We first define a permutation on $V(\Gamma)$ as follows:

$$g : \begin{array}{lll} (a^{3r})_0 \mapsto (a^{3r})_0, & (a^{3r})_1 \mapsto (ba^{3r})_0, & (a^{3r+1})_0 \mapsto (ba^{3r+1})_1, \\ (a^{3r+1})_1 \mapsto (a^{24\ell+3r+1})_1, & (a^{3r+2})_i \mapsto (ba^{12\ell+3r+2})_{i+1}, & (ba^{3r})_0 \mapsto (a^{3r})_1, \\ (ba^{3r})_1 \mapsto (ba^{24\ell+3r})_1, & (ba^{3r+1})_0 \mapsto (ba^{3r+1})_0, & (ba^{3r+1})_1 \mapsto (a^{3r+1})_0, \\ (ba^{3r+2})_i \mapsto (a^{-12\ell+3r+2})_{i+1}, & & \end{array}$$

where $r \in \mathbb{Z}_{16\ell}$, $i \in \mathbb{Z}_2$.

It is easy to check that g is an involution, and furthermore, for any $t \in \mathbb{Z}_{16\ell}$, we have

$$\begin{aligned} \Gamma((a^{3r})_0)^g &= \{(a^{3r})_1, (ba^{3r})_0, (ba^{3r+1})_0\} = \Gamma((a^{3r})_0), \\ \Gamma((a^{3r})_1)^g &= \{(ba^{3r})_1, (a^{3r})_0, (a^{3r-1})_0\} = \Gamma((ba^{3r})_0), \\ \Gamma((ba^{3r})_1)^g &= \{(ba^{24\ell+3r})_0, (a^{3r})_1, (a^{12\ell+3r+1})_1\} = \Gamma((ba^{24\ell+3r})_1), \\ \Gamma((a^{3r+1})_0)^g &= \{(ba^{3r+1})_0, (a^{24\ell+3r+1})_1, (a^{36\ell+3r+2})_1\} = \Gamma((ba^{3r+1})_1), \\ \Gamma((a^{3r+1})_1)^g &= \{(a^{24\ell+3r+1})_0, (ba^{3r+1})_1, (ba^{36\ell+3r})_1\} = \Gamma((a^{24\ell+3r+1})_1), \\ \Gamma((ba^{3r+1})_0)^g &= \{(ba^{3r+1})_1, (a^{3r+1})_0, (a^{3r})_0\} = \Gamma((ba^{3r+1})_0), \\ \Gamma((a^{3r+2})_0)^g &= \{(ba^{12\ell+3r+2})_0, (a^{36\ell+3r+2})_1, (a^{3r+3})_1\} = \Gamma((ba^{12\ell+3r+2})_1), \\ \Gamma((a^{3r+2})_1)^g &= \{(ba^{12\ell+3r+2})_1, (a^{12\ell+3r+2})_0, (a^{12\ell+3r+1})_0\} = \Gamma((ba^{12\ell+3r+2})_0). \end{aligned}$$

This implies that g is an automorphism of Γ . Observing that g maps 1_1 to b_0 , it follows that $\langle \mathcal{R}(H), g \rangle$ is transitive on $V(\Gamma)$, and so Γ is a vertex-transitive graph.

Below, we shall first prove the following claim.

Claim. $\text{Aut}(\Gamma)_{1_0} = \langle g \rangle$.

Let $A = \text{Aut}(\Gamma)$. It is easy to see that g fixes 1_0 , and so $g \in A_{1_0}$. To prove the Claim, it suffices to prove that $|A_{1_0}| = 2$.

Note that the neighborhood $\Gamma(1_0)$ of 1_0 in Γ is $\{1_1, b_0, (ba)_0\}$. By a direct computation, we find that in Γ there is a unique 8-cycle passing through $1_0, 1_1$ and b_0 , that is,

$$C_0 = (1_0, 1_1, (ba^{24\ell})_1, (ba^{24\ell})_0, (a^{24\ell})_0, (a^{24\ell})_1, b_1, b_0, 1_0).$$

Furthermore, in Γ there is no 8-cycle passing through 1_0 and $(ba)_0$. So A_{1_0} fixes $(ba)_0$.

If A_{1_0} also fixes 1_1 and b_0 , then A_{1_0} will fix every neighbor of 1_0 , and the connectedness and vertex-transitivity of Γ give that $A_{1_0} = 1$, a contradiction. Therefore, A_{1_0} swaps 1_1 and b_0 , and $(ba)_0$ is the unique neighbor of 1_0 such that $A_{1_0} = A_{(ba)_0}$. It follows that $\{1_0, (ba)_0\}$ is a block of imprimitivity of A acting on $V(\Gamma)$. Since Γ is vertex-transitive, every $v \in V(\Gamma)$ has a unique neighbor, say u such that $A_u = A_v$. Then the set

$$\mathcal{B} = \{\{u, v\} \in E(\Gamma) \mid A_u = A_v\}$$

forms an A -invariant partition of $V(\Gamma)$. Clearly, $\{1_0, (ba)_0\} \in \mathcal{B}$. Similarly, since C_0 is also the unique 8-cycle of Γ passing through $1_0, 1_1$ and b_0 , A_{1_1} swaps 1_0 and b_0 , and

$(ba^{12\ell-1})_1$ is the unique neighbor of 1_1 such that $A_{1_1} = A_{(ba^{12\ell-1})_1}$. So $\{1_1, (ba^{12\ell-1})_1\} \in \mathcal{B}$. Set

$$\mathcal{B}_0 = \{\{1_0, (ba)_0\}^{\mathcal{R}(h)} \mid h \in H\} \text{ and } \mathcal{B}_1 = \{\{1_1, (ba^{12\ell-1})_1\}^{\mathcal{R}(h)} \mid h \in H\}.$$

Clearly, $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$.

Now we consider the quotient graph $\Gamma_{\mathcal{B}}$ of Γ relative to \mathcal{B} . It is easy to see that $\langle \mathcal{R}(a) \rangle$ acts semiregularly on \mathcal{B} with \mathcal{B}_0 and \mathcal{B}_1 as its two orbits. So $\Gamma_{\mathcal{B}}$ is isomorphic to a bi-Cayley graph over $\langle a \rangle$. Set $B_0 = \{1_0, (ba)_0\}$ and $B_1 = \{1_1, (ba^{12\ell-1})_1\}$. Then one may see that the neighbors of B_0 in $\Gamma_{\mathcal{B}}$ are: $B_0^{\mathcal{R}(a)}, B_0^{\mathcal{R}(a^{-1})}, B_1, B_1^{\mathcal{R}(a^{-12\ell+2})}$, and the neighbors of B_1 in $\Gamma_{\mathcal{B}}$ are: $B_1^{\mathcal{R}(a^{12\ell+1})}, B_1^{\mathcal{R}(a^{-12\ell-1})}, B_0, B_0^{\mathcal{R}(a^{12\ell-2})}$. So

$$\Gamma_{\mathcal{B}} \cong \Gamma' = \text{BiCay}(\langle a \rangle, \{a, a^{-1}\}, \{a^{12\ell+1}, a^{-12\ell-1}\}, \{1, a^{-12\ell+2}\}).$$

Observe that there is one and only one edge of Γ between B_0 and any one of its neighbors in $\Gamma_{\mathcal{B}}$. Clearly, A acts transitively on $V(\Gamma_{\mathcal{B}})$, so there is one and only one edge of Γ between every two adjacent blocks of \mathcal{B} . It follows that A acts faithfully on $V(\Gamma_{\mathcal{B}})$, and hence we may view A as a subgroup of $\text{Aut}(\Gamma_{\mathcal{B}})$. Recall that $g \in A_{1_0} = A_{(ba)_0}$. Moreover, g swaps the two neighbors 1_1 and b_0 of 1_0 . Clearly, $1_1 \in B_1$ and $b_0 \in B_0^{\mathcal{R}(a^{-1})}$, so g swaps the two blocks B_1 and $B_0^{\mathcal{R}(a^{-1})}$. Similarly, g swaps the two neighbors $(ba)_1$ and a_0 of $(ba)_0$. Clearly, $(ba)_1 \in B_1^{\mathcal{R}(a^{-12\ell+2})}$ and $a_0 \in B_0^{\mathcal{R}(a)}$, so g swaps the two blocks $B_1^{\mathcal{R}(a^{-12\ell+2})}$ and $B_0^{\mathcal{R}(a)}$. Note that $\mathcal{R}(ab)$ swaps the two vertices in B_0 . So $\langle g, \mathcal{R}(ab) \rangle$ acts transitively on the neighborhood of B_0 in $\Gamma_{\mathcal{B}}$. This implies that A acts transitively on the arcs of $\Gamma_{\mathcal{B}}$, and so Γ' is a tetravalent arc-transitive bi-circulant. In [11], a characterization of tetravalent edge-transitive bi-circulants is given. It is easy to see that our graph Γ' belongs to Class 1(c) of [11, Theorem 1.1]. By checking [11, Theorem 4.1], we see that the stabilizer $\text{Aut}(\Gamma')_u$ of $u \in V(\Gamma')$ has order 4. This implies that $|A| = 4|V(\Gamma_{\mathcal{B}})| = 8n$. Consequently, $|A_{1_0}| = 2$ and so our claim holds.

Now we are ready to finish the proof. Suppose to the contrary that Γ is a Cayley graph. By Proposition 2.1, A contains a subgroup, say J acting regularly on $V(\Gamma)$. By Claim, J has index 2 in A , and since $g \in A_{1_0}$, one has $A = J \rtimes \langle g \rangle$. It is easy to check that $\mathcal{R}(a), \mathcal{R}(b)$ and g satisfy the following relations:

$$(g\mathcal{R}(b))^4 = \mathcal{R}(a^{24\ell}), g\mathcal{R}(a^3) = \mathcal{R}(a^3)g, g\mathcal{R}(ba) = \mathcal{R}(ba)g, g = \mathcal{R}(a)(g\mathcal{R}(b))^2\mathcal{R}(a^{12\ell-1}).$$

Suppose that $\mathcal{R}(H) \not\leq J$. Then $A = J\mathcal{R}(H)$. Since $|J|/|\mathcal{R}(H)| = 2$, it follows that $|\mathcal{R}(H) : J \cap \mathcal{R}(H)| = 2$. Thus, $J \cap \mathcal{R}(H) = \langle \mathcal{R}(a) \rangle$ or $\langle \mathcal{R}(a^2), \mathcal{R}(b) \rangle$. If $\mathcal{R}(H) \cap J = \langle \mathcal{R}(a) \rangle$, then we have $\mathcal{R}(b) \notin J, \mathcal{R}(a) \in J$, and hence $A = J \cup J\mathcal{R}(b) = J \cup Jg$, implying that $J\mathcal{R}(b) = Jg$. It follows that $g\mathcal{R}(b) \in J$, and then $g = \mathcal{R}(a)(g\mathcal{R}(b))^2\mathcal{R}(a^{12\ell-1}) \in J$ due to $\mathcal{R}(a) \in J$, a contradiction. If $\mathcal{R}(H) \cap J = \langle \mathcal{R}(a^2), \mathcal{R}(b) \rangle$, then $\mathcal{R}(a) \notin J$, and again we have $A = J \cup J\mathcal{R}(a) = J \cup Jg$, implying that $J\mathcal{R}(a) = Jg$. So, $\mathcal{R}(a)g, g\mathcal{R}(a^{-1}) \in J$. Then

$$g = \mathcal{R}(a)g\mathcal{R}(b)g\mathcal{R}(b)\mathcal{R}(a^{12\ell-1}) = (\mathcal{R}(a)g)\mathcal{R}(b)(g\mathcal{R}(a^{-1}))\mathcal{R}(ba^{12\ell-2}) \in J,$$

a contradiction.

Suppose that $\mathcal{R}(H) \leq J$. Then $|J : \mathcal{R}(H)| = 2$ and $\mathcal{R}(H) \trianglelefteq J$. Since J is regular on $V(\Gamma)$, by Proposition 2.4, there exists a $\delta_{\alpha, x, y} \in J$ such that $1_0^{\delta_{\alpha, x, y}} = 1_1$, where $\alpha \in$

$\text{Aut}(H)$ and $x, y \in H$. By the definition of $\delta_{\alpha, x, y}$, we have $1_1 = 1_0^{\delta_{\alpha, x, y}} = (x \cdot 1^\alpha)_1 = x_1$, implying that $x = 1$. Furthermore, we have the following relations:

$$R^\alpha = x^{-1}Lx, L^\alpha = y^{-1}Ry, S^\alpha = y^{-1}S^{-1}x,$$

where $R = \{b, ba\}$, $L = \{ba^{24\ell}, ba^{12\ell-1}\}$, $S = \{1\}$. In particular, the last equality implies that $x = y$ due to $S = \{1\}$. So we have $x = y = 1$. From the proof of Claim we know that $B_0 = \{1_0, (ba)_0\}$ and $B_1 = \{1_1, (ba^{12\ell-1})_1\}$ are two blocks of imprimitivity of A acting on $V(\Gamma)$. So we have $((ba)_0)^{\delta_{\alpha, 1, 1}} = (ba^{12\ell-1})_1$. It follows that $(ba)^\alpha = ba^{12\ell-1}$, and then from $R^\alpha = L$ we obtain that $b^\alpha = ba^{24\ell}$. Consequently, we have $a^\alpha = a^{36\ell-1}$. On the other hand, we have $\{b, ba\} = R = L^\alpha = \{b, ba^{24\ell+1}\}$. This forces that $ba = ba^{24\ell+1}$, which is clearly impossible. \square

6 Two families of trivalent Cayley bi-dihedrants

In this section, we shall prove two lemmas which will be used the proof of Theorem 1.3.

Lemma 6.1. *Let $H = \langle a, b \mid a^{12m} = b^2 = 1, a^b = a^{-1} \rangle$ be a dihedral group with m odd. Then for each $i \in \mathbb{Z}_{12m}$, $\Gamma = \text{BiCay}(H, \{b, ba^i\}, \{ba^{6m}, ba^{3m-i}\}, \{1\})$ is a Cayley graph whenever $\langle a^i, a^{3m} \rangle = \langle a \rangle$.*

Proof. Let g be a permutation of $V(\Gamma)$ defined as follows:

$$g : \begin{array}{ll} (a^{6km+3ri})_j \mapsto (ba^{6(k+1)m+3ri})_{j+1}, & (ba^{6km+3ri})_j \mapsto (a^{6km+3ri})_{j+1}, \\ (a^{3km+(3r+1)i})_0 \mapsto (a^{3(k+1)m+(3r+1)i})_0, & (ba^{3km+(3r+1)i})_0 \mapsto (a^{3(k+1)m+(3r+1)i})_1, \\ (a^{3km+(3r+1)i})_1 \mapsto (ba^{3(k+1)m+(3r+1)i})_0, & (ba^{3km+(3r+1)i})_1 \mapsto (ba^{3(k-1)m+(3r+1)i})_1, \\ (a^{3km+(3r+2)i})_0 \mapsto (ba^{3(k+1)m+(3r+2)i})_1, & (ba^{3km+(3r+2)i})_0 \mapsto (ba^{3(k+1)m+(3r+2)i})_0, \\ (a^{3km+(3r+2)i})_1 \mapsto (a^{3(k-1)m+(3r+2)i})_1, & (ba^{3km+(3r+2)i})_1 \mapsto (a^{3(k+1)m+(3r+2)i})_0, \end{array}$$

where $r \in \mathbb{Z}_m$, $k \in \mathbb{Z}_4$ and $j \in \mathbb{Z}_2$.

It is easy to check that $g \in \text{Aut}(\Gamma)$. Furthermore, one may check that g and $\mathcal{R}(a^2)$ satisfy the following relations:

$$\begin{aligned} \mathcal{R}(a^{12m}) &= g^4 = 1, \quad g^2 = \mathcal{R}(a^{6m}), \quad \mathcal{R}(a^6)g = g\mathcal{R}(a^6), \\ \mathcal{R}(b^{-1})g\mathcal{R}(b) &= g\mathcal{R}(a^{6m}), \quad \mathcal{R}(a^2)g = g\mathcal{R}(a^4)g\mathcal{R}(a^{-2}). \end{aligned}$$

By the last equality, we have

$$(\mathcal{R}(a^2)g)^3 = [g\mathcal{R}(a^4)g\mathcal{R}(a^{-2})]\mathcal{R}(a^2)g\mathcal{R}(a^2)g = g\mathcal{R}(a^4)g^2\mathcal{R}(a^2)g.$$

It then follows from the second and third equalities that

$$g\mathcal{R}(a^4)g^2\mathcal{R}(a^2)g = g\mathcal{R}(a^{6+6m})g = g^2\mathcal{R}(a^{6+6m}) = \mathcal{R}(a^6).$$

Therefore, $(\mathcal{R}(a^2)g)^3 = \mathcal{R}(a^6)$.

Let $G = \langle \mathcal{R}(a^2), \mathcal{R}(b), g \rangle$ and $T = \langle \mathcal{R}(a^6) \rangle$. Then $T \trianglelefteq G$ and

$$\begin{aligned} G/T &= \langle \mathcal{R}(a^2)T, \mathcal{R}(b)T, gT \rangle \\ &= \langle \mathcal{R}(a^2)T, gT \mid \mathcal{R}(a^2)^3T = g^2T = (\mathcal{R}(a^2)g)^3T = T \rangle \rtimes \langle \mathcal{R}(b)T \rangle \\ &\cong A_4 \rtimes \mathbb{Z}_2. \end{aligned}$$

So $|G| = 48m$.

Let

$$\begin{aligned} \Omega_{00} &= \{t_0 \mid t \in \langle a^2, b \rangle\}, & \Omega_{01} &= \{t_1 \mid t \in \langle a^2, b \rangle\}, \\ \Omega_{10} &= \{(at)_0 \mid t \in \langle a^2, b \rangle\}, & \Omega_{11} &= \{(at)_1 \mid t \in \langle a^2, b \rangle\}. \end{aligned}$$

Then Ω_{ij} 's ($0 \leq i, j \leq 1$) are orbits of T and $V(\Gamma) = \bigcup_{0 \leq i, j \leq 1} \Omega_{ij}$. Since $1_0^g = (ba^{6m})_1 \in \Omega_{01}$, $a_0^g = (a^{3m+1})_0 \in \Omega_{00}$ and $a_1^g = (ba^{3m+1})_1 \in \Omega_{01}$, it follows that G is transitive, and so regular on $V(\Gamma)$. By Proposition 2.1, Γ is a Cayley graph on G , as required. \square

Lemma 6.2. *Let $H = \langle a, b \mid a^{12m} = b^2 = 1, a^b = a^{-1} \rangle$ be a dihedral group with m even and $4 \nmid m$. Then the following two bi-Cayley graphs:*

$$\begin{aligned} \Gamma_1 &= \text{BiCay}(H, \{b, ba\}, \{ba^{6m}, ba^{3m-1}\}, \{1\}), \\ \Gamma_2 &= \text{BiCay}(H, \{b, ba\}, \{ba^{6m}, ba^{9m-1}\}, \{1\}) \end{aligned}$$

are both Cayley graphs.

Proof. Let $V = H_0 \cup H_1$. Then $V(\Gamma_1) = V(\Gamma_2) = V$. We first define two permutations on V as follows:

$$\begin{aligned} g_1 : \quad & (a^{4r})_i \mapsto (ba^{6m+4r})_{i+1}, & (ba^{4r})_i & \mapsto (a^{4r})_{i+1}, \\ & (a^{4r+1})_i \mapsto (ba^{9m+4r+1})_{i+1}, & (ba^{4r+1})_i & \mapsto (a^{3m+4r+1})_{i+1}, \\ & (a^{4r+2})_i \mapsto (ba^{4r+2})_{i+1}, & (ba^{4r+2})_i & \mapsto (a^{6m+4r+2})_{i+1}, \\ & (a^{4r+3})_i \mapsto (ba^{3m+4r+3})_{i+1}, & (ba^{4r+3})_i & \mapsto (a^{9m+4r+3})_{i+1}, \\ \\ g_2 : \quad & (a^{4r})_i \mapsto (ba^{6m+4r})_{i+1}, & (ba^{4r})_i & \mapsto (a^{4r})_{i+1}, \\ & (a^{4r+1})_i \mapsto (ba^{3m+4r+1})_{i+1}, & (ba^{4r+1})_i & \mapsto (a^{9m+4r+1})_{i+1}, \\ & (a^{4r+2})_i \mapsto (ba^{4r+2})_{i+1}, & (ba^{4r+2})_i & \mapsto (a^{6m+4r+2})_{i+1}, \\ & (a^{4r+3})_i \mapsto (ba^{9m+4r+3})_{i+1}, & (ba^{4r+3})_i & \mapsto (a^{3m+4r+3})_{i+1}, \end{aligned}$$

where $r \in \mathbb{Z}_{3m}$ and $i \in \mathbb{Z}_2$.

It is easy to check that $g_j \in \text{Aut}(\Gamma_j)$ for $j = 1$ or 2 . Furthermore, $\mathcal{R}(a^2), \mathcal{R}(b)$ and g_j ($j = 1$ or 2) satisfy the following relations:

$$\begin{aligned} \mathcal{R}(a^{12m}) &= \mathcal{R}(b^2) = g_j^4 = 1, \mathcal{R}(b)\mathcal{R}(a^2)\mathcal{R}(b) = \mathcal{R}(a^{-2}), \\ g_j^2 &= \mathcal{R}(a^{6m}), \mathcal{R}(b)g_j\mathcal{R}(b) = g_j^{-1}, \\ g_1^{-1}\mathcal{R}(a)g_1 &= \mathcal{R}(a^{3m+1}), g_2^{-1}\mathcal{R}(a)g_2 = \mathcal{R}(a^{9m+1}). \end{aligned}$$

For $j = 1$ or 2 , let $G_j = \langle \mathcal{R}(a), \mathcal{R}(b), g_j \rangle$. From the above relations it is easy to see that

$$G_j = (\langle \mathcal{R}(a) \rangle \langle g_j \rangle) \rtimes \langle \mathcal{R}(b) \rangle$$

has order at most $48m$. Observe that $1_0^{g_j} = (ba^{6m})_1 \in H_1$ for $j = 1$ or 2 . It follows that G_j is transitive on $V(\Gamma_j)$, and so G_j acts regularly on $V(\Gamma_j)$. By Proposition 2.1, each Γ_j is a Cayley graph. \square

7 Vertex-transitive trivalent bi-dihedrants

In this section, we shall give a complete classification of trivalent vertex-transitive non-Cayley bi-dihedrants. For convenience of the statement, throughout this section, we shall make the following assumption.

Assumption I.

- H : the dihedral group $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle (n \geq 3)$,
- $\Gamma = \text{BiCay}(H, R, L, \{1\})$: a connected trivalent 2-type vertex-transitive bi-Cayley graph over the group H (in this case, $|R| = |L| = 2$),
- G : a minimum group of automorphisms of Γ subject to that $\mathcal{R}(H) \leq G$ and G is transitive on the vertices but intransitive on the arcs of Γ .

The following lemma given in [20] shows that the group G must be solvable.

Lemma 7.1 ([20, Lemma 6.2]). $G = \mathcal{R}(H)P$ is solvable, where P is a Sylow 2-subgroup of G .

7.1 H_0 and H_1 are blocks of imprimitivity of G

The case where H_0 and H_1 are blocks of imprimitivity of G has been considered in [20], and the main result is the following proposition.

Proposition 7.2 ([20, Theorem 1.3]). *If H_0 and H_1 are blocks of imprimitivity of G on $V(\Gamma)$, then either Γ is Cayley or one of the following occurs:*

- (1) $(R, L, S) \equiv (\{b, ba^{\ell+1}\}, \{ba, ba^{\ell^2+\ell+1}\}, \{1\})$, where $n \geq 5$, $\ell^3 + \ell^2 + \ell + 1 \equiv 0 \pmod{n}$, $\ell^2 \not\equiv 1 \pmod{n}$;
- (2) $(R, L, S) \equiv (\{ba^{-\ell}, ba^{\ell}\}, \{a, a^{-1}\}, \{1\})$, where $n = 2k$ and $\ell^2 \equiv -1 \pmod{k}$.
Furthermore, Γ is also a bi-Cayley graph over an abelian group $\mathbb{Z}_n \times \mathbb{Z}_2$.

Furthermore, all of the graphs arising from (1)-(2) are vertex-transitive non-Cayley.

In particular, it is proved in [20] that if n is odd and Γ is not a Cayley graph, then H_0 and H_1 are blocks of imprimitivity of G on $V(\Gamma)$. Consequently, we can get a classification of trivalent vertex-transitive non-Cayley bi-Cayley graphs over a dihedral group D_{2n} with n odd.

Proposition 7.3 ([20, Proposition 6.4]). *If n is odd, then either Γ is a Cayley graph, or H_0 and H_1 are blocks of imprimitivity of G on $V(\Gamma)$.*

7.2 H_0 and H_1 are not blocks of imprimitivity of G

In this subsection, we shall consider the case where H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$. We begin by citing a lemma from [20].

Lemma 7.4 ([20, Lemma 6.3]). *Suppose that H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$. Let N be a normal subgroup of G , and let K be the kernel of G acting on $V(\Gamma_N)$. Let Δ be an orbit of N . If N fixes H_0 setwise, then one of the following holds:*

- (1) $\Gamma[\Delta]$ has valency 1, $|V(\Gamma_N)| \geq 3$ and Γ is a Cayley graph;
- (2) $\Gamma[\Delta]$ has valency 0, Γ_N has valency 3, and $K = N$ is semiregular.

The following lemma deals with the case where $\text{Core}_G(\mathcal{R}(H)) = 1$, and in this case we shall see that Γ is just the cross ladder graph.

Lemma 7.5. *Suppose that H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$. If $\text{Core}_G(\mathcal{R}(H)) = \bigcap_{g \in G} \mathcal{R}(H) = 1$, then Γ is isomorphic to the cross ladder graph CL_{4n} with n odd, and furthermore, for any minimal normal subgroup N of G , we have the following:*

- (1) N is a 2-group which is non-regular on $V(\Gamma)$;
- (2) N does not fix H_0 setwise;
- (3) every orbit of N consists of two non-adjacent vertices.

Proof. Let N be a minimal normal subgroup of G . By Lemma 7.1, G is solvable. It follows that N is an elementary abelian r -subgroup for some prime divisor r of $|G|$. Clearly, $N \not\leq \mathcal{R}(H)$ due to $\text{Core}_G(\mathcal{R}(H)) = 1$. Then $|N\mathcal{R}(H)|/|\mathcal{R}(H)| \mid |G|/|\mathcal{R}(H)|$. From Lemma 7.1 it follows that $|G|/|\mathcal{R}(H)|$ is a power of 2, and hence N is a 2-group.

Suppose that N is regular on $V(\Gamma)$. Then $N\mathcal{R}(H)$ is transitive on $V(\Gamma)$ and $\mathcal{R}(H)$ is also a 2-group. Therefore, $N\mathcal{R}(H)$ is not transitive on the arcs of Γ . The minimality of G gives that $G = N\mathcal{R}(H)$. Since n is even, $\mathcal{R}(a^{\frac{n}{2}})$ is in the center of $\mathcal{R}(H)$. Set $Q = N\langle \mathcal{R}(a^{\frac{n}{2}}) \rangle$. Then $Q \trianglelefteq G$ and then $1 \neq N \cap Z(Q) \trianglelefteq G$. Since N is a minimal normal subgroup of G , one has $N \leq Z(Q)$, and hence Q is abelian. It follows that $\langle \mathcal{R}(a^{\frac{n}{2}}) \rangle \trianglelefteq G$, contrary to the assumption that $\text{Core}_G(\mathcal{R}(H)) = 1$. Thus, N is not regular on $V(\Gamma)$. (1) is proved.

For (2), by way of contradiction, suppose that N fixes H_0 setwise. Consider the quotient graph Γ_N of Γ relative to N , and let K be the kernel of G acting on $V(\Gamma_N)$. Take Δ to be an orbit of N on $V(\Gamma)$. Then either (1) or (2) of Lemma 7.4 happens.

For the former, $\Gamma[\Delta]$ has valency 1 and $|V(\Gamma_N)| \geq 3$. Then Γ_N is a cycle. Moreover, any two neighbors of $u \in \Delta$ are in different orbits of N . It follows that the stabilizer N_v of v in N fixes every neighbor of u . The connectedness of Γ implies that $N_v = 1$. Thus, $K = N$ is semiregular and Γ_N is a cycle of length $\ell = 2|\mathcal{R}(H)|/|N|$. So $G/N \leq \text{Aut}(\Gamma_N) \cong D_{2\ell}$. If $G/N < \text{Aut}(\Gamma_N)$, then $|G : N| = \ell$ and so $|G| = 2|\mathcal{R}(H)|$. This implies that $\mathcal{R}(H) \trianglelefteq G$, contrary to the assumption that $\text{Core}_G(\mathcal{R}(H)) = 1$. If $G/N = \text{Aut}(\Gamma_N)$, then $|G : \mathcal{R}(H)| = 4$. Since $N \not\leq \mathcal{R}(H)$ and since N fixes H_0 setwise, one has $|G : \mathcal{R}(H)N| = 2$. It follows that $\mathcal{R}(H)N \trianglelefteq G$. Clearly, H_0 and H_1 are just two orbits of $\mathcal{R}(H)N$, and they are also two blocks of imprimitivity of G on $V(\Gamma)$, a contradiction.

For the latter, $\Gamma[\Delta]$ has valency 0, Γ_N has valency 3 and $N = K$ is semiregular. Let \bar{H}_i be the set of orbits of N contained in H_i with $i = 1, 2$. Then $\Gamma_N[\bar{H}_0]$ and $\Gamma_N[\bar{H}_1]$ are of valency 2 and the edges between \bar{H}_0 and \bar{H}_1 form a perfect matching. Without loss of generality, we may assume that $1_0 \in \Delta$. Since $\mathcal{R}(H)$ acts on H_0 by right multiplication, we have the subgroup of $\mathcal{R}(H)$ fixing Δ setwise is just $\mathcal{R}(H)_\Delta = \{\mathcal{R}(h) \mid h_0 \in \Delta\}$. If $\mathcal{R}(H)_\Delta \leq \langle \mathcal{R}(a) \rangle$, then $\mathcal{R}(H)_\Delta \trianglelefteq \mathcal{R}(H)$, and the transitivity of $\mathcal{R}(H)$ on H_0 implies that $\mathcal{R}(H)_\Delta$ will fix all orbits of N contained in H_0 . Since the edges between \bar{H}_0 and \bar{H}_1 are independent, $\mathcal{R}(H)_\Delta$ fixes all orbits of N . It follows that $\mathcal{R}(H)_\Delta \leq N$, namely, $\mathcal{R}(H)N/N$ acts regularly on \bar{H}_0 . Then $|\mathcal{R}(H)/(\mathcal{R}(H) \cap N)| = |\mathcal{R}(H)N/N| = |H_0/N|$, and so $|N| = |\mathcal{R}(H) \cap N|$, forcing $N \leq \mathcal{R}(H)$, a contradiction. Thus, $\mathcal{R}(H)_\Delta \not\leq \langle \mathcal{R}(a) \rangle$, and so $\langle \mathcal{R}(a) \rangle \mathcal{R}(H)_\Delta = \mathcal{R}(H)$. This implies that $\langle \mathcal{R}(a), N \rangle / N$ is transitive and so regular on \bar{H}_0 . Similarly, $\langle \mathcal{R}(a), N \rangle / N$ is also regular on \bar{H}_1 . Thus, Γ_N is a trivalent 2-type bi-Cayley graph over $\langle \mathcal{R}(a), N \rangle / N$. By [24, Lemma 5.3], \bar{H}_0 and \bar{H}_1 are blocks of imprimitivity of G/N , and so H_0 and H_1 are blocks of imprimitivity of G , a contradiction.

So far, we have completed the proof of (2). Then N does not fix H_0 setwise, and then $N\mathcal{R}(H)$ is transitive on $V(\Gamma)$. The minimality of G gives that $G = N\mathcal{R}(H)$. Let P and P_1 be Sylow 2-subgroups of G and $\mathcal{R}(H)$, respectively, such that $P_1 \leq P$. Then $N \leq P$ and $P = NP_1$.

If n is even, then by a similar argument to the second paragraph, a contradiction occurs. Thus, n is odd. As $H \cong D_{2n}$, $P_1 \cong \mathbb{Z}_2$ and P_1 is non-normal in $\mathcal{R}(H)$. So $N \cap \mathcal{R}(H) = 1$. Clearly, $|V(\Gamma)| = 4n$. If N is semiregular on $V(\Gamma)$, then $N \cong \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, and then $|G| = |\mathcal{R}(H)||N| = 2|\mathcal{R}(H)|$ or $4|\mathcal{R}(H)|$. Since $\text{Core}_G(\mathcal{R}(H)) = 1$, we must have $|G : \mathcal{R}(H)| = 4$ and $G \lesssim \text{Sym}(4)$. Since n is odd, one has $n = 3$ and $H \cong \text{Sym}(3)$. So $G \cong \text{Sym}(4)$ and hence $G_{1_0} \cong \mathbb{Z}_2$. Then all involutions of $G(\cong \text{Sym}(4))$ not contained in N are conjugate. Take $1 \neq g \in G_{1_0}$. Then g is an involution which is not contained in N because N is semiregular on $V(\Gamma)$. Since $\mathcal{R}(H) \cap N = 1$, every involution in $\mathcal{R}(H)$ would be conjugate to g . This is clearly impossible because $\mathcal{R}(H)$ is semiregular on $V(\Gamma)$. Thus, N is not semiregular on $V(\Gamma)$. (3) is proved.

Since n is odd, we have $|V(\Gamma_N)| > 2$. Since N is not semiregular on $V(\Gamma)$, Γ_N has valency 2 and $\Gamma[\Delta]$ has valency 0. This implies that the subgraph induced by any two adjacent two orbits of N is either a union of several cycles or a perfect matching. Thus, Γ_N has even order. As Γ has order $4n$ with n odd, every orbit of N has length 2. It is easy to see that Γ is isomorphic to the cross ladder graph CL_{4n} . \square

The following is the main result of this section.

Theorem 7.6. *Suppose that H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$. Then $\Gamma = \text{BiCay}(H, R, L, S)$ is vertex-transitive non-Cayley if and only if one of the followings occurs:*

- (1) $(R, L, S) \equiv (\{b, ba\}, \{b, ba^{2m}\}, \{1\})$, where $n = 2(2m + 1)$, $m \not\equiv 1 \pmod{3}$, and the corresponding graph is isomorphic the multi-cross ladder graph $\text{MCL}_{4m,2}$;
- (2) $(R, L, S) \equiv (\{b, ba\}, \{ba^{24\ell}, ba^{12\ell-1}\}, \{1\})$, where $n = 48\ell$ and $\ell \geq 1$.

Proof. The sufficiency can be obtained from Theorem 1.2 and Lemma 5.1. We shall prove the necessity in the following subsection by a series of lemmas. \square

7.3 Proof of the necessity of Theorem 7.6

The purpose of this subsection is to prove the necessity of Theorem 7.6. Throughout this subsection, we shall always assume that H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$ and that $\Gamma = \text{BiCay}(H, R, L, S)$ is vertex-transitive non-Cayley. In this subsection, we shall always use the following notation.

Assumption II. Let $N = \text{Core}_G(\mathcal{R}(H))$.

Our first lemma gives some properties of the group N .

Lemma 7.7. $1 < N < \langle \mathcal{R}(a) \rangle$, $|\langle \mathcal{R}(a) \rangle : N| = n/|N|$ is odd and the quotient graph Γ_N of Γ relative to N is isomorphic to the cross ladder graph $\text{CL}_{4n/|N|}$.

Proof. If $N = 1$, then from Lemma 7.5 it follows that $\Gamma \cong \text{CL}_{4n}$ which is a Cayley graph by Theorem 1.1, a contradiction. Thus, $N > 1$. Since H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$, one has $N < \mathcal{R}(H)$.

Consider the quotient graph Γ_N . Clearly, N fixes H_0 setwise. Recall that H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$ and that Γ is non-Cayley. Applying Lemma 7.4, we see that Γ_N is a trivalent 2-type bi-Cayley graph over $\mathcal{R}(H)/N$. This implies that $|\mathcal{R}(H) : N| > 2$, and since H is a dihedral group, one has $N < \langle \mathcal{R}(a) \rangle$.

Again, by Lemma 7.4, $\mathcal{R}(H)/N$ acts semiregularly on $V(\Gamma_N)$ with two orbits, \bar{H}_0 and \bar{H}_1 , where \bar{H}_i is the set of orbits of N contained in H_i with $i = 1, 0$. Furthermore, N is just the kernel of G acting on $V(\Gamma_N)$ and N acts semiregularly on $V(\Gamma)$. Then G/N is also a minimal vertex-transitive automorphism group of Γ_N containing $\mathcal{R}(H)/N$. If \bar{H}_0 and \bar{H}_1 are blocks of imprimitivity of G/N on $V(\Gamma_N)$, then H_0 and H_1 will be blocks of imprimitivity of G on $V(\Gamma)$, which is impossible by our assumption. Thus, \bar{H}_0 and \bar{H}_1 are not blocks of imprimitivity of G/N on $V(\Gamma_N)$. Since $N = \text{Core}_G(\mathcal{R}(H))$, $\text{Core}_{G/N}(\mathcal{R}(H)/N)$ is trivial. Then from Lemma 7.5 it follows that $\Gamma_N \cong \text{CL}_{\frac{4n}{|N|}}$, where $\frac{n}{|N|}$ is odd. □

Next, we introduce another notation which will be used in the proof.

Assumption III. Take M/N to be a minimal normal subgroup of G/N .

We shall first consider some basic properties of the quotient graph Γ_M of Γ relative to M .

Lemma 7.8. *The quotient graph Γ_M of Γ relative to M is a cycle of length $n/|N|$. Furthermore, every orbit of M on $V(\Gamma)$ is a union of an orbit of N on H_0 and an orbit of N on H_1 , and these two orbits of N are non-adjacent.*

Proof. Applying Lemma 7.5 to Γ_N and G/N , we obtain the following facts:

- (a) M/N is an elementary abelian 2-group which is not regular on $V(\Gamma_N)$,
- (b) M/N does not fix \bar{H}_0 setwise,
- (c) every orbit of M/N on $V(\Gamma_N)$ consists of two non-adjacent vertices of Γ_N .

From (b) and (c) it follows that every orbit of M on $V(\Gamma)$ is just a union of an orbit of N on H_0 and an orbit of N on H_1 , and these two orbits are non-adjacent. Since every orbit of N on $V(\Gamma)$ is an independent subset of $V(\Gamma)$, each orbit of M on $V(\Gamma)$ is also an independent subset.

Recall that $\Gamma_N \cong \text{CL}_{4m}$ where $m = \frac{n}{|N|}$ is odd. The quotient graph of Γ_N relative to M/N is just a cycle of length m , and so the quotient graph Γ_M of Γ relative to M is also a cycle of length m . □

By Lemma 7.8, each orbit of M on $V(\Gamma)$ is an independent subset. It follows that the subgraph induced by any two adjacent orbits of M is either a perfect matching or a union of several cycles. For convenience of the statement, the following notations will be used in the remainder of the proof:

Assumption IV.

- (1) Let Δ and Δ' be two adjacent orbits of M on $V(\Gamma)$ such that $\Gamma[\Delta \cup \Delta']$ is a union of several cycles.
- (2) Let $\Delta = \Delta_0 \cup \Delta_1$ and $\Delta' = \Delta'_0 \cup \Delta'_1$, where $\Delta_0, \Delta'_0 \subseteq H_0$ and $\Delta_1, \Delta'_1 \subseteq H_1$ are four orbits of N on $V(\Gamma)$.

(3) $1_0 \in \Delta_0$.

Since $\Gamma[\Delta]$ and $\Gamma[\Delta']$ are both null graphs and since $\Gamma[\Delta \cup \Delta']$ is a union of several cycles, we have the following easy observation.

Lemma 7.9. $\Gamma[\Delta_i \cup \Delta'_j]$ is a perfect matching for any $0 \leq i, j \leq 1$.

The following lemma tells us the possibility of R (Recall that we assume that $\Gamma = \text{BiCay}(H, R, L, \{1\})$).

Lemma 7.10. Up to graph isomorphism, we may assume that $R = \{b, ba^i\}$ with $i \in \mathbb{Z}_n \setminus \{0\}$ and that $b_0 \in \Delta'_0$. Furthermore, we have

$$\begin{aligned}\Delta_0 &= \{h_0 \mid \mathcal{R}(h) \in N\}, \Delta'_0 = \{(bh)_0 \mid \mathcal{R}(h) \in N\}, \\ \Delta_1 &= \{h_1 \mid \mathcal{R}(h) \in N\}, \Delta_1 = \{(bh)_1 \mid \mathcal{R}(h) \in N\},\end{aligned}$$

and 1_1 is adjacent to $(ba^l)_1 \in \Delta_1$ for some $\mathcal{R}(a^l) \in N$.

Proof. Recall that N is a proper subgroup of $\langle \mathcal{R}(a) \rangle$ and that $n/|N|$ is odd. Since n is even by Proposition 7.3, it follows that N is of even order, and so the unique involution $\mathcal{R}(a^{n/2})$ of $\langle \mathcal{R}(a) \rangle$ is contained in N . As $1_0 \in \Delta_0$ and $N \leq \langle \mathcal{R}(a) \rangle$ acts on H_0 by right multiplication, one has $\Delta_0 = \{h_0 \mid h \in N\}$. Since $\Gamma[\Delta_0]$ is an empty graph, one has $a^{n/2} \notin R$. By Proposition 2.3 (1), we have $\langle R \cup L \rangle = H$, and since R and L are both self-inverse, either $R \subseteq b\langle a \rangle$ or $L \subseteq b\langle a \rangle$. By Proposition 2.3 (4), we may assume that $R \subseteq b\langle a \rangle$.

Recall that $\Gamma[\Delta_i \cup \Delta'_j]$ is a perfect matching for any $0 \leq i, j \leq 1$. Then 1_0 is adjacent to $r_0 \in \Delta'_0$ for some $r \in R$. Since $R \subseteq b\langle a \rangle$ and $\text{Aut}(H)$ is transitive on $b\langle a \rangle$, by Proposition 2.3 (3), we may assume that $r = b$. So 1_0 is adjacent to $b_0 \in \Delta'_0$. Since $N \leq \langle \mathcal{R}(a) \rangle$ acts on H_i with $i = 0$ or 1 by right multiplication, we see that the two orbits Δ_0, Δ'_0 of N are just the form as given in the lemma. Since $S = \{1\}$, the edges between H_0 and H_1 form a perfect matching. This enables us to obtain another two orbits Δ_1, Δ'_1 of N which have the form as given in the lemma.

By Lemma 7.9, $\Gamma[\Delta_1 \cup \Delta'_1]$ is a perfect matching. So we may assume that 1_1 is adjacent to $(ba^l)_1 \in \Delta_1$ for some $\mathcal{R}(a^l) \in N$. \square

Now we shall introduce some new notations which will be used in the following.

Assumption V.

(1) Let $T = \langle \mathcal{R}(a^l) \rangle$ be of order t , where a^l is given in the above lemma.

(2) Let

$$\begin{aligned}\Omega_0 &= \{(a^{i\frac{t}{2}})_0 \mid 0 \leq i \leq t-1\}, & \Omega_1 &= \{(ba^{i\frac{t}{2}})_1 \mid 0 \leq i \leq t-1\}, \\ \Omega'_0 &= \{(ba^{i\frac{t}{2}})_0 \mid 0 \leq i \leq t-1\}, & \Omega'_1 &= \{(a^{i\frac{t}{2}})_1 \mid 0 \leq i \leq t-1\}.\end{aligned}$$

(3) $\mathcal{B} = \{B^{\mathcal{R}(h)} \mid h \in H\}$, where $B = \Omega_0 \cup \Omega_1$.

(4) Let $B' = \Omega'_0 \cup \Omega'_1$. Then $B' = B^{\mathcal{R}(b)}$.

Lemma 7.11. The followings hold.

(1) $T \leq N$.

- (2) $\Omega_0, \Omega_1, \Omega'_0, \Omega'_1$ are four orbits of T .
- (3) $\Gamma[\Omega_0 \cup \Omega_1 \cup \Omega'_0 \cup \Omega'_1]$ is a cycle of length $4t$.
- (4) \mathcal{B} is a G -invariant partition of $V(\Gamma)$.

Proof. By Lemma 7.10, we see that $\mathcal{R}(a^l) \in N$, and so $T \leq N$. (1) holds. Since $T = \langle \mathcal{R}(a^l) \rangle$ is assumed to be of order t , one has $T = \langle \mathcal{R}(a^{n/t}) \rangle$, and then one can obtain (2). By the adjacency rule of bi-Cayley graph, we can obtain (3).

Set $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega'_0 \cup \Omega'_1$ and $B = \Omega_0 \cup \Omega_1$. By Lemma 7.8, $\Gamma[\Delta]$ is a null graph, and so $B = \Delta \cap \Omega$. Since Γ has valency 3, it follows that $\Delta \cup \Delta'$ is a block of imprimitivity of G on $V(\Gamma)$, and hence Ω is also a block of imprimitivity of G on $V(\Gamma)$ since $\Gamma[\Omega]$ is a component of $\Gamma[\Delta \cup \Delta']$. Since Δ is also a block of imprimitivity of G on $V(\Gamma)$, $B (= \Delta \cap \Omega)$ is a block of imprimitivity of G on $V(\Gamma)$. Then $\mathcal{B} = \{B^{\mathcal{R}(h)} \mid h \in H\}$ is a G -invariant partition of $V(\Gamma)$. □

Lemma 7.12. $T < N$ and the quotient graph $\Gamma_{\mathcal{B}}$ of Γ relative to \mathcal{B} is isomorphic to the cross ladder graph $CL_{\frac{4n}{2t}}$. Moreover, T is the kernel of G acting on \mathcal{B} .

Proof. Let $K_{\mathcal{B}}$ be the kernel of G acting on \mathcal{B} . Clearly, $T \leq K_{\mathcal{B}}$. Let $B' = \Omega'_0 \cup \Omega'_1$. Then $B' = B^{\mathcal{R}(b)} \in \mathcal{B}$. Let $B^{\mathcal{R}(h)} \in \mathcal{B}$ be adjacent to B and $B^{\mathcal{R}(h)} \neq B'$.

Suppose that $\Gamma[B \cup B^{\mathcal{R}(h)}]$ is a perfect matching. Since G is transitive on \mathcal{B} , $\Gamma_{\mathcal{B}}$ is a cycle of length $\frac{2n}{t}$. Clearly, $G/K_{\mathcal{B}}$ is vertex-transitive but not edge-transitive on $\Gamma_{\mathcal{B}}$, so $G/K_{\mathcal{B}} \cong D_{2n/t}$. If $t = 1$, then it is easy to see that $\Gamma \cong CL_{4n}$ which is a Cayley graph by Theorem 1.1, a contradiction. If $t > 1$, then since $\Gamma[\Omega] = \Gamma[B \cup B']$ is a cycle of length $4t$, $K_{\mathcal{B}}$ acts faithfully on B , and so $K_{\mathcal{B}} \leq \text{Aut}(\Gamma[B \cup B']) \cong D_{8t}$. Since $K_{\mathcal{B}}$ fixes B , one has $|K_{\mathcal{B}}| \mid 4t$, implying that $|G| = |K_{\mathcal{B}}| \cdot \frac{2n}{t} \mid 8n$. As $|R(H)| = 2n$ and $\mathcal{R}(H)$ is non-normal in G , one has $|K_{\mathcal{B}}| = 4t$ due to $T \leq K_{\mathcal{B}}$. In view of the fact that $K_{\mathcal{B}} \lesssim D_{8t}$, $K_{\mathcal{B}}$ has a characteristic cyclic subgroup, say J , of order $2t$. Then we have $J \trianglelefteq G$ because $K_{\mathcal{B}} \trianglelefteq G$. Clearly, J is regular on B and $J \cap N = T$, so $J\mathcal{R}(H)$ is regular on $V(\Gamma)$. It follows from Proposition 2.1 that Γ is a Cayley graph, a contradiction.

Therefore, $\Gamma[B \cup B^{\mathcal{R}(h)}]$ is not a perfect matching. If $N = T$, then $B = \Delta$ and $B' = \Delta'$ are orbits of M , and then $\Gamma[B \cup B^{\mathcal{R}(h)}]$ will be a perfect matching, a contradiction. Thus, $N > T$.

Now we are going to prove that $\Gamma_{\mathcal{B}} \cong CL_{\frac{n}{2t}}$. Since B is adjacent to $B^{\mathcal{R}(h)}$, Ω_i is adjacent to $\Omega_j^{\mathcal{R}(h)}$ for some $i, j \in \{0, 1\}$. Then because Ω_i and $\Omega_j^{\mathcal{R}(h)}$ are orbits of T , $\Gamma[\Omega_i \cup \Omega_j^{\mathcal{R}(h)}]$ is a perfect matching. This implies that $\Gamma_{\mathcal{B}}$ is of valency 3, and so $K_{\mathcal{B}}$ is intransitive on B . As every $B^h \in \mathcal{B}$ is a union of two orbits of T on $V(\Gamma)$, $K_{\mathcal{B}}$ fixes every orbit of T . Since N is cyclic, the normality of N in G implies that $T \trianglelefteq G$. Clearly, Ω_0 is adjacent to three pair-wise different orbits of T , so the quotient graph Γ_T of Γ relative to T is of valency 3. Consequently, the kernel of G acting on $V(\Gamma_T)$ is T . Then $K_{\mathcal{B}} = T$. Now $\mathcal{R}(H)/T \cong D_{2n/t}$ is regular on \mathcal{B} , and so $\Gamma_{\mathcal{B}}$ is a Cayley graph over $\mathcal{R}(H)/T$. Furthermore, G/T is not arc-transitive on $\Gamma_{\mathcal{B}}$. Since $\mathcal{R}(H)/T$ is non-normal in G/T , $\Gamma_{\mathcal{B}}$ is a non-normal Cayley graph over $\mathcal{R}(H)/T$. If $\Gamma_{\mathcal{B}}$ is arc-transitive, then by [13, Theorem 1], either $|\text{Aut}(\Gamma_{\mathcal{B}})| = 3k|\mathcal{R}(H)/T|$ with $k \leq 2$, or $\Gamma_{\mathcal{B}}$ has order $2 \cdot p$ with $p = 3$ or 7 . For the former, since G/T is not arc-transitive on $\Gamma_{\mathcal{B}}$, one has $|G/T : \mathcal{R}(H)/T| \leq 2$, implying $\mathcal{R}(H) \trianglelefteq G$, a contradiction. For the latter, we have $\frac{2n}{t} = 6$ or 14 , implying $\frac{n}{t} = 3$ or 7 . It follows that T is a maximal subgroup of $\langle \mathcal{R}(a) \rangle$, and so $T = N$, a contradiction.

Therefore, $\Gamma_{\mathcal{B}}$ is not arc-transitive. Since $\mathcal{R}(H)/T$ is non-normal in G/T , by Theorem 1.1, one has $\Gamma_{\mathcal{B}} \cong \text{CL}_{\frac{4n}{2t}}$, as required. \square

Proof of Theorem 7.6. By Lemma 7.12, we have $\Gamma_{\mathcal{B}} \cong \text{CL}_{\frac{4n}{2t}}$. By the definition of $\text{CL}_{\frac{4n}{2t}}$, we may partition the vertex set of $\Gamma_{\mathcal{B}}$ in the following way:

$$V(\Gamma_{\mathcal{B}}) = V_0 \cup V_1 \cup \cdots \cup V_{\frac{2n}{2t}-2} \cup V_{\frac{2n}{2t}-1}, \text{ where } V_i = \{B_i^0, B_i^1\}, i \in \mathbb{Z}_{\frac{2n}{2t}}$$

and

$$E(\Gamma_{\mathcal{B}}) = \{\{B_{2i}^r, B_{2i+1}^r\}, \{B_{2i+1}^r, B_{2i+2}^s\} \mid i \in \mathbb{Z}_{\frac{n}{2t}}, r, s \in \mathbb{Z}_2\}.$$

Assume that $B_0^0 = B$ and $B_1^0 = B'$. Recall that $B = \Omega_0 \cup \Omega_1$ and $B' = \Omega'_0 \cup \Omega'_1 = B^{\mathcal{R}(b)}$. Moreover, $\Omega_0, \Omega_1, \Omega'_0$ and Ω'_1 are four orbits of T . Then every $B_i^j \in \mathcal{B}$ is just a union of two orbits of T . For convenience, we may let

$$B_i^j = \Omega_{i0}^j \cup \Omega_{i1}^j, i \in \mathbb{Z}_{\frac{2n}{2t}}, j \in \mathbb{Z}_2,$$

where $\Omega_{i0}^j, \Omega_{i1}^j$ are two orbits of T . For $B = B_0^0$, we let $\Omega_0 = \Omega_{00}^0$ and $\Omega_1 = \Omega_{01}^0$, and for $B' = B_1^0$, we let $\Omega'_0 = \Omega_{10}^0$ and $\Omega'_1 = \Omega_{11}^0$.

For convenience, in the remainder of the proof, we shall use C_{4t} to denote a cycle of length $4t$, and we also call C_{4t} a $4t$ -cycle. Recall that $\Gamma[B \cup B'] = \Gamma[B_0^0 \cup B_1^0] \cong C_{4t}$, and that the edges between $\Omega_{0i}^0 (= \Omega_i)$ and $\Omega_{1j}^0 (= \Omega'_j)$ form a perfect matching for all $i, j \in \mathbb{Z}_2$. Since $T \triangleleft G$, the quotient graph Γ_T of Γ relative to T has valency 3. So the edges between any two adjacent orbits of T form a perfect matching.

From the construction of $\Gamma_{\mathcal{B}}$, one may see that there exists $g \in G$ such that $\{V_0, V_1\}^g = \{V_{2i}, V_{2i+1}\}$ for each $i \in \mathbb{Z}_{\frac{n}{2t}}$. So for each $i \in \mathbb{Z}_{\frac{n}{2t}}, r \in \mathbb{Z}_2$, we may assume that $\Gamma[B_{2i}^r \cup B_{2i+1}^r] \cong C_{4t}$, and $\Omega_{(2i)s}^r \sim \Omega_{(2i+1)t}^r$ for all $s, t \in \mathbb{Z}_2$. (Here $\Omega_{(2i)s}^r \sim \Omega_{(2i+1)t}^r$ means that $\Omega_{(2i)s}^r$ and $\Omega_{(2i+1)t}^r$ are adjacent in $\Gamma_{\mathcal{B}}$.) Again, from the construction of $\Gamma_{\mathcal{B}}$, we may assume that

$$\Omega_{(2i+2)0}^0 \sim \Omega_{(2i+1)0}^0, \Omega_{(2i+2)1}^0 \sim \Omega_{(2i+1)0}^1, \Omega_{(2i+2)1}^1 \sim \Omega_{(2i+1)0}^1, \Omega_{(2i+2)1}^1 \sim \Omega_{(2i+1)1}^0,$$

for each $i \in \mathbb{Z}_{\frac{n}{2t}}$. We draw a local subgraph of $\Gamma_{\mathcal{B}}$ in Figure 3. Observing that every

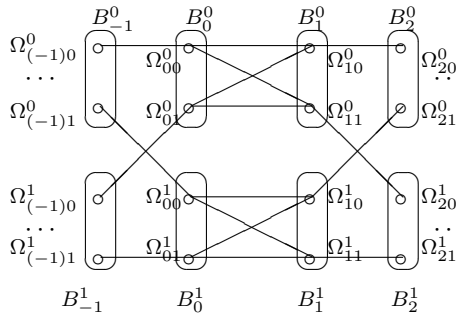


Figure 3: The sketch graph of $\Gamma_{\mathcal{B}}$

$V_i = \{B_i^0, B_i^1\}$ with $i \in \mathbb{Z}_{\frac{2n}{2t}}$ is a block of imprimitivity of $G/K_{\mathcal{B}}$ acting on $V(\Gamma_{\mathcal{B}})$. So every $B_i^0 \cup B_i^1$ with $i \in \mathbb{Z}_{\frac{2n}{2t}}$ is a block of imprimitivity of G acting on $V(\Gamma)$. Let E be the kernel of G acting on the block system $\Lambda = \{B_i^0 \cup B_i^1 \mid i \in \mathbb{Z}_{\frac{2n}{2t}}\}$. Then $G/E \cong D_{\frac{n}{t}}$

acts regularly on Λ . Clearly, $\mathcal{R}(H)$ is also transitive on Ω , so $G/E = \mathcal{R}(H)E/E$. By Lemma 7.12, T is the kernel of G acting on \mathcal{B} . So E/T is an elementary 2-group. From $\mathcal{R}(H)/(\mathcal{R}(H) \cap E) \cong D_{\frac{n}{2t}}$ it follows that $\mathcal{R}(H) \cap E = \langle \mathcal{R}(a^{\frac{n}{2t}}) \rangle \cong \mathbb{Z}_{2t}$, and so $(\mathcal{R}(H) \cap E)/T$ is a normal subgroup of G/T of order 2. This implies that $B_i^1 = (B_i^0)^{\mathcal{R}(a^{\frac{n}{2t}})}$ for $i \in \mathbb{Z}_{\frac{2n}{2t}}$. We may further assume that $\Omega_{01}^1 = (\Omega_{00}^0)^{\mathcal{R}(a^{\frac{n}{2t}})} \subseteq B_0^1$. So $\Omega_{00}^0 \cup \Omega_{01}^1$ is just the orbit of $\langle \mathcal{R}(a^{\frac{n}{2t}}) \rangle$ containing 1_0 .

Observing that $\Omega_{10}^0 \sim \Omega_{20}^0$ and the edges between them are of the form $\{g_0, (ba^i g_0)\}$ with $g_0 \in \Omega_{10}^0$, one has $\Omega_{20}^0 = ba^i \Omega_{10}^0 = ba^i (\Omega_{00}^0)^{\mathcal{R}(b)} = (\Omega_{00}^0)^{\mathcal{R}(a^{-i})}$. So $\Omega_{20}^1 \subseteq (B_0^1)^{\mathcal{R}(a^{-i})}$.

Since $B_1^0 = B' = B^{\mathcal{R}(b)} = (B_0^0)^{\mathcal{R}(b)}$, one has $B_1^1 = (B_0^1)^{\mathcal{R}(b)}$. Recall that $1_1 \in \Omega_{11}^0 = \Omega_1'$ and 1_1 is adjacent to $1_0 \in \Omega_{00}^0 = \Omega_0$ and $(ba^t)_1 \in \Omega_{01}^0 = \Omega_1$. As we assume that $\Omega_{11}^0 \sim \Omega_{20}^1$, 1_1 is adjacent to some vertex in Ω_{20}^1 . So $\Omega_{20}^1 \subseteq H_1$ and hence

$$\begin{aligned} \Omega_{20}^1 &= (\Omega_{00}^1)^{\mathcal{R}(a^{-i})} \\ &= (\Omega_{01}^0)^{\mathcal{R}(a^{\frac{n}{2t}})\mathcal{R}(a^{-i})} \\ &= (\Omega_{01}^0)^{\mathcal{R}(a^{\frac{n}{2t}-i})} \\ &= \{(ba^{k\frac{n}{t}})_1 \mid 0 \leq k \leq t-1\}^{\mathcal{R}(a^{\frac{n}{2t}-i})}. \end{aligned}$$

So we have the following claim.

Claim 1 $L = \{ba^l, ba^{k\frac{n}{t} + \frac{n}{2t}-i}\}$ and $R = \{ba^i, b\}$, where $|\mathcal{R}(a^l)| = t$, $i \in \mathbb{Z}_n$ and $0 \leq k \leq t-1$.

Let $G_{1_0}^*$ be the kernel of G_{1_0} acting on the neighborhood of 1_0 in Γ . Then $G_{1_0}^* \leq E_{1_0}$. Recall that for each $i \in \mathbb{Z}_{\frac{n}{2t}}$, $r \in \mathbb{Z}_2$, $\Gamma[B_{2i}^r \cup B_{2i+1}^r] \cong C_{4t}$ and the edges between $B_{2i+1}^0 \cup B_{2i+1}^1$ and $B_{2i+2}^0 \cup B_{2i+2}^1$ form a perfect matching. It follows that E acts faithfully on each $B_i^0 \cup B_i^1$. Clearly, $G_{1_0}^* \leq E_{1_0}$, so $G_{1_0}^*$ acts faithfully on each $B_i^0 \cup B_i^1$.

Claim 2 If $t > 2$ then $G_{1_0}^* = 1$, and if $t = 2$ then $G_{1_0}^* \leq \mathbb{Z}_2$ and $3 \mid n$.

Assume that $t \geq 2$. Since $\Gamma[B_0^0 \cup B_1^0] \cong C_{4t}$, $G_{1_0}^*$ fixes every vertex in B_0^0 , and so fixes every vertex in $\Omega_{(-1)0}^0$ since $\Omega_{(-1)0}^0 \sim \Omega_{00}^0$ (see Figure 3). This implies that $G_{1_0}^*$ fixes $\Omega_{(-1)1}^0$ setwise, and so fixes Ω_{00}^1 setwise since $\Omega_{(-1)1}^0 \sim \Omega_{00}^1$. Consequently, $G_{1_0}^*$ also fixes Ω_{01}^1 setwise. Similarly, by considering the edges between $B_1^0 \cup B_1^1$ and $B_2^0 \cup B_2^1$, we see that $G_{1_0}^*$ fixes both Ω_{10}^1 and Ω_{11}^1 setwise. Recall that the edges between Ω_{0i}^1 and Ω_{1j}^1 form a perfect matching for $i, j \in \mathbb{Z}_2$. As $\Gamma[B_1^0 \cup B_1^1] \cong C_{4t}$, $G_{1_0}^*$ acts faithfully on Ω_{00}^1 (or Ω_{01}^1), and so $G_{1_0}^* \leq \mathbb{Z}_2$.

If $t > 2$, then since $\Gamma[B_{-2}^0 \cup B_{-1}^0] \cong C_{4t}$, $G_{1_0}^*$ will fix every vertex in this cycle, and in particular, $G_{1_0}^*$ will fix every vertex in $\Omega_{(-1)1}^0$. As $\Omega_{(-1)1}^0 \sim \Omega_{00}^1$, $G_{1_0}^*$ will fix every vertex in Ω_{00}^1 . Since $G_{1_0}^*$ acts faithfully on Ω_{00}^1 , one has $G_{1_0}^* = 1$.

Let $t = 2$. We shall show that $3 \mid n$. Then $T = \langle \mathcal{R}(a^{\frac{n}{2}}) \rangle$. Recall that $(\mathcal{R}(H) \cap E)/T$ is a normal subgroup of G/T of order 2. Let $M = \mathcal{R}(H) \cap E$. Then M is a normal subgroup of G of order 4. Since $\mathcal{R}(H)$ is dihedral, one has $M = \langle \mathcal{R}(a^{\frac{n}{4}}) \rangle$. Let $C = C_G(M)$. Then $\mathcal{R}(a) \in C$ and $\mathcal{R}(b) \notin C$. It follows that C is a proper subgroup of G . Since G/E acts regularly on Λ , C_{1_0} fixes every element in Λ . Since C_{1_0} centralizes M , C_{1_0} fixes every vertex in the orbit $\Omega_{00}^0 \cup \Omega_{01}^1$ of M containing 1_0 . Clearly, $C_{1_0} \leq G_{1_0}$, so $C_{1_0}/(C_{1_0} \cap G_{1_0}^*) \leq \mathbb{Z}_2$. As we have shown that $G_{1_0}^*$ acts faithfully on Ω_{01}^1 , it follows that

$C_{1_0} \cap G_{1_0}^* = 1$ since C_{1_0} fixes Ω_{01}^1 pointwise, and hence $C_{1_0} \leq \mathbb{Z}_2$. On the other hand, as $G_{1_0}^* \leq \mathbb{Z}_2$, one has $|G| \mid 4 \cdot 4n = 16n$. Since $C < G$ and $\mathcal{R}(a) \in C$, one has $|C| = kn$ with $k \mid 8$.

Suppose that $3 \nmid n$. For any odd prime divisor p of n , let P be a Sylow p -subgroup of $\langle \mathcal{R}(a) \rangle$. Then P is also a Sylow p -subgroup of C . If P is not normal in C , then by Sylow's theorem, we have $|C : N_C(P)| = k'p + 1 \mid 8$ for some integer k' . Since $p \neq 3$, one has $p = 7$ and $k' = 1$. This implies that $|C| = 8|N_C(P)|$, and so $|C| = 8n$ due to $\mathcal{R}(a) \in C$ and $C < G$. Since $C_{1_0} \leq \mathbb{Z}_2$, one has $|C : C_{1_0}| \geq 4n$, and so C is transitive on $V(\Gamma)$. Moreover, we have $C_C(P) = N_C(P) = \langle \mathcal{R}(a) \rangle$. By Burnside theorem, C has a normal subgroup M such that $C = M \rtimes P$. Then the quotient graph Γ_M of Γ relative to M would be a cycle of length $|P|$, and the subgraph induced by each orbit of M is just a perfect matching. This implies that M is just the kernel of G acting on $V(\Gamma_M)$. Furthermore, C/M is a vertex-transitive subgroup of $\text{Aut}(\Gamma_M)$. Since Γ_M is a cycle, C/M must contain a subgroup, say B/M acting regularly on $V(\Gamma_M)$. Then B will be regular on $V(\Gamma)$, and so by Proposition 2.1, Γ is a Cayley graph, a contradiction. Therefore, $P \trianglelefteq C$, and since $C \trianglelefteq G$, one has $P \trianglelefteq G$, implying $P \leq N$. By the arbitrariness of P , $n/|N|$ must be even, contrary to Lemma 7.7. Thus, $3 \mid n$, as claimed.

The following claim shows that $t = 1$ or 2 .

Claim 3 $t \leq 2$.

By way of contradiction, suppose that $t > 2$. Let $C = C_G(T)$. Then $\langle \mathcal{R}(a) \rangle \leq C$ and $\mathcal{R}(H) \not\leq C$ since $|T| = t > 2$. Clearly, $C_{1_0} \leq E_{1_0}$. As C_{1_0} centralizes T , C_{1_0} will fix every vertex in Ω_{00}^0 since Ω_{00}^0 is an orbit of T containing 1_0 . Since $\Gamma[B_0^0 \cup B_1^0] \cong C_{4t}$, C_{1_0} fixes every vertex in this $4t$ -cycle, and so $C_{1_0} \leq G_{1_0}^* = 1$ (by Claim 2). Thus, C acts semiregularly on $V(\Gamma)$. If $C = \langle \mathcal{R}(a) \rangle$, then by N/C-theorem, we have $G/\langle \mathcal{R}(a) \rangle = G/C \leq \text{Aut}(T)$. Since $T \leq N \leq \langle \mathcal{R}(a) \rangle$ is cyclic, $\text{Aut}(T)$ is abelian. It then follows that $\mathcal{R}(H)/C \trianglelefteq G/C$, and hence $\mathcal{R}(H) \trianglelefteq G$, a contradiction. If $C > \langle \mathcal{R}(a) \rangle$, then $|C| = 2n$ because Γ is non-Cayley. Since H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$, C does not fix H_0 setwise, and so $\mathcal{R}(H)C$ is transitive on $V(\Gamma)$. Clearly, $\mathcal{R}(H) \cap C = \langle \mathcal{R}(a) \rangle$, so $|\mathcal{R}(H)C| = |\mathcal{R}(H)||C|/|\langle \mathcal{R}(a) \rangle| = 4n$. It follows that $\mathcal{R}(H)C$ is regular on $V(\Gamma)$, contradicting that Γ is non-Cayley.

By Claim 3, we only need to consider the following two cases:

Case 1 $t = 1$.

In this case, by Claim 1, we have $R = \{b, ba^i\}$ and $L = \{b, ba^{\frac{n}{2}-i}\}$. For convenience, we let $n = 2\ell$. Then $R = \{b, ba^i\}$ and $L = \{b, ba^{\ell-i}\}$.

By Proposition 2.3 (1), the connectedness of Γ implies that $\langle a^i, a^\ell \rangle = \langle a \rangle$. Then either $(i, 2\ell) = 1$, or $i = 2k$ with $(k, 2\ell) = 1$ and ℓ is odd. Recall that $H = \langle a, b \mid a^{2\ell} = b^2 = 1, bab = a^{-1} \rangle$. For any $\lambda \in \mathbb{Z}_{2\ell}^*$, let α_λ be the automorphism of H induced by the map

$$a^\lambda \mapsto a, b \mapsto b.$$

So if $(i, 2\ell) = 1$, then we have

$$(R, L)^{\alpha_i} = (\{b, ba\}, \{b, ba^{\ell-1}\}),$$

and if $i = 2k$ with $(k, 2\ell) = 2$ and ℓ is odd, then we have

$$(R, L)^{\alpha_k} = (\{b, ba^2\}, \{b, ba^{\ell-2}\}).$$

So by Proposition 2.3 (3), we have

$$(R, L, S) \equiv (\{b, ba\}, \{b, ba^{\ell-1}\}, \{1\}) \text{ or } (\{b, ba^2\}, \{b, ba^{\ell-2}\}, \{1\}) (\ell \text{ is odd}).$$

Suppose that ℓ is even. Then $(R, L, S) \equiv (\{b, ba\}, \{b, ba^{\ell-1}\}, \{1\})$. Since ℓ is even, one has $(2\ell, \ell + 1) = 1$ and $(\ell + 1)^2 \equiv 1 \pmod{2\ell}$. Then it is easy to check that $\alpha_{\ell+1}$ is an automorphism of H of order 2 that swaps $\{b, ba\}$ and $\{b, ba^{\ell-1}\}$. By Proposition 2.4, we have $\delta_{\alpha_{\ell+1}, 1, 1} \in I$, and then $\Gamma \cong \text{BiCay}(H, \{b, ba\}, \{b, ba^{\ell+1}\}, \{1\})$ is a Cayley graph, a contradiction.

Now we assume that $n = 2\ell$ with $\ell = 2m + 1$ for some integer m . Let

$$\Gamma_1 = \text{BiCay}(H, \{b, ba\}, \{b, ba^{2m}\}, \{1\}), \Gamma_2 = \text{BiCay}(H, \{b, ba^2\}, \{b, ba^{2m-1}\}, \{1\}).$$

Direct calculation shows that $(n, 2m - 1) = 1$, and $2m(2m - 1) \equiv 2 \pmod{n}$. Then the automorphism $\alpha_{2m-1} : a \mapsto a^{2m-1}, b \mapsto b$ maps the pair of two subsets $(\{b, ba\}, \{b, ba^{2m}\})$ to $(\{b, ba^{2m-1}\}, \{b, ba^2\})$. So, we have $(R, L, S) \equiv (\{b, ba\}, \{b, ba^{2m}\}, \{1\})$. By Lemma 4.1 and Theorem 1.2, $\Gamma \cong \text{MCL}(4m, 2)$ and Γ is non-Cayley if and only if $3 \nmid (2m + 1)$. Note that $3 \nmid (2m + 1)$ is equivalent to $m \not\equiv 1 \pmod{3}$. So we obtain the first family of graphs in Theorem 7.6.

Case 2 $t = 2$.

In this case, by Claim 1, we have $R = \{b, ba^i\}$ and $L = \{ba^{\frac{n}{2}}, ba^{\frac{3n}{4}-i}\}$ or $\{ba^{\frac{n}{2}}, ba^{\frac{n}{4}-i}\}$. We still use the following notation: For any $\lambda \in \mathbb{Z}_{2\ell}^*$, let α_λ be the automorphism of H induced by the map

$$a^\lambda \mapsto a, b \mapsto b.$$

Note that

$$(\{b, ba^i\}, \{ba^{\frac{n}{2}}, ba^{\frac{3n}{4}-i}\})^{\alpha_{-1}} = (\{b, ba^{-i}\}, \{ba^{\frac{n}{2}}, ba^{\frac{n}{4}-(-i)}\}).$$

By replacing $-i$ by i , we may always assume that

$$(R, L) = (\{b, ba^i\}, \{ba^{\frac{n}{2}}, ba^{\frac{n}{4}-i}\}).$$

By Claim 2, we have $3 \mid n$. So we may assume that $n = 12m$ for some integer m . Then we have

$$(R, L) = (\{b, ba^i\}, \{ba^{6m}, ba^{3m-i}\}).$$

Since Γ is connected, by Proposition 2.3, we have $\langle a^i, a^{3m} \rangle = \langle a \rangle$. If m is odd, by Lemma 6.1, Γ will be a Cayley graph which is impossible. Thus, m is even. It then follows that $\langle a^i \rangle \cap \langle a^{3m} \rangle > 1$ since $\langle a^i, a^{3m} \rangle = \langle a^i \rangle \langle a^{3m} \rangle = \langle a \rangle$. Since $\langle a^{3m} \rangle \cong \mathbb{Z}_4$, one has $|\langle a^i \rangle \cap \langle a^{3m} \rangle| = 2$ or 4 . For the former, we would have $|\langle a^i \rangle| = 6m$, and since m is even, one has $4 \mid |\langle a^i \rangle|$, and hence $a^{3m} \in \langle a^i \rangle$, a contradiction. Thus, we have $|\langle a^i \rangle \cap \langle a^{3m} \rangle| = 4$, that is, $\langle a^i \rangle = \langle a \rangle$. So $(i, 12m) = 1$, and then $\alpha_i \in \text{Aut}(H)$ which maps $(\{b, ba^i\}, \{ba^{6m}, ba^{3m-i}\})$ to $(\{b, ba\}, \{ba^{6m}, ba^{3m-1}\})$ or $(\{b, ba\}, \{ba^{6m}, ba^{-3m-1}\})$. Then

$$(R, L, S) \equiv (\{b, ba\}, \{ba^{6m}, ba^{3m-1}\}, \{1\}) \text{ or } (\{b, ba\}, \{ba^{6m}, ba^{-3m-1}\}, \{1\}).$$

If $m \equiv 2 \pmod{4}$, then by Lemma 6.2, we see that Γ will be a Cayley graph, a contradiction. Thus, $m \equiv 0 \pmod{4}$. Clearly, $(3m - 1, 12m) = 1$, and hence the map $a \mapsto a^{3m-1}, b \mapsto ba^{6m}$ induces an automorphism, say β of H . It is easy to check that

$$(\{b, ba\}, \{ba^{6m}, ba^{3m-1}\})^\beta = (\{ba^{6m}, ba^{-3m-1}\}, \{b, ba\}).$$

Thus,

$$(R, L, S) \equiv (\{b, ba\}, \{ba^{6m}, ba^{3m-1}\}, \{1\}).$$

By Proposition 5.1, Γ is a non-Cayley graph. Let $m = 4\ell$ for some integer ℓ . Then $n = 48\ell$ and then we get the second family of graphs in Theorem 7.6. This completes the proof of Theorem 7.6. \square

7.4 Proof of Theorem 1.3

By [20, Theorem 1.2], if Γ is 0- or 1-type, then Γ is a Cayley graph. Let Γ be of 2-type. Suppose that Γ is a non-Cayley graph. Let $G \leq \text{Aut}(\Gamma)$ be minimal subject to that $\mathcal{R}(H) \leq G$ and G is transitive on $V(\Gamma)$. If Γ is arc-transitive or H_0 and H_1 are blocks of imprimitivity of G on $V(\Gamma)$, then by [20, Theorem 1.1] and Proposition 7.2, we obtain the graphs in part (1)–(3) of Theorem 1.3. Otherwise, Γ is not arc-transitive and H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$, by Theorem 7.6, we obtain the last two families of graphs of Theorem 1.3. \square

ORCID iDs

Mi-Mi Zhang  <https://orcid.org/0000-0002-8719-5888>

Jin-Xin Zhou  <https://orcid.org/0000-0002-8353-896X>

References

- [1] M. Alaeiyan, M. Ghasemi and G. R. Safakish, The normality of cubic Cayley graphs for dihedral groups, *Vietnam J. Math.* **37** (2009), 43–48, http://www.math.ac.vn/publications/vjm/VJM_37/43.htm.
- [2] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, 1993.
- [3] J. Bondy and U. Murty, *Graph Theory with Applications*, Elsevier Science Ltd/North-Holland, 1976.
- [4] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.* **24** (1997), 235–265, doi:10.1006/jsco.1996.0125, computational algebra and number theory (London, 1993).
- [5] P. Cameron, J. Sheehan and P. Spiga, Semiregular automorphisms of vertex-transitive cubic graphs, *European J. Combin.* **27** (2006), 924–930, doi:10.1016/j.ejc.2005.04.008.
- [6] E. Dobson, A. Malnič, D. Marušič and L. A. Nowitz, Semiregular automorphisms of vertex-transitive graphs of certain valencies, *J. Combin. Theory Ser. B* **97** (2007), 371–380, doi:10.1016/j.jctb.2006.06.004.
- [7] S. Du, A. Malnič and D. Marušič, Classification of 2-arc-transitive dihedrants, *J. Combin. Theory Ser. B* **98** (2008), 1349–1372, doi:10.1016/j.jctb.2008.02.007.
- [8] R. Frucht, J. E. Graver and M. E. Watkins, The groups of the generalized Petersen graphs, *Proc. Cambridge Philos. Soc.* **70** (1971), 211–218, doi:10.1017/s0305004100049811.
- [9] C. D. Godsil, On the full automorphism group of a graph, *Combinatorica* **1** (1981), 243–256, doi:10.1007/bf02579330.
- [10] C. D. Godsil, The automorphism groups of some cubic Cayley graphs, *European J. Combin.* **4** (1983), 25–32, doi:10.1016/s0195-6698(83)80005-5.
- [11] I. Kovács, B. Kuzman, A. Malnič and S. Wilson, Characterization of edge-transitive 4-valent bicirculants, *J. Graph Theory* **69** (2012), 441–463, doi:10.1002/jgt.20594.

- [12] D. Marušič, On 2-arc-transitivity of Cayley graphs, *J. Combin. Theory Ser. B* **87** (2003), 162–196, doi:10.1016/s0095-8956(02)00033-3, dedicated to Crispin St. J. A. Nash-Williams.
- [13] D. Marušič and T. Pisanski, Symmetries of hexagonal molecular graphs on the torus, *Croat. Chem. Acta* **73** (2000), 969–981, <https://hrcak.srce.hr/131972>.
- [14] R. Nedela and M. Škoviera, Which generalized Petersen graphs are Cayley graphs?, *J. Graph Theory* **19** (1995), 1–11, doi:10.1002/jgt.3190190102.
- [15] T. Pisanski, A classification of cubic bicirculants, *Discrete Math.* **307** (2007), 567–578, doi:10.1016/j.disc.2005.09.053.
- [16] P. Potočnik, P. Spiga and G. Verret, Cubic vertex-transitive graphs on up to 1280 vertices, *J. Symbolic Comput.* **50** (2013), 465–477, doi:10.1016/j.jsc.2012.09.002.
- [17] M. E. Watkins, A theorem on Tait colorings with an application to the generalized Petersen graphs, *J. Combinatorial Theory* **6** (1969), 152–164.
- [18] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964.
- [19] M.-Y. Xu, Automorphism groups and isomorphisms of Cayley digraphs, *Discrete Math.* **182** (1998), 309–319, doi:10.1016/s0012-365x(97)00152-0, graph theory (Lake Bled, 1995).
- [20] M.-M. Zhang and J.-X. Zhou, Trivalent vertex-transitive bi-dihedrants, *Discrete Math.* **340** (2017), 1757–1772, doi:10.1016/j.disc.2017.03.017.
- [21] W.-J. Zhang, Y.-Q. Feng and J.-X. Zhou, Cubic vertex-transitive non-Cayley graphs of order $12p$, *Sci. China Math.* **61** (2018), 1153–1162, doi:10.1007/s11425-016-9095-8.
- [22] C. Zhou and Y.-Q. Feng, Automorphism groups of connected cubic Cayley graphs of order $4p$, *Algebra Colloq.* **14** (2007), 351–359, doi:10.1142/s100538670700034x.
- [23] J.-X. Zhou and Y.-Q. Feng, Cubic vertex-transitive non-Cayley graphs of order $8p$, *Electron. J. Combin.* **19** (2012), Paper 53, 13, doi:10.37236/2087.
- [24] J.-X. Zhou and Y.-Q. Feng, Cubic bi-Cayley graphs over abelian groups, *European J. Combin.* **36** (2014), 679–693, doi:10.1016/j.ejc.2013.10.005.
- [25] J.-X. Zhou and Y.-Q. Feng, The automorphisms of bi-Cayley graphs, *J. Combin. Theory Ser. B* **116** (2016), 504–532, doi:10.1016/j.jctb.2015.10.004.
- [26] J.-X. Zhou and M. Ghasemi, Automorphisms of a family of cubic graphs, *Algebra Colloq.* **20** (2013), 495–506, doi:10.1142/s1005386713000461.