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Trivalent dihedrants and bi-dihedrants*

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Abstract

A Cayley (resp. bi-Cayley) graph on a dihedral group is called a *dihedrant* (resp. *bi-dihedrant*). In 2000, a classification of trivalent arc-transitive dihedrants was given by Marušič and Pisanski, and several years later, trivalent non-arc-transitive dihedrants of order $4p$ or $8p$ (p a prime) were classified by Feng et al. As a generalization of these results, our first result presents a classification of trivalent non-arc-transitive dihedrants. Using this, a complete classification of trivalent vertex-transitive non-Cayley bi-dihedrants is given, thus completing the study of trivalent bi-dihedrants initiated in our previous paper [Discrete Math. 340 (2017) 1757–1772]. As a by-product, we generalize a theorem in [The Electronic Journal of Combinatorics 19 (2012) #P53].

Keywords: Cayley graph, non-Cayley, bi-Cayley, dihedral group, dihedrant, bi-dihedrant.

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1 Introduction

In this paper we describe an investigation of trivalent Cayley graphs on dihedral groups as well as vertex-transitive trivalent bi-Cayley graphs over dihedral groups. To be brief, we shall say that a Cayley (resp. bi-Cayley) graph on a dihedral group a *dihedrant* (resp. *bi-dihedrant*).

Cayley graphs are usually defined in the following way. Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is a

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graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. For any $g \in G$, $R(g)$ is the permutation of G defined by $R(g) : x \mapsto xg$ for $x \in G$. Set $R(G) := \{R(g) \mid g \in G\}$. It is well-known that $R(G)$ is a subgroup of $\mathrm{Aut}\,(\mathrm{Cay}(G, S))$. We say that the Cayley graph $\mathrm{Cay}(G, S)$ is *normal* if $R(G)$ is normal in $\mathrm{Aut}\,(\mathrm{Cay}(G, S))$ (see [19]).

In 2000, Marušič and Pisanski [13] initiated the study of automorphisms of dihedrants, and they gave a classification of trivalent arc-transitive dihedrants. Following this work, highly symmetrical dihedrants have been extensively studied, and one of the remarkable achievements is the complete classification of 2-arc-transitive dihedrants (see [6, 12]). In contrast, however, relatively little is known about the automorphisms of non-arc-transitive dihedrants. In [1], the authors claimed that every trivalent non-arc-transitive dihedrant is normal. However, this is not true. There exist non-arc-transitive and non-normal dihedrants. Actually, in [22, 25], the automorphism groups of trivalent dihedrants of order $4p$ and $8p$ are determined for each prime p, and the result reveals that every non-arc-transitive trivalent dihedrant of order $4p$ or $8p$ is either a normal Cayley graph, or isomorphic to the so-called cross ladder graph. For an integer $m \geq 2$, the *cross ladder graph*, denoted by CL_{4m}, is a trivalent graph of order $4m$ with vertex set $V_0 \cup V_1 \cup \ldots V_{2m-2} \cup V_{2m-1}$, where $V_i = \{x_i^0, x_i^1\}$, and edge set $\{\{x_{2i}^r, x_{2i+1}^r\}, \{x_{2i+1}^r, x_{2i+2}^s\} \mid i \in \mathbb{Z}_m, r, s \in \mathbb{Z}_2\}$ (see Fig. 1 for CL_{4m}). It is worth mentioning that the cross ladder graph plays an important role in the

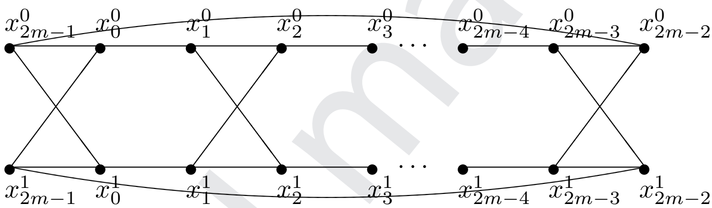

Figure 1: The cross ladder graph CL_{4m}

study of automorphisms of trivalent graphs (see, for example, [5, 25, 21]). Motivated by the above mentioned facts, we shall focus on trivalent non-arc-transitive dihedrants. Our first theorem generalizes the results in [22, 25] to all trivalent dihedrants.

Theorem 1.1. *Let* $\Sigma = \mathrm{Cay}(H, S)$ *be a connected trivalent Cayley graph, where* $H = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle (n \geq 3)$*. If* Σ *is non-arc-transitive and non-normal, then* n *is even and* $\Gamma \cong \mathrm{CL}_{4\cdot\frac{n}{2}}$ *and* $S^\alpha = \{b, ba, ba^{\frac{n}{2}}\}$ *for some* $\alpha \in \mathrm{Aut}\,(H)$.

Recall that for an integer $m \geq 2$, the cross ladder graph CL_{4m} has vertex set $V_0 \cup V_1 \cup \ldots V_{2m-2} \cup V_{2m-1}$, where $V_i = \{x_i^0, x_i^1\}$. The *multi-cross ladder graph*, denoted by $\mathrm{MCL}_{4m,2}$, is the graph obtained from CL_{4m} by blowing up each vertex x_i^r of CL_{4m} into two vertices $x_i^{r,0}$ and $x_i^{r,1}$. The edge set is $\{\{x_{2i}^{r,s}, x_{2i+1}^{r,t}\}, \{x_{2i+1}^{r,s}, x_{2i+2}^{s,r}\} \mid i \in \mathbb{Z}_m, r, s, t \in \mathbb{Z}_2\}$ (see Fig. 2 for $\mathrm{MCL}_{20,2}$).

Note that the multi-cross ladder graph $\mathrm{MCL}_{4m,2}$ is just the graph given in [23, Definition 7]. From [7, Proposition 3.3] we know that every $\mathrm{MCL}_{4m,2}$ is vertex-transitive. However, not all multi-cross ladder graphs are Cayley graphs. Actually, in [23, Theorem 9], it is proved that $\mathrm{MCL}_{4p,2}$ is a vertex-transitive non-Cayley graph for each prime $p > 7$. Our second theorem generalizes this result to all multi-cross ladder graphs.

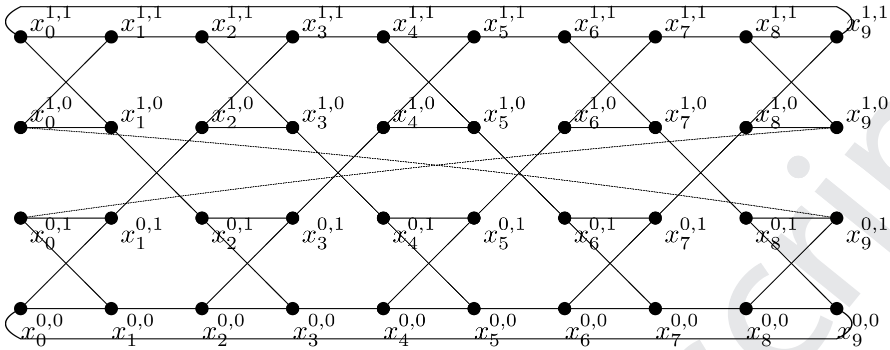

Figure 2: The multi-cross ladder graph $\mathrm{MCL}_{20,2}$

Theorem 1.2. *The multi-cross ladder graph* $\mathrm{MCL}_{4m,2}$ *is a Cayley graph if and only if either* m *is even, or* m *is odd and* $3 \mid m$*.*

Both of the above two theorems are crucial in attacking the problem of classification of trivalent vertex-transitive non-Cayley bi-dihedrants. Before proceeding, we give some background to this topic, and set some notation.

Let R, L and S be subsets of a group H such that $R = R^{-1}$, $L = L^{-1}$ and $R \cup L$ does not contain the identity element of H. The *bi-Cayley graph* $\mathrm{BiCay}(H, R, L, S)$ over H relative to R, L, S is a graph having vertex set the union of the *right part* $H_0 = \{h_0 \mid h \in H\}$ and the *left part* $H_1 = \{h_1 \mid h \in H\}$, and edge set the union of the *right edges* $\{\{h_0, g_0\} \mid gh^{-1} \in R\}$, the *left edges* $\{\{h_1, g_1\} \mid gh^{-1} \in L\}$ and the *spokes* $\{\{h_0, g_1\} \mid gh^{-1} \in S\}$. If $|R| = |L| = s$, then $\mathrm{BiCay}(H,\ R,\ L,\ S)$ is said to be an *s-type bi-Cayley graph*.

In [20] we initiated a program to investigate the automorphism groups of the trivalent vertex-transitive bi-dihedrants. This was partially motivated by the following facts. As one of the most important finite graphs, the Petersen graph is a bi-circulant, but it is not a Cayley graph. Note that a *bi-circulant* is a bi-Cayley graph over a cyclic group. The Petersen graph is the initial member of a family of graphs $P(n, t)$, known now as the *generalized Petersen graphs* (see [17]), which can be also constructed as bi-circulants. Let $n \geq 3$, $1 \leq t < n/2$ and set $H = \langle a \rangle \cong \mathbb{Z}_n$. The generalized Petersen graph $P(n, t)$ is isomorphic to the bi-circulant $\mathrm{BiCay}(H,\ \{a, a^{-1}\},\ \{a^t, a^{-t}\},\ \{1\})$. The complete classification of vertex-transitive generalized Petersen graphs has been worked out in [8, 14]. Latter, this was generalized by Marušič et al. in [13, 15] where all trivalent vertex-transitive bi-circulants were classified in [13, 15], and more recently, all trivalent vertex-transitive bi-Cayley graphs over abelian groups were classified in [24]. The characterization of trivalent vertex-transitive bi-dihedrants is the next natural step.

Another motivation for us to consider trivalent vertex-transitive bi-dihedrants comes from the excellent work in a highly cited article [16], where the authors give a census of trivalent vertex-transitive graphs of order up to 1000. This is very important in the study of trivalent vertex-transitive graphs. Actually, by checking this census, we find out that there are 981 non-Cayley graphs, and among these graphs, 233 graphs are non-Cayley bi-dihedrants. This may suggest bi-dihedrants form an important class of trivalent vertex-transitive non-Cayley graphs.

In [20], we gave a classification of trivalent arc-transitive bi-dihedrants, and we also proved that every trivalent vertex-transitive 0- or 1-type bi-dihedrant is a Cayley graph, and gave a classification of trivalent vertex-transitive non-Cayley bi-dihedrants of order $4n$ with n odd. The goal of this paper is to complete the classification of trivalent vertex-transitive non-Cayley bi-dihedrants.

Before stating the main result, we need the following concepts. For a bi-Cayley graph $\Gamma = \mathrm{BiCay}(H,\ R,\ L,\ S)$ over a group H, we can assume that the identity 1 of H is in S (see Proposition 2.3 (2)). The triple (R, L, S) of three subsets R, L, S of a group H is called *bi-Cayley triple* if $R = R^{-1}, L = L^{-1}$, and $1 \in S$. Two bi-Cayley triples (R, L, S) and (R', L', S') of a group H are said to be *equivalent*, denoted by $(R, L, S) \equiv (R', L', S')$, if either $(R', L', S') = (R, L, S)^\alpha$ or $(R', L', S') = (L, R, S^{-1})^\alpha$ for some automorphism α of H. The bi-Cayley graphs corresponding to two equivalent bi-Cayley triples of the same group are isomorphic (see Proposition 2.3 (3)-(4)).

Theorem 1.3. *Let $\Gamma = \mathrm{BiCay}(R, L, S)$ be a trivalent vertex-transitive bi-dihedrant where $H = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ is a dihedral group. Then either Γ is a Cayley graph or one of the following occurs:*

(1) $(R, L, S) \equiv (\{b,\ ba^{\ell+1}\}, \{ba,\ ba^{\ell^2+\ell+1}\}, \{1\})$, *where* $n \geq 5$, $\ell^3 + \ell^2 + \ell + 1 \equiv 0 \pmod n$, $\ell^2 \not\equiv 1 \pmod n$.

(2) $(R, L, S) \equiv (\{ba^{-\ell},\ ba^{\ell}\}, \{a,\ a^{-1}\}, \{1\})$, *where* $n = 2m$ *and* $\ell^2 \equiv -1 \pmod m$. *Furthermore,* Γ *is also a bi-Cayley graph over an abelian group* $\mathbb{Z}_n \times \mathbb{Z}_2$.

(3) $(R, L, S) \equiv (\{b, ba\}, \{b, ba^{2m}\}, \{1\})$, *where* $n = 2(2m + 1)$, $m \not\equiv 1 \pmod 3$, *and the corresponding graph is isomorphic the multi-cross ladder graph* $\mathrm{MCL}_{4m,2}$.

(4) $(R, L, S) \equiv (\{b, ba\}, \{ba^{24\ell}, ba^{12\ell-1}\}, \{1\})$, *where* $n = 48\ell$ *and* $\ell \geq 1$.

Moreover, all of the graphs arising from (1)-(4) *are vertex-transitive non-Cayley.*

2 Preliminaries

All groups considered in this paper are finite, and all graphs are finite, connected, simple and undirected. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [3, 18].

2.1 Definitions and notations

For a positive integer, let $\mathbb{Z}_n$ be the cyclic group of order n and $\mathbb{Z}_n^*$ be the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n. For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. For a subgroup H of a group G, denote $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H of G. Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular.

For a finite, simple and undirected graph Γ, we use $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$, $\mathrm{Aut}\,(\Gamma)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. For any subset B of $V(\Gamma)$, the subgraph of Γ induced by B will be denoted by $\Gamma[B]$. For any $v \in V(\Gamma)$ and a positive integer i no more than the diameter of Γ, denote by $\Gamma_i(v)$ be the set of vertices at distance i from v. Clearly, $\Gamma_1(v)$ is just the neighborhood of v. We shall often abuse the notation by using $\Gamma(v)$ to replace $\Gamma_1(v)$.

A graph Γ is said to be *vertex-transitive*, and *arc-transitive* (or *symmetric*) if $\mathrm{Aut}\,(\Gamma)$ acts transitively on $V(\Gamma)$ and $A(\Gamma)$, respectively. Let Γ be a connected vertex-transitive graph, and let $G \leq \text{Aut}(\Gamma)$ be vertex-transitive on Γ. For a G-invariant partition $\mathcal{B}$ of $V(\Gamma)$, the *quotient graph* $\Gamma_{\mathcal{B}}$ is defined as the graph with vertex set $\mathcal{B}$ such that, for any two different vertices $B, C \in \mathcal{B}$, B is adjacent to C if and only if there exist $u \in B$ and $v \in C$ which are adjacent in Γ. Let N be a normal subgroup of G. Then the set $\mathcal{B}$ of orbits of N in $V(\Gamma)$ is a G-invariant partition of $V(\Gamma)$. In this case, the symbol $\Gamma_{\mathcal{B}}$ will be replaced by Γ_N. The original graph Γ is said to be a *N-cover* of Γ_N if Γ and Γ_N have the same valency.

2.2 Cayley graphs

Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph on G with respect to S. Then Γ is vertex-transitive due to $R(G) \leq \text{Aut}(\Gamma)$. In general, we have the following proposition.

Proposition 2.1. [2, Lemma 16.3] *A vertex-transitive graph Γ is isomorphic to a Cayley graph on a group G if and only if its automorphism group has a subgroup isomorphic to G, acting regularly on the vertex set of Γ.*

In 1981, Godsil [9] proved that the normalizer of $R(G)$ in $\text{Aut}(\text{Cay}(G, S))$ is $R(G) \rtimes \text{Aut}(G, S)$, where $\text{Aut}(G, S)$ is the group of automorphisms of G fixing the set S set-wise. This result has been successfully used in characterizing various families of Cayley graphs $\text{Cay}(G, S)$ such that $R(G) = \text{Aut}(\text{Cay}(G, S))$ (see, for example, [9, 10]). Recall that a Cayley graph $\text{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\text{Aut}(\text{Cay}(G, S))$ (see [19]).

Proposition 2.2. [19, Proposition 1.5] *The Cayley graph $\Gamma = \text{Cay}(G, S)$ is normal if and only if $A_1 = \text{Aut}(G, S)$, where A_1 is the stabilizer of the identity 1 of G in $\text{Aut}(\Gamma)$.*

2.3 Basic properties of bi-Cayley graphs

In this subsection, we let Γ be a connected bi-Cayley graph $\text{BiCay}(H, R, L, S)$ over a group H. It is easy to prove some basic properties of such a Γ, as in [24, Lemma 3.1].

Proposition 2.3. *The following hold.*

(1) *H is generated by $R \cup L \cup S$.*

(2) *Up to graph isomorphism, S can be chosen to contain the identity of H.*

(3) *For any automorphism α of H,* $\text{BiCay}(H,\ R,\ L,\ S) \cong \text{BiCay}(H,\ R^\alpha,\ L^\alpha,\ S^\alpha)$.

(4) $\text{BiCay}(H,\ R,\ L,\ S) \cong \text{BiCay}(H,\ L,\ R,\ S^{-1})$.

Next, we collect several results about the automorphisms of bi-Cayley graph $\Gamma = \text{BiCay}(H,\ R,\ L,\ S)$. For each $g \in H$, define a permutation as follows:

$$\mathcal{R}(g) : h_i \mapsto (hg)_i, \quad \forall i \in \mathbb{Z}_2,\ h \in H. \tag{2.1}$$

Set $\mathcal{R}(H) = \{\mathcal{R}(g) \mid g \in H\}$. Then $\mathcal{R}(H)$ is a semiregular subgroup of $\text{Aut}(\Gamma)$ with H_0 and H_1 as its two orbits.

For an automorphism α of H and $x, y, g \in H$, define two permutations of $V(\Gamma) = H_0 \cup H_1$ as follows:

$$\begin{aligned} \delta_{\alpha,x,y} &: \ h_0 \mapsto (xh^\alpha)_1,\ h_1 \mapsto (yh^\alpha)_0,\ \forall h \in H, \\ \sigma_{\alpha,g} &: \ h_0 \mapsto (h^\alpha)_0,\ h_1 \mapsto (gh^\alpha)_1,\ \forall h \in H. \end{aligned} \tag{2.2}$$

Set

$$\begin{aligned} &I=\{\delta_{\alpha,x,y}\mid \alpha\in \mathrm{Aut}\,(H)\ s.t.\ R^{\alpha}=x^{-1}Lx,\ L^{\alpha}=y^{-1}Ry,\ S^{\alpha}=y^{-1}S^{-1}x\},\\ &F=\{\sigma_{\alpha,g}\mid \alpha\in \mathrm{Aut}\,(H)\ s.t.\ R^{\alpha}=R,\ L^{\alpha}=g^{-1}Lg,\ S^{\alpha}=g^{-1}S\}.\end{aligned} \tag{2.3}$$

Proposition 2.4. [26, Theorem 1.1] *Let $\Gamma=\mathrm{BiCay}(H,R,L,S)$ be a connected bi-Cayley graph over the group H. Then $N_{\mathrm{Aut}\,(\Gamma)}(\mathcal{R}(H))=\mathcal{R}(H)\rtimes F$ if $I=\emptyset$ and $N_{\mathrm{Aut}\,(\Gamma)}(\mathcal{R}(H))=\mathcal{R}(H)\langle F,\ \delta_{\alpha,x,y}\rangle$ if $I\neq\emptyset$ and $\delta_{\alpha,x,y}\in I$. Furthermore, for any $\delta_{\alpha,x,y}\in I$, we have the following:*

(1) *$\langle\mathcal{R}(H),\ \delta_{\alpha,x,y}\rangle$ acts transitively on $V(\Gamma)$;*

(2) *if α has order 2 and $x=y=1$, then Γ is isomorphic to the Cayley graph Cay($\bar{H},\ R\cup\alpha S$), where $\bar{H}=H\rtimes\langle\alpha\rangle$.*

3 Cross ladder graphs

The goal of this section is to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that $\Sigma=\mathrm{Cay}(H,S)$ is a connected trivalent Cayley graph which is neither normal nor arc-transitive, where $H=\langle a,b\mid a^n=b^2=1, bab=a^{-1}\rangle(n\geq 3)$. Then S is a generating subset of H and $|S|=3$. So S must contain an involution of H outside $\langle a\rangle$. As $\mathrm{Aut}\,(H)$ is transitive on the coset $b\langle a\rangle$, we may assume that $S=\{b,x,y\}$ for $x,y\in H\setminus\langle b\rangle$.

Suppose first that x is not an involution. Then we must have $y=x^{-1}$. Since S generates H, one has $\langle a\rangle=\langle x\rangle$, and so $bxb=x^{-1}$. Then there exists an automorphism of H sending b,x to b,a respectively. So we may assume that $S=\{b,a,a^{-1}\}$. Now it is easy to check that Σ is isomorphic to the generalized Petersen graph $P(n,1)$. Since Σ is not arc-transitive, by [8, 14], we have $|\mathrm{Aut}\,(\Sigma)|=2|H|$, and so Σ would be a normal Cayley graph of H, a contradiction.

Therefore, both x and y must be involutions. Suppose that $x\in\langle a\rangle$. Then n is even and $x=a^{n/2}$. Again since S generates H, one has $y=ba^j$, where $1\leq j\leq n-1$ and either $(j,n)=1$ or $(j,n)=2$ and $\frac{n}{2}$ is odd. Note that the subgroup of $\mathrm{Aut}\,(H)$ fixing b is transitive on the set of generators of $\langle a\rangle$ and that $\langle a^{n/2}\rangle$ is the center of H. There exists $\alpha\in\mathrm{Aut}\,(H)$ such that

$$S^{\alpha}=\{b,ba,a^{\frac{n}{2}}\}\text{ or }\{b,ba^2,a^{\frac{n}{2}}\}.$$

Without loss of generality, we may assume that $S=\{b,ba,a^{\frac{n}{2}}\}$ or $\{b,ba^2,a^{\frac{n}{2}}\}$. If $S=\{b,ba^2,a^{\frac{n}{2}}\}$, we shall prove that $\Sigma\cong P(n,1)$. Note that the generalized Petersen graph $P(n,1)$ has vertex set $\{u_i,v_i\mid i\in\mathbb{Z}_n\}$ and edge set $\{\{u_i,u_{i+1}\},\{v_i,v_{i+1}\},\{u_i,v_i\}\mid i\in\mathbb{Z}_n\}$. Define a map from $V(\Sigma)$ to $V(P(n,1))$ as follows:

$$\varphi:\quad \begin{array}{ll} a^{2i}\mapsto u_{2i}, & a^{2i+\frac{n}{2}}\mapsto v_{2i},\\ ba^{2i}\mapsto u_{2i-1}, & ba^{2i+\frac{n}{2}}\mapsto v_{2i-1},\end{array}$$

where $0\leq i\leq\frac{n}{2}-1$. It is easy to see that φ is an isomorphism form Σ to $P(n,1)$. Since Σ is not arc-transitive, by [8, 14], we have $|\mathrm{Aut}\,(\Sigma)|=2|H|$, and so Σ would be a normal Cayley graph of H, a contradiction. If $S=\{b,ba,a^{\frac{n}{2}}\}$, then Σ has a connected subgraph $\Sigma_1=\mathrm{Cay}(H,\{b,ba\})$ which is a cycle of length $2n$, and Σ is just the graph obtained from

Σ_1 by adding a 1-factor such that each vertex g of Σ_1 is adjacent to its antipodal vertex $a^{\frac{n}{2}}g$. Then $R(H) \rtimes \mathbb{Z}_2 \cong \mathrm{Aut}\,(\Sigma_1) \leq \mathrm{Aut}\,(\Sigma)$, and then since Σ is assumed to be not arc-transitive, $\mathrm{Aut}\,(\Sigma)$ will fix the 1-factor $\{\{g, a^{\frac{n}{2}}g\} \mid g \in H\}$ setwise. This implies that $\mathrm{Aut}\,(\Sigma) \leq \mathrm{Aut}\,(\Sigma_1)$ and so $\mathrm{Aut}\,(\Sigma) = \mathrm{Aut}\,(\Sigma_1)$. Consequently, we have Σ is a normal Cayley graph of H, a contradiction.

Similarly, we have $y \notin \langle a \rangle$. Then we may assume that $x = ba^i$ and $y = ba^j$ for some $1 \leq i, j \leq n-1$ and $i \neq j$. Then $S = \{b, ba^i, ba^j\} \subseteq b\langle a \rangle$. This implies that Σ is a bipartite graph with $\langle a \rangle$ and $b\langle a \rangle$ as its two partition sets. Since Σ is not arc-transitive, $\mathrm{Aut}\,(\Sigma)_1$ is intransitive on the neighbourhood S of 1, and since Σ is not a normal Cayley graph of H, there exists a unique element, say $s \in S$, such that $\mathrm{Aut}\,(\Sigma)_1 = \mathrm{Aut}\,(\Sigma)_s$. Considering the fact that $\mathrm{Aut}\,(H)$ is transitive on $b\langle a \rangle$, without loss of generality, we may assume that $\mathrm{Aut}\,(\Sigma)_1 = \mathrm{Aut}\,(\Sigma)_b$ and $\mathrm{Aut}\,(\Sigma)_1$ swaps ba^i and ba^j. Then for any $h \in H$, we have

$$\mathrm{Aut}\,(\Sigma)_h = (\mathrm{Aut}\,(\Sigma)_1)^{R(h)} = (\mathrm{Aut}\,(\Sigma)_b)^{R(h)} = \mathrm{Aut}\,(\Sigma)_{bh}.$$

Direct computation shows that

$$\Sigma_2(1) = \{a^{-i}, a^{-j}, a^i, a^{i-j}, a^j, a^{j-i}\},$$
$$\Sigma_3(1) = \{ba^{-i}, ba^{j-i}, ba^{-j}, ba^{i-j}, ba^{2i}, ba^{j+i}, ba^{2i-j}, ba^{2j}, ba^{2j-i}\}.$$

Let $\mathrm{Aut}\,(\Sigma)_1^*$ be the kernel of $\mathrm{Aut}\,(\Sigma)_1$ acting on S. Take an $\alpha \in \mathrm{Aut}\,(\Sigma)_1^*$. Then α fixes every element in S. As $\mathrm{Aut}\,(\Sigma)_h = \mathrm{Aut}\,(\Sigma)_{bh}$ for any $h \in H$, α will fix $b(ba^i) = a^i$ and $b(ba^j) = a^j$. Note that $\Sigma(ba^i) \setminus \{1, a^i\} = \{a^{i-j}\}$ and $\Sigma(ba^j) \setminus \{1, a^j\} = \{a^{j-i}\}$. Then α also fixes a^{i-j} and a^{j-i}, and then α also fixes ba^{i-j} and ba^{j-i}.

If $|\Sigma_2(1)| = 6$, then it is easy to check that a^{-i} is the unique common neighbor of b and ba^{j-i}. So α also fixes a^{-i}. Now one can see that α fixes every vertex in $\Sigma_2(1)$. If $|\Sigma_2(1)| < 6$ and either $|\Sigma_1(b) \cap \Sigma_1(ba^i)| > 1$ or $|\Sigma_1(b) \cap \Sigma_1(ba^j)| > 1$, then α also fixes every vertex in $\Sigma_2(1)$. In the above two cases, by the connectedness and vertex-transitivity of Σ, α would fix all vertices of Σ, implying that $\alpha = 1$. Hence, $\mathrm{Aut}\,(\Sigma)_1^* = 1$ and $\mathrm{Aut}\,(\Sigma)_1 \cong \mathbb{Z}_2$. This forces that Σ is a normal Cayley graph of H, a contradiction.

Thus, we have $|\Sigma_2(1)| < 6$ and $|\Sigma_1(b) \cap \Sigma_1(ba^i)| = |\Sigma_1(b) \cap \Sigma_1(ba^j)| = 1$. This implies that $\Sigma_1(ba^i) \cap \Sigma_1(ba^j) = \{1, a^{i-j}\} = \{1, a^{j-i}\}$, and so $a^{i-j} = a^{j-i}$. It follows that a^{i-j} is an involution, and hence n is even and $a^{i-j} = a^{n/2}$. So $S = \{b, ba^i, ba^{i+n/2}\}$. As S generates H, one has $\langle a^i, a^{n/2} \rangle = \langle a \rangle$. So either $(i, n) = 1$ or $(i, n) = 2$ and $\frac{n}{2}$ is odd. Note that the subgroup of $\mathrm{Aut}\,(H)$ fixing b is transitive on the set of generators of $\langle a \rangle$ and that $\langle a^{n/2} \rangle$ is the center of H. There exists $\alpha \in \mathrm{Aut}\,(H)$ such that

$$S^\alpha = \{b, ba, ba^{1+\frac{n}{2}}\} \text{ or } \{b, ba^2, ba^{2+\frac{n}{2}}\}.$$

Let β_ϵ be the automorphism of H induced by the map $a \mapsto a^{-1}, b \mapsto ba^\epsilon$, where $\epsilon \in \mathbb{Z}_2$. Then

$$\{b, ba, ba^{1+\frac{n}{2}}\}^{\beta_1} = \{b, ba, ba^{\frac{n}{2}}\}, \text{and } \{b, ba^2, ba^{2+\frac{n}{2}}\}^{\beta_2} = \{b, ba^2, ba^{\frac{n}{2}}\}.$$

If $\frac{n}{2}$ is odd, then the map $\eta : a \mapsto a^{2+\frac{n}{2}}, b \mapsto ba^{\frac{n}{2}}$ induces an automorphism of H, and $\{b, ba, ba^{\frac{n}{2}}\}^\eta = \{b, ba^2, ba^{\frac{n}{2}}\}$. So there always exists $\gamma \in \mathrm{Aut}\,(H)$ such that $S^\gamma = \{b, ba, ba^{\frac{n}{2}}\}$, completing the proof of the first part of our theorem.

Finally, we shall prove $\Sigma \cong \mathrm{CL}_{4\cdot\frac{n}{2}}$. Without loss of generality, assume that $S = \{b, ba, ba^{\frac{n}{2}}\}$. Recall that $V(\mathrm{CL}_{4\cdot\frac{n}{2}}) = \{x_i^r \mid i \in \mathbb{Z}_{2n}, r \in \mathbb{Z}_2\}$ and $E(\mathrm{CL}_{4\cdot\frac{n}{2}}) =$

$\{\{x_i^r, x_{i+1}^r\}, \{x_{2i}^r, x_{2i+1}^{r+1}\}, \mid i \in \mathbb{Z}_{2n}, r \in \mathbb{Z}_2\}$. Let ϕ be a map from $V(\Sigma)$ to $V(\mathrm{CL}_{4\cdot\frac{n}{2}})$ as following:

$$\phi: \begin{array}{ll} a^i \mapsto x_{2i}^0, & a^{i+\frac{n}{2}} \mapsto x_{2i}^1, \\ ba^j \mapsto x_{2j-1}^0, & ba^{j+\frac{n}{2}} \mapsto x_{2j-1}^1, \end{array}$$

where $0 \le i \le \frac{n}{2}-1$ and $1 \le j \le \frac{n}{2}$. It is easy to check that ϕ is an isomorphism from Σ and $X(\mathrm{CL}_{4\cdot\frac{n}{2}})$, as desired. □

4 Multi-cross ladder graphs

The goal of this section is to prove Theorem 1.2. We first show that each $\mathrm{MCL}_{4m,2}$ is a bi-Cayley graph.

Lemma 4.1. *The multi-cross ladder graph* $\mathrm{MCL}_{4m,2}$ *is isomorphic to the bi-Cayley graph* $\mathrm{BiCay}(H, \{c, ca\}, \{ca, ca^2b\}, \{1\})$*, where*

$$H = \langle a, b, c \mid a^m = b^2 = c^2 = 1, a^b = a, a^c = a^{-1}, b^c = b\rangle.$$

Proof. For convenience, let Γ be the bi-Cayley graph given in our lemma, and let $X = \mathrm{MCL}_{4m,2}$. Let ϕ be a map from $V(X)$ to $V(\Gamma)$ defined by the following rule:

$$\phi: \begin{array}{llll} x_{2t}^{1,1} \mapsto (a^t)_0, & x_{2t+1}^{1,1} \mapsto (ca^{t+1})_0, & x_{2t}^{1,0} \mapsto (ca^{t+1})_1, & x_{2t+1}^{1,0} \mapsto (a^t)_1, \\ x_{2t}^{0,1} \mapsto (ca^{t+1}b)_1, & x_{2t+1}^{0,1} \mapsto (a^tb)_1, & x_{2t}^{0,0} \mapsto (a^tb)_0, & x_{2t+1}^{0,0} \mapsto (ca^{t+1}b)_0, \end{array}$$

where $t \in \mathbb{Z}_m$.

It is easy to see that ϕ is an adjacency preserving isomorphism from X to Γ. □

Remark 1 Let m be odd, let $e = ab$ and $f = ca$. Then the group given in Lemma 4.1 has the following presentation:

$$H = \langle e, f \mid e^{2m} = f^2 = 1, e^f = e^{-1}\rangle.$$

Clearly, in this case, H is a dihedral group. Furthermore, the corresponding bi-Cayley graph given in Lemma 4.1 will be

$$\mathrm{BiCay}(H, \{f, fe\}, \{f, fe^{m-1}\}, \{1\}).$$

Proof of Theorem 1.2. By Lemma 4.1, we may let $\Gamma = \mathrm{MCL}_{4m,2}$ be just the bi-Cayley graph $\mathrm{BiCay}(H, R, L, S)$, where

$$\begin{array}{l} H = \langle a, b, c \mid a^m = b^2 = c^2 = 1, a^b = a, a^c = a^{-1}, b^c = b\rangle, \\ R = \{c, ca\}, L = \{ca, ca^2b\}, S = \{1\}. \end{array}$$

We first prove the sufficiency. Assume first that m is even. Then the map

$$a \mapsto ab, b \mapsto b, c \mapsto cb$$

induces an automorphism, say α of H of order 2. Furthermore, $R^\alpha = \{c, ca\}^\alpha = caLca$, $L^\alpha = \{ca, ca^2b\}^\alpha = caRca$ and $S^\alpha = \{1\}^\alpha = ca\{1\}ca = S^{-1}$. By Proposition 2.4, $\delta_{\alpha,ca,ca} \in \mathrm{Aut}(\Gamma)$ and $\mathcal{R}(H) \rtimes \langle\delta_{\alpha,ca,ca}\rangle$ acts regularly on $V(\Gamma)$. Consequently, by Proposition 2.1, Γ is a Cayley graph.

Assume now that m is odd and $3 \mid m$. In this case, we shall use the bi-Cayley presentation for Γ as in Remark 5.1, that is,

$$\Gamma = \mathrm{BiCay}(H, \{f, fe\}, \{f, fe^{m-1}\}, \{1\}),$$

where

$$H = \langle e, f \mid e^{2m} = f^2 = 1, e^f = e^{-1} \rangle.$$

Let β be a permutation of $V(\Gamma)$ defined as following:

$$\beta: \begin{array}{ll} (f^i e^{3t+1})_i \leftrightarrow (f^i e^{m+3t+1})_i, & (f^{i+1} e^{3t+1})_i \leftrightarrow (f^i e^{m+3t+1})_{i+1}, \\ (f^{i+1} e^{3t+2})_i \leftrightarrow (f^{i+1} e^{m+3t+2})_i, & (f^i e^{3t+2})_i \leftrightarrow (f^{i+1} e^{m+3t+2})_{i+1}, \\ (e^{3t})_i \leftrightarrow (fe^{3t})_{i+1}, & (e^{m+3t})_i \leftrightarrow (fe^{m+3t})_{i+1}, \end{array}$$

where $t \in \mathbb{Z}_{\frac{m}{3}}$ and $i \in \mathbb{Z}_2$. It is easy to check that β is an automorphism of Γ of order 2. Furthermore, $\mathcal{R}(e), \mathcal{R}(f)$ and β satisfy the following relations:

$$\begin{array}{l} \mathcal{R}(e)^{2m} = \mathcal{R}(f)^2 = \beta^2 = 1,\ \mathcal{R}(f)^{-1}\mathcal{R}(e)\mathcal{R}(f) = \mathcal{R}(e)^{-1},\ \mathcal{R}(f)^{-1}\beta\mathcal{R}(f) = \beta, \\ \mathcal{R}(e)^6\beta = \beta\mathcal{R}(e)^6,\ \mathcal{R}(e)^2\beta = \beta\mathcal{R}(e)^4\beta\mathcal{R}(e)^{-2}. \end{array}$$

Let $G = \langle \mathcal{R}(e^2), \mathcal{R}(f), \beta \rangle$ and $P = \langle \mathcal{R}(e^2), \beta \rangle$. Then $\mathcal{R}(f) \notin P$ and $G = P\langle \mathcal{R}(f) \rangle$. Since $\mathcal{R}(e)^6\beta = \beta\mathcal{R}(e)^6$, we have $\mathcal{R}(e^6) \in Z(P)$. Since $\mathcal{R}(e)^2\beta = \beta\mathcal{R}(e)^4\beta\mathcal{R}(e)^{-2}$, it follows that

$$(\mathcal{R}(e)^2\beta)^3 = \mathcal{R}(e)^2\beta[\beta\mathcal{R}(e)^4\beta\mathcal{R}(e)^{-2}]\mathcal{R}(e)^2\beta = \mathcal{R}(e^6).$$

Let $N = \langle \mathcal{R}(e^6) \rangle$. Clearly, N is a normal subgroup of G. Furthermore,

$$P/N = \langle \mathcal{R}(e^2)N, \beta N \mid \mathcal{R}(e^2)^3 N = \beta^2 N = (\mathcal{R}(e^2)\beta)^3 N = N \rangle \cong A_4.$$

Therefore, $|P| = 4m$ and $|G| \leq 8m$.

Let

$$\begin{array}{ll} \Delta_{00} = \{x_0 \mid x \in \langle e^2, f \rangle\}, & \Delta_{10} = \{(ex)_0 \mid x \in \langle e^2, f \rangle\}, \\ \Delta_{01} = \{x_1 \mid x \in \langle e^2, f \rangle\}, & \Delta_{11} = \{(ex)_1 \mid x \in \langle e^2, f \rangle\}. \end{array}$$

Then Δ_{ij}'s $(i, j \in \mathbb{Z}_2)$ are four orbits of $\langle \mathcal{R}(e^2), \mathcal{R}(f) \rangle$. Moreover,

$$1_0^{\beta\mathcal{R}(f)} = 1_1 \in \Delta_{01},\ e_0^{\beta} = (e^{m+1})_0 \in \Delta_{00},\ e_1^{\beta} = (fe^{m+1})_0 \in \Delta_{00}.$$

This implies that G is transitive on $V(\Gamma)$. Hence, $|G| = 8m$ and so G is regular on $V(\Gamma)$, and by Proposition 2.1, Γ is a Cayley graph.

To prove the necessity, it suffices to prove that if m is odd and $3 \nmid m$, then Γ is a non-Cayley graph. In this case, we shall use the original definition of $\Gamma = \mathrm{MCL}_{4m,2}$. Suppose that m is odd and $3 \nmid m$. We already know from [7, Proposition 3.3] that Γ is vertex-transitive. Let $A = \mathrm{Aut}(\Gamma)$. For $m = 5$ or 7, using Magma [4], Γ is a non-Cayley graph. In what follows, we assume that $m \geq 11$.

For each $j \in \mathbb{Z}_m$, $\mathrm{C}_j^0 = (x_{2j}^{0,0}, x_{2j+1}^{0,0}, x_{2j}^{0,1}, x_{2j+1}^{0,1})$ and $\mathrm{C}_j^1 = (x_{2j}^{1,1}, x_{2j+1}^{1,1}, x_{2j}^{1,0}, x_{2j+1}^{1,0})$ are two 4-cycles. Set $\mathcal{F} = \{\mathrm{C}_j^i \mid i \in \mathbb{Z}_2, j \in \mathbb{Z}_m\}$. From the construction of $\Gamma = \mathrm{MCL}_{4m,2}$, it is easy to see that in $\Gamma = \mathrm{MCL}_{4m,2}$ passing each vertex there is exactly one 4-cycle, which belongs to $\mathcal{F}$. Clearly, any two distinct 4-cycles in $\mathcal{F}$ are vertex-disjoint. This implies that $\Delta = \{V(\mathrm{C}_j^i) \mid i \in \mathbb{Z}_2, j \in \mathbb{Z}_m\}$ is an A-invariant partition of $V(\Gamma)$. Consider the quotient graph Γ_Δ, and let T be the kernel of A acting on Δ. Then $\Gamma_\Delta \cong \mathrm{C}_m[2K_1]$, the lexicographic product of a cycle of length m and an empty graph of order 2. Hence $A/T \leq \mathrm{Aut}\,(\mathrm{C}_m[2K_1]) \cong \mathbb{Z}_2^m \rtimes D_{2m}$. Note that between any two adjacent vertices of Γ_Δ there is exactly one edge of $\Gamma = \mathrm{MCL}_{4m,2}$. Then T fixes each vertex of Γ and hence $T = 1$. So we may view A as a subgroup of $\mathrm{Aut}\,(\Gamma_\Delta) \cong \mathrm{Aut}\,(\mathrm{C}_m[2K_1]) \cong \mathbb{Z}_2^m \rtimes D_{2m}$.

For convenience, we will simply use the C_j^i's to represent the vertices of Γ_Δ. Then Γ_Δ has vertex set

$$\{\mathrm{C}_j^0, \mathrm{C}_j^1 \mid j \in \mathbb{Z}_m\}$$

and edge set

$$\{\{\mathrm{C}_j^0, \mathrm{C}_{j+1}^0\}, \{\mathrm{C}_j^1, \mathrm{C}_{j+1}^1\}, \{\mathrm{C}_j^0, \mathrm{C}_{j+1}^1\}, \{\mathrm{C}_j^1, \mathrm{C}_{j+1}^0\} \mid j \in \mathbb{Z}_m\}.$$

Let $\mathcal{B} = \{\{\mathrm{C}_j^0, \mathrm{C}_j^1\} \mid j \in \mathbb{Z}_m\}$. Then $\mathcal{B}$ is an $\mathrm{Aut}\,(\Gamma_\Delta)$-invariant partition of $V(\Gamma_\Delta)$. Let K be the kernel of $\mathrm{Aut}\,(\Gamma_\Delta)$ acting on $\mathcal{B}$. Then $K = \langle k_0 \rangle \times \langle k_2 \rangle \times \cdots \times \langle k_{m-1} \rangle$, where we use k_i to denote the transposition $(\mathrm{C}_j^0\ \mathrm{C}_j^1)$ for $j \in \mathbb{Z}_m$. Clearly, K is the maximal normal 2-subgroup of $\mathrm{Aut}\,(\Gamma_\Delta)$.

Suppose to the contrary that $\Gamma = \mathrm{MCL}_{4m,2}$ is a Cayley graph. By Proposition 2.1, A has a subgroup, say G acting regularly on $V(\Gamma)$. Then G has order $8m$, and

$$G/(G \cap K) \cong GK/K \leq \mathrm{Aut}\,(\Gamma_\Delta)/K \lesssim D_{2m}.$$

Since m odd, it follows that $|G \cap K| = 4$ or 8, and so $G \cap K \cong \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$.

If $G \cap K \cong \mathbb{Z}_2^2$, then $|GK/K| = 2m$ and $GK/K = \mathrm{Aut}\,(\Gamma_\Delta)/K \cong D_{2m}$. So $GK = \mathrm{Aut}\,(\Gamma_\Delta) \cong \mathbb{Z}_2^m \rtimes D_{2m}$. Let M be a Hall $2'$-subgroup of G. Then $M \cong \mathbb{Z}_m$ and M is also a Hall $2'$-subgroup of $\mathrm{Aut}\,(\Gamma_\Delta)$. Clearly, $\mathrm{Aut}\,(\Gamma_\Delta)$ is solvable, so all Hall $2'$-subgroups of $\mathrm{Aut}\,(\Gamma_\Delta)$ are conjugate. Without loss of generality, we may let $M = \langle \alpha \rangle$, where α is the following permutation on $V(\Gamma_\Delta)$:

$$\alpha = (\mathrm{C}_0^0\ \mathrm{C}_1^0 \ldots \mathrm{C}_{m-1}^0)(\mathrm{C}_0^1\ \mathrm{C}_1^1 \ldots \mathrm{C}_{m-1}^1).$$

Then $K \rtimes \langle \alpha \rangle$ acts transitively on $V(\Gamma_\Delta)$. Clearly, $C_K(\alpha)$ is contained in the center of $K \rtimes \langle \alpha \rangle$. So $C_K(\alpha)$ is semiregular on $V(\Gamma_\Delta)$. This implies that

$$C_K(\alpha) = \langle k_0 k_1 \ldots k_{m-1} \rangle \cong \mathbb{Z}_2.$$

On the other hand, let $L = (G \cap K)M$. Clearly, $G \cap K \trianglelefteq G$, so L is a subgroup of G of order $4m$. For any odd prime factor p of m, let P be a Sylow p-subgroup of M. Then P is also a Sylow p-subgroup of L, and since M is cyclic, one has $M \leq N_L(P)$. By Sylow theorem, we have $|L : N_L(P)| = kp + 1 \mid 4$ for some integer k. Since $3 \nmid m$, one has $L = N_L(P)$. It follows that $M \trianglelefteq L$ and so $L = M \times (G \cap K)$. This implies that $G \cap K \leq C_K(M) = C_K(\alpha) \cong \mathbb{Z}_2$, a contradiction.

If $G \cap K \cong \mathbb{Z}_2^3$, then $|GK/K| = m$. Furthermore, $GK/K \cong \mathbb{Z}_m$ and GK/K acts on $\mathcal{B}$ regularly. Since G is transitive on $V(\Gamma)$, there exists $g \in G$ such that $(x_0^{1,1})^g = x_1^{1,1}$, where $x_0^{1,1}, x_1^{1,1} \in \mathrm{C}_0^1$. As $V(\Gamma_\Delta) = \{\mathrm{C}_j^i \mid i \in \mathbb{Z}_2, j \in \mathbb{Z}_m\}$, g fixes the 4-cycle $\mathrm{C}_0^1 = (x_0^{1,1}, x_1^{1,1}, x_0^{1,0}, x_1^{1,0})$. Since $\mathcal{B} = \{\{\mathrm{C}_j^0, \mathrm{C}_j^1\} \mid j \in \mathbb{Z}_m\}$ is also A-invariant, g fixes $\{\mathrm{C}_0^0, \mathrm{C}_0^1\}$ setwise. Since GK/K acts on $\mathcal{B}$ regularly, g fixes $\{\mathrm{C}_j^0, \mathrm{C}_j^1\}$ setwise for every $j \in \mathbb{Z}_m$. Observe that $\{x_0^{1,1}, x_{2m-1}^{1,1}\}$ and $\{x_1^{1,1}, x_2^{1,1}\}$ are the unique edges of Γ between C_0^1 and C_{m-1}^1, C_0^1 and C_2^1, respectively. This implies that g will map C_{m-1}^1 to C_2^1, contradicting that g fixes $\{\mathrm{C}_j^0, \mathrm{C}_j^1\}$ setwise for every $j \in \mathbb{Z}_m$. □

5 A family of trivalent VNC bi-dihedrants

The goal of this section is to prove the following lemma which gives a new family of trivalent vertex-transitive non-Cayley bi-dihedrants. To be brief, a vertex-transitive non-Cayley graph is sometimes simply called a *VNC graph*.

Lemma 5.1. *Let $H = \langle a, b \mid a^n = b^2 = 1, a^b = a^{-1}\rangle$ be a dihedral group, where $n = 48\ell$ and $\ell \geq 1$. Then $\Gamma = \mathrm{BiCay}(H, \{b, ba\}, \{ba^{24\ell}, ba^{12\ell-1}\}, \{1\})$ is a VNC dihedrant.*

Proof. We first define a permutation on $V(\Gamma)$ as follows:

$$\begin{array}{llll} g: & (a^{3r})_0 \mapsto (a^{3r})_0, & (a^{3r})_1 \mapsto (ba^{3r})_0, & (a^{3r+1})_0 \mapsto (ba^{3r+1})_1, \\ & (a^{3r+1})_1 \mapsto (a^{24\ell+3r+1})_1, & (a^{3r+2})_i \mapsto (ba^{12\ell+3r+2})_{i+1}, & (ba^{3r})_0 \mapsto (a^{3r})_1, \\ & (ba^{3r})_1 \mapsto (ba^{24\ell+3r})_1, & (ba^{3r+1})_0 \mapsto (ba^{3r+1})_0, & (ba^{3r+1})_1 \mapsto (a^{3r+1})_0, \\ & (ba^{3r+2})_i \mapsto (a^{-12\ell+3r+2})_{i+1}, & & \end{array}$$

where $r \in \mathbb{Z}_{16\ell},\ i \in \mathbb{Z}_2$.

It is easy to check that g is an involution, and furthermore, for any $t \in \mathbb{Z}_{16\ell}$, we have

$$\begin{array}{l} \Gamma((a^{3r})_0)^g = \{(a^{3r})_1, (ba^{3r})_0, (ba^{3r+1})_0\} = \Gamma((a^{3r})_0), \\ \Gamma((a^{3r})_1)^g = \{(ba^{3r})_1, (a^{3r})_0, (a^{3r-1})_0\} = \Gamma((ba^{3r})_0), \\ \Gamma((ba^{3r})_1)^g = \{(ba^{24\ell+3r})_0, (a^{3r})_1, (a^{12\ell+3r+1})_1\} = \Gamma((ba^{24\ell+3r})_1), \\ \Gamma((a^{3r+1})_0)^g = \{(ba^{3r+1})_0, (a^{24\ell+3r+1})_1, (a^{36\ell+3r+2})_1\} = \Gamma((ba^{3r+1})_1), \\ \Gamma((a^{3r+1})_1)^g = \{(a^{24\ell+3r+1})_0, (ba^{3r+1})_1, (ba^{36\ell+3r})_1\} = \Gamma((a^{24\ell+3r+1})_1), \\ \Gamma((ba^{3r+1})_0)^g = \{(ba^{3r+1})_1, (a^{3r+1})_0, (a^{3r})_0\} = \Gamma((ba^{3r+1})_0), \\ \Gamma((a^{3r+2})_0)^g = \{(ba^{12\ell+3r+2})_0, (a^{36\ell+3r+2})_1, (a^{3r+3})_1\} = \Gamma((ba^{12\ell+3r+2})_1), \\ \Gamma((a^{3r+2})_1)^g = \{(ba^{12\ell+3r+2})_1, (a^{12\ell+3r+2})_0, (a^{12\ell+3r+1})_0\} = \Gamma((ba^{12\ell+3r+2})_0). \end{array}$$

This implies that g is an automorphism of Γ. Observing that g maps 1_1 to b_0, it follows that $\langle \mathcal{R}(H), g\rangle$ is transitive on $V(\Gamma)$, and so Γ is a vertex-transitive graph.

Below, we shall first prove the following claim.

Claim. $\mathrm{Aut}\,(\Gamma)_{1_0} = \langle g\rangle$.

Let $A = \mathrm{Aut}\,(\Gamma)$. It is easy to see that g fixes 1_0, and so $g \in A_{1_0}$. To prove the Claim, it suffices to prove that $|A_{1_0}| = 2$.

Note that the neighborhood $\Gamma(1_0)$ of 1_0 in Γ is $= \{1_1, b_0, (ba)_0\}$. By a direct computation, we find that in Γ there is a unique 8-cycle passing through 1_0, 1_1 and b_0, that is,

$$\mathrm{C}_0 = (1_0, 1_1, (ba^{24\ell})_1, (ba^{24\ell})_0, (a^{24\ell})_0, (a^{24\ell})_1, b_1, b_0, 1_0).$$

Furthermore, in Γ there is no 8-cycle passing through 1_0 and $(ba)_0$. So A_{1_0} fixes $(ba)_0$.

If A_{1_0} also fixes 1_1 and b_0, then A_{1_0} will fix every neighbor of 1_0, and the connectedness and vertex-transitivity of Γ give that $A_{1_0} = 1$, a contradiction. Therefore, A_{1_0} swaps 1_1 and b_0, and $(ba)_0$ is the unique neighbor of 1_0 such that $A_{1_0} = A_{(ba)_0}$. It follows that $\{1_0, (ba)_0\}$ is a block of imprimitivity of A acting on $V(\Gamma)$. Since Γ is vertex-transitive, every $v \in V(\Gamma)$ has a unique neighbor, say u such that $A_u = A_v$. Then the set

$$\mathcal{B} = \{\{u, v\} \in E(\Gamma) \mid A_u = A_v\}$$

forms an A-invariant partition of $V(\Gamma)$. Clearly, $\{1_0, (ba)_0\} \in \mathcal{B}$. Similarly, since C_0 is also the unique 8-cycle of Γ passing through 1_0, 1_1 and b_0, A_{1_1} swaps 1_0 and b_0, and

$(ba^{12\ell-1})_1$ is the unique neighbor of 1_1 such that $A_{1_1} = A_{(ba^{12\ell-1})_1}$. So $\{1_1, (ba^{12\ell-1})_1\} \in \mathcal{B}$. Set

$$\mathcal{B}_0 = \{\{1_0, (ba)_0\}^{\mathcal{R}(h)} \mid h \in H\} \text{ and } \mathcal{B}_1 = \{\{1_1, (ba^{12\ell-1})_1\}^{\mathcal{R}(h)} \mid h \in H\}.$$

Clearly, $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$.

Now we consider the quotient graph $\Gamma_{\mathcal{B}}$ of Γ relative to $\mathcal{B}$. It is easy to see that $\langle \mathcal{R}(a)\rangle$ acts semiregularly on $\mathcal{B}$ with $\mathcal{B}_0$ and $\mathcal{B}_1$ as its two orbits. So $\Gamma_{\mathcal{B}}$ is isomorphic to a bi-Cayley graph over $\langle a\rangle$. Set $B_0 = \{1_0, (ba)_0\}$ and $B_1 = \{1_1, (ba^{12\ell-1})_1\}$. Then one may see that the neighbors of B_0 in $\Gamma_{\mathcal{B}}$ are: $B_0^{\mathcal{R}(a)}, B_0^{\mathcal{R}(a^{-1})}, B_1, B_1^{\mathcal{R}(a^{-12\ell+2})}$, and the neighbors of B_1 in $\Gamma_{\mathcal{B}}$ are: $B_1^{\mathcal{R}(a^{12\ell+1})}, B_1^{\mathcal{R}(a^{-12\ell-1})}, B_0, B_0^{\mathcal{R}(a^{12\ell-2})}$. So

$$\Gamma_{\mathcal{B}} \cong \Gamma' = \mathrm{BiCay}(\langle a\rangle, \{a, a^{-1}\}, \{a^{12\ell+1}, a^{-12\ell-1}\}, \{1, a^{-12\ell+2}\}).$$

Observe that there is one and only one edge of Γ between B_0 and any one of its neighbors in $\Gamma_{\mathcal{B}}$. Clearly, A acts transitively on $V(\Gamma_{\mathcal{B}})$, so there is one and only one edge of Γ between every two adjacent blocks of $\mathcal{B}$. It follows that A acts faithfully on $V(\Gamma_{\mathcal{B}})$, and hence we may view A as a subgroup of $\mathrm{Aut}\,(\Gamma_{\mathcal{B}})$. Recall that $g \in A_{1_0} = A_{(ba)_0}$. Moreover, g swaps the two neighbors 1_1 and b_0 of 1_0. Clearly, $1_1 \in B_1$ and $b_0 \in B_0^{\mathcal{R}(a^{-1})}$, so g swaps the two blocks B_1 and $B_0^{\mathcal{R}(a^{-1})}$. Similarly, g swaps the two neighbors $(ba)_1$ and a_0 of $(ba)_0$. Clearly, $(ba)_1 \in B_1^{\mathcal{R}(a^{-12\ell+2})}$ and $a_0 \in B_0^{\mathcal{R}(a)}$, so g swaps the two blocks $B_1^{\mathcal{R}(a^{-12\ell+2})}$ and $B_0^{\mathcal{R}(a)}$. Note that $\mathcal{R}(ab)$ swaps the two vertices in B_0. So $\langle g, \mathcal{R}(ab)\rangle$ acts transitively on the neighborhood of B_0 in $\Gamma_{\mathcal{B}}$. This implies that A acts transitively on the arcs of $\Gamma_{\mathcal{B}}$, and so Γ' is a tetravalent arc-transitive bi-circulant. In [11], a characterization of tetravalent edge-transitive bi-circulants is given. It is easy to see that our graph Γ' belongs to Class 1(c) of [11, Theorem 1.1]. By checking [11, Theorem 4.1], we see that the stabilizer $\mathrm{Aut}\,(\Gamma')_u$ of $u \in V(\Gamma')$ has order 4. This implies that $|A| = 4|V(\Gamma_{\mathcal{B}})| = 8n$. Consequently, $|A_{1_0}| = 2$ and so our claim holds.

Now we are ready to finish the proof. Suppose to the contrary that Γ is a Cayley graph. By Proposition 2.1, A contains a subgroup, say J acting regularly on $V(\Gamma)$. By Claim, J has index 2 in A, and since $g \in A_{1_0}$, one has $A = J \rtimes \langle g\rangle$. It is easy to check that $\mathcal{R}(a), \mathcal{R}(b)$ and g satisfy the following relations:

$$(g\mathcal{R}(b))^4 = \mathcal{R}(a^{24\ell}),\ g\mathcal{R}(a^3) = \mathcal{R}(a^3)g,\ g\mathcal{R}(ba) = \mathcal{R}(ba)g,\ g = \mathcal{R}(a)(g\mathcal{R}(b))^2\mathcal{R}(a^{12\ell-1}).$$

Suppose that $\mathcal{R}(H) \nleq J$. Then $A = J\mathcal{R}(H)$. Since $|J|/|\mathcal{R}(H)| = 2$, it follows that $|\mathcal{R}(H) : J \cap \mathcal{R}(H)| = 2$. Thus, $J \cap \mathcal{R}(H) = \langle \mathcal{R}(a)\rangle$ or $\langle \mathcal{R}(a^2), \mathcal{R}(b)\rangle$. If $\mathcal{R}(H) \cap J = \langle \mathcal{R}(a)\rangle$, then we have $\mathcal{R}(b) \notin J, \mathcal{R}(a) \in J$, and hence $A = J \cup J\mathcal{R}(b) = J \cup Jg$, implying that $J\mathcal{R}(b) = Jg$. It follows that $g\mathcal{R}(b) \in J$, and then $g = \mathcal{R}(a)(g\mathcal{R}(b))^2\mathcal{R}(a^{12\ell-1}) \in J$ due to $\mathcal{R}(a) \in J$, a contradiction. If $\mathcal{R}(H) \cap J = \langle \mathcal{R}(a^2), \mathcal{R}(b)\rangle$, then $\mathcal{R}(a) \notin J$, and again we have $A = J \cup J\mathcal{R}(a) = J \cup Jg$, implying that $J\mathcal{R}(a) = Jg$. So, $\mathcal{R}(a)g, g\mathcal{R}(a^{-1}) \in J$. Then

$$g = \mathcal{R}(a)g\mathcal{R}(b)g\mathcal{R}(b)\mathcal{R}(a^{12\ell-1}) = (\mathcal{R}(a)g)\mathcal{R}(b)(g\mathcal{R}(a^{-1}))\mathcal{R}(ba^{12\ell-2}) \in J,$$

a contradiction.

Suppose that $\mathcal{R}(H) \leq J$. Then $|J : \mathcal{R}(H)| = 2$ and $\mathcal{R}(H) \trianglelefteq J$. Since J is regular on $V(\Gamma)$, by Proposition 2.4, there exists a $\delta_{\alpha,x,y} \in J$ such that $1_0^{\delta_{\alpha,x,y}} = 1_1$, where $\alpha \in$

$\mathrm{Aut}\,(H)$ and $x, y \in H$. By the definition of $\delta_{\alpha,x,y}$, we have $1_1 = 1_0^{\delta_{\alpha,x,y}} = (x \cdot 1^\alpha)_1 = x_1$, implying that $x = 1$. Furthermore, we have the following relations:

$$R^\alpha = x^{-1}Lx, L^\alpha = y^{-1}Ry, S^\alpha = y^{-1}S^{-1}x,$$

where $R = \{b, ba\}, L = \{ba^{24\ell}, ba^{12\ell-1}\}, S = \{1\}$. In particular, the last equality implies that $x = y$ due to $S = \{1\}$. So we have $x = y = 1$. From the proof of Claim we know that $B_0 = \{1_0, (ba)_0\}$ and $B_1 = \{1_1, (ba^{12\ell-1})_1\}$ are two blocks of imprimitivity of A acting on $V(\Gamma)$. So we have $((ba)_0)^{\delta_{\alpha,1,1}} = (ba^{12\ell-1})_1$. It follows that $(ba)^\alpha = ba^{12\ell-1}$, and then from $R^\alpha = L$ we obtain that $b^\alpha = ba^{24\ell}$. Consequently, we have $a^\alpha = a^{36\ell-1}$. One the other hand, we have $\{b, ba\} = R = L^\alpha = \{b, ba^{24\ell+1}\}$. This forces that $ba = ba^{24\ell+1}$, which is clearly impossible. □

6 Two families of trivalent Cayley bi-dihedrants

In this section, we shall prove two lemmas which will be used the proof of Theorem 1.3.

Lemma 6.1. *Let $H = \langle a, b \mid a^{12m} = b^2 = 1, a^b = a^{-1}\rangle$ be a dihedral group with m odd. Then for each $i \in \mathbb{Z}_{12m}$, $\Gamma = \mathrm{BiCay}(H, \{b, ba^i\}, \{ba^{6m}, ba^{3m-i}\}, \{1\})$ is a Cayley graph whenever $\langle a^i, a^{3m}\rangle = \langle a\rangle$.*

Proof. Let g be a permutation of $V(\Gamma)$ defined as follows:

$$g:\ \begin{array}{ll} (a^{6km+3ri})_j \mapsto (ba^{6(k+1)m+3ri})_{j+1}, & (ba^{6km+3ri})_j \mapsto (a^{6km+3ri})_{j+1},\\ (a^{3km+(3r+1)i})_0 \mapsto (a^{3(k+1)m+(3r+1)i})_0, & (ba^{3km+(3r+1)i})_0 \mapsto (a^{3(k+1)m+(3r+1)i})_1,\\ (a^{3km+(3r+1)i})_1 \mapsto (ba^{3(k+1)m+(3r+1)i})_0, & (ba^{3km+(3r+1)i})_1 \mapsto (ba^{3(k-1)m+(3r+1)i})_1,\\ (a^{3km+(3r+2)i})_0 \mapsto (ba^{3(k+1)m+(3r+2)i})_1, & (ba^{3km+(3r+2)i})_0 \mapsto (ba^{3(k+1)m+(3r+2)i})_0,\\ (a^{3km+(3r+2)i})_1 \mapsto (a^{3(k-1)m+(3r+2)i})_1, & (ba^{3km+(3r+2)i})_1 \mapsto (a^{3(k+1)m+(3r+2)i})_0, \end{array}$$

where $r \in \mathbb{Z}_m$, $k \in \mathbb{Z}_4$ and $j \in \mathbb{Z}_2$.

It is easy to check that $g \in \mathrm{Aut}\,(\Gamma)$. Furthermore, one may check that g and $\mathcal{R}(a^2)$ satisfy the following relations:

$$\begin{array}{l} \mathcal{R}(a^{12m}) = g^4 = 1,\ g^2 = \mathcal{R}(a^{6m}),\ \mathcal{R}(a^6)g = g\mathcal{R}(a^6),\\ \mathcal{R}(b^{-1})g\mathcal{R}(b) = g\mathcal{R}(a^{6m}),\ \mathcal{R}(a^2)g = g\mathcal{R}(a^4)g\mathcal{R}(a^{-2}). \end{array}$$

By the last equality, we have

$$(\mathcal{R}(a^2)g)^3 = [g\mathcal{R}(a^4)g\mathcal{R}(a^{-2}))]\mathcal{R}(a^2)g\mathcal{R}(a^2)g = g\mathcal{R}(a^4)g^2\mathcal{R}(a^2)g.$$

It then follows from the second and third equalities that

$$g\mathcal{R}(a^4)g^2\mathcal{R}(a^2)g = g\mathcal{R}(a^{6+6m})g = g^2\mathcal{R}(a^{6+6m}) = \mathcal{R}(a^6).$$

Therefore, $(\mathcal{R}(a^2)g)^3 = \mathcal{R}(a^6)$.

Let $G = \langle \mathcal{R}(a^2), \mathcal{R}(b), g\rangle$ and $T = \langle \mathcal{R}(a^6)\rangle$. Then $T \trianglelefteq G$ and

$$\begin{aligned} G/T &= \langle \mathcal{R}(a^2)T, \mathcal{R}(b)T, gT\rangle\\ &= \langle \mathcal{R}(a^2)T, gT \mid \mathcal{R}(a^2)^3T = g^2T = (\mathcal{R}(a^2)g)^3T = T\rangle \rtimes \langle \mathcal{R}(b)T\rangle\\ &\cong A_4 \rtimes \mathbb{Z}_2. \end{aligned}$$

So $|G| = 48m$.

Let

$$\Omega_{00} = \{t_0 \mid t \in \langle a^2, b\rangle\}, \qquad \Omega_{01} = \{t_1 \mid t \in \langle a^2, b\rangle\},$$
$$\Omega_{10} = \{(at)_0 \mid t \in \langle a^2, b\rangle\}, \quad \Omega_{11} = \{(at)_1 \mid t \in \langle a^2, b\rangle\}.$$

Then Ω_{ij}'s $(0 \le i, j \le 1)$ are orbits of T and $V(\Gamma) = \bigcup_{0\le i,j\le 1} \Omega_{ij}$. Since $1_0^g = (ba^{6m})_1 \in \Omega_{01}$, $a_0^g = (a^{3m+1})_0 \in \Omega_{00}$ and $a_1^g = (ba^{3m+1})_1 \in \Omega_{01}$, it follows that G is transitive, and so regular on $V(\Gamma)$. By Proposition 2.1, Γ is a Cayley graph on G, as required. □

Lemma 6.2. *Let $H = \langle a, b \mid a^{12m} = b^2 = 1, a^b = a^{-1}\rangle$ be a dihedral group with m even and $4 \nmid m$. Then the following two bi-Cayley graphs:*

$$\Gamma_1 = \mathrm{BiCay}(H, \{b, ba\}, \{ba^{6m}, ba^{3m-1}\}, \{1\}),$$
$$\Gamma_2 = \mathrm{BiCay}(H, \{b, ba\}, \{ba^{6m}, ba^{9m-1}\}, \{1\})$$

are both Cayley graphs.

Proof. Let $V = H_0 \cup H_1$. Then $V(\Gamma_1) = V(\Gamma_2) = V$. We first define two permutations on V as follows:

$$g_1: \begin{array}{ll} (a^{4r})_i \mapsto (ba^{6m+4r})_{i+1}, & (ba^{4r})_i \mapsto (a^{4r})_{i+1}, \\ (a^{4r+1})_i \mapsto (ba^{9m+4r+1})_{i+1}, & (ba^{4r+1})_i \mapsto (a^{3m+4r+1})_{i+1}, \\ (a^{4r+2})_i \mapsto (ba^{4r+2})_{i+1}, & (ba^{4r+2})_i \mapsto (a^{6m+4r+2})_{i+1}, \\ (a^{4r+3})_i \mapsto (ba^{3m+4r+3})_{i+1}, & (ba^{4r+3})_i \mapsto (a^{9m+4r+3})_{i+1}, \end{array}$$

$$g_2: \begin{array}{ll} (a^{4r})_i \mapsto (ba^{6m+4r})_{i+1}, & (ba^{4r})_i \mapsto (a^{4r})_{i+1}, \\ (a^{4r+1})_i \mapsto (ba^{3m+4r+1})_{i+1}, & (ba^{4r+1})_i \mapsto (a^{9m+4r+1})_{i+1}, \\ (a^{4r+2})_i \mapsto (ba^{4r+2})_{i+1}, & (ba^{4r+2})_i \mapsto (a^{6m+4r+2})_{i+1}, \\ (a^{4r+3})_i \mapsto (ba^{9m+4r+3})_{i+1}, & (ba^{4r+3})_i \mapsto (a^{3m+4r+3})_{i+1}, \end{array}$$

where $r \in \mathbb{Z}_{3m}$ and $i \in \mathbb{Z}_2$.

It is easy to check that $g_j \in \mathrm{Aut}(\Gamma_j)$ for $j = 1$ or 2. Furthermore, $\mathcal{R}(a^2), \mathcal{R}(b)$ and g_j $(j = 1$ or $2)$ satisfy the following relations:

$$\mathcal{R}(a^{12m}) = \mathcal{R}(b^2) = g_j^4 = 1, \mathcal{R}(b)\mathcal{R}(a^2)\mathcal{R}(b) = \mathcal{R}(a^{-2}),$$
$$g_j^2 = \mathcal{R}(a^{6m}), \mathcal{R}(b)g_j\mathcal{R}(b) = g_j^{-1},$$
$$g_1^{-1}\mathcal{R}(a)g_1 = \mathcal{R}(a^{3m+1}), g_2^{-1}\mathcal{R}(a)g_2 = \mathcal{R}(a^{9m+1}).$$

For $j = 1$ or 2, let $G_j = \langle \mathcal{R}(a), \mathcal{R}(b), g_j\rangle$. From the above relations it is east to see that

$$G_j = (\langle \mathcal{R}(a)\rangle\langle g_j\rangle) \rtimes \langle \mathcal{R}(b)\rangle$$

has order at most $48m$. Observe that $1_0^{g_j} = (ba^{6m})_1 \in H_1$ for $j = 1$ or 2. It follows that G_j is transitive on $V(\Gamma_j)$, and so G_j acts regularly on $V(\Gamma_j)$. By Proposition 2.1, each Γ_j is a Cayley graph. □

7 Vertex-transitive trivalent bi-dihedrants

In this section, we shall give a complete classification of trivalent vertex-transitive non-Cayley bi-dihedrants. For convenience of the statement, throughout this section, we shall make the following assumption.

Assumption I.

- H: the dihedral group $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle (n \geq 3)$,
- $\Gamma = \mathrm{BiCay}(H,\ R,\ L,\ \{1\})$: a connected trivalent 2-type vertex-transitive bi-Cayley graph over the group H (in this case, $|R| = |L| = 2$),
- G: a minimum group of automorphisms of Γ subject to that $\mathcal{R}(H) \leq G$ and G is transitive on the vertices but intransitive on the arcs of Γ.

The following lemma given in [20] shows that the group G must be solvable.

Lemma 7.1. [20, Lemma 6.2] *$G = \mathcal{R}(H)P$ is solvable, where P is a Sylow 2-subgroup of G.*

7.1 H_0 and H_1 are blocks of imprimitivity of G

The case where H_0 and H_1 are blocks of imprimitivity of G has been considered in [20], and the main result is the following proposition.

Proposition 7.2. [20, Theorem 1.3] *If H_0 and H_1 are blocks of imprimitivity of G on $V(\Gamma)$, then either Γ is Cayley or one of the following occurs:*

(1) $(R, L, S) \equiv (\{b,\ ba^{\ell+1}\}, \{ba,\ ba^{\ell^2+\ell+1}\}, \{1\})$, *where* $n \geq 5$, $\ell^3 + \ell^2 + \ell + 1 \equiv 0 \pmod n$, $\ell^2 \not\equiv 1 \pmod n$*;*

(2) $(R, L, S) \equiv (\{ba^{-\ell},\ ba^{\ell}\}, \{a,\ a^{-1}\}, \{1\})$, *where* $n = 2k$ *and* $\ell^2 \equiv -1 \pmod k$. *Furthermore, Γ is also a bi-Cayley graph over an abelian group $\mathbb{Z}_n \times \mathbb{Z}_2$.*

Furthermore, all of the graphs arising from (1)-(2) are vertex-transitive non-Cayley.

In particular, it is proved in [20] that if n is odd and Γ is not a Cayley graph, then H_0 and H_1 are blocks of imprimitivity of G on $V(\Gamma)$. Consequently, we can get a classification of trivalent vertex-transitive non-Cayley bi-Cayley graphs over a dihedral group D_{2n} with n odd.

Proposition 7.3. [20, Proposition 6.4] *If n is odd, then either Γ is a Cayley graph, or H_0 and H_1 are blocks of imprimitivity of G on $V(\Gamma)$.*

7.2 H_0 and H_1 are not blocks of imprimitivity of G

In this subsection, we shall consider the case where H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$. We begin by citing a lemma from [20].

Lemma 7.4. [20, Lemma 6.3] *Suppose that H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$. Let N be a normal subgroup of G, and let K be the kernel of G acting on $V(\Gamma_N)$. Let Δ be an orbit of N. If N fixes H_0 setwise, then one of the following holds:*

(1) $\Gamma[\Delta]$ *has valency* 1, $|V(\Gamma_N)| \geq 3$ *and* Γ *is a Cayley graph;*

(2) $\Gamma[\Delta]$ *has valency* 0, Γ_N *has valency* 3, *and* $K = N$ *is semiregular.*

The following lemma deals with the case where $\mathrm{Core}_G(\mathcal{R}(H)) = 1$, and in this case we shall see that Γ is just the cross ladder graph.

Lemma 7.5. *Suppose that H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$. If $\mathrm{Core}_G(\mathcal{R}(H)) = \bigcap_{g\in G} \mathcal{R}(H) = 1$, then Γ is isomorphic to the cross ladder graph CL_{4n} with n odd, and furthermore, for any minimal normal subgroup N of G, we have the following:*

(1) *N is a 2-group which is non-regular on $V(\Gamma)$;*

(2) *N does not fix H_0 setwise;*

(3) *every orbit of N consists of two non-adjacent vertices.*

Proof. Let N be a minimal normal subgroup of G. By Lemma 7.1, G is solvable. It follows that N is an elementary abelian r-subgroup for some prime divisor r of $|G|$. Clearly, $N \nleq \mathcal{R}(H)$ due to $\mathrm{Core}_G(\mathcal{R}(H)) = 1$. Then $|N\mathcal{R}(H)|/|\mathcal{R}(H)| \mid |G|/|\mathcal{R}(H)|$. From Lemma 7.1 it follows that $|G|/|\mathcal{R}(H)|$ is a power of 2, and hence N is a 2-group.

Suppose that N is regular on $V(\Gamma)$. Then $N\mathcal{R}(H)$ is transitive on $V(\Gamma)$ and $\mathcal{R}(H)$ is also a 2-group. Therefore, $N\mathcal{R}(H)$ is not transitive on the arcs of Γ. The minimality of G gives that $G = N\mathcal{R}(H)$. Since n is even, $\mathcal{R}(a^{\frac{n}{2}})$ is in the center of $\mathcal{R}(H)$. Set $Q = N\langle\mathcal{R}(a^{\frac{n}{2}})\rangle$. Then $Q \trianglelefteq G$ and then $1 \neq N \cap Z(Q) \trianglelefteq G$. Since N is a minimal normal subgroup of G, one has $N \leq Z(Q)$, and hence Q is abelian. It follows that $\langle\mathcal{R}(a^{\frac{n}{2}})\rangle \trianglelefteq G$, contrary to the assumption that $\mathrm{Core}_G(\mathcal{R}(H)) = 1$. Thus, N is not regular on $V(\Gamma)$. (1) is proved.

For (2), by way of contradiction, suppose that N fixes H_0 setwise. Consider the quotient graph Γ_N of Γ relative to N, and let K be the kernel of G acting on $V(\Gamma_N)$. Take Δ to be an orbit of N on $V(\Gamma)$. Then either (1) or (2) of Lemma 7.4 happens.

For the former, $\Gamma[\Delta]$ has valency 1 and $|V(\Gamma_N)| \geq 3$. Then Γ_N is a cycle. Moreover, any two neighbors of $u \in \Delta$ are in different orbits of N. It follows that the stabilizer N_v of v in N fixes every neighbor of u. The connectedness of Γ implies that $N_v = 1$. Thus, $K = N$ is semiregular and Γ_N is a cycle of length $\ell = 2|\mathcal{R}(H)|/|N|$. So $G/N \leq \mathrm{Aut}\,(\Gamma_N) \cong D_{2\ell}$. If $G/N < \mathrm{Aut}\,(\Gamma_N)$, then $|G : N| = \ell$ and so $|G| = 2|\mathcal{R}(H)|$. This implies that $\mathcal{R}(H) \trianglelefteq G$, contrary to the assumption that $\mathrm{Core}_G(\mathcal{R}(H)) = 1$. If $G/N = \mathrm{Aut}\,(\Gamma_N)$, then $|G : \mathcal{R}(H)| = 4$. Since $N \nleq \mathcal{R}(H)$ and since N fixes H_0 setwise, one has $|G : \mathcal{R}(H)N| = 2$. It follows that $\mathcal{R}(H)N \trianglelefteq G$. Clearly, H_0 and H_1 are just two orbits of $\mathcal{R}(H)N$, and they are also two blocks of imprimitivity of G on $V(\Gamma)$, a contradiction.

For the latter, $\Gamma[\Delta]$ has valency 0, Γ_N has valency 3 and $N = K$ is semiregular. Let $\bar{H}_i$ be the set of orbits of N contained in H_i with $i = 1, 2$. Then $\Gamma_N[\bar{H}_0]$ and $\Gamma_N[\bar{H}_1]$ are of valency 2 and the edges between $\bar{H}_0$ and $\bar{H}_1$ form a perfect matching. Without loss of generality, we may assume that $1_0 \in \Delta$. Since $\mathcal{R}(H)$ acts on H_0 by right multiplication, we have the subgroup of $\mathcal{R}(H)$ fixing Δ setwise is just $\mathcal{R}(H)_\Delta = \{\mathcal{R}(h) \mid h_0 \in \Delta\}$. If $\mathcal{R}(H)_\Delta \leq \langle\mathcal{R}(a)\rangle$, then $\mathcal{R}(H)_\Delta \trianglelefteq \mathcal{R}(H)$, and the transitivity of $\mathcal{R}(H)$ on H_0 implies that $\mathcal{R}(H)_\Delta$ will fix all orbits of N contained in H_0. Since the edges between $\bar{H}_0$ and $\bar{H}_1$ are independent, $\mathcal{R}(H)_\Delta$ fixes all orbits of N. It follows that $\mathcal{R}(H)_\Delta \leq N$, namely, $\mathcal{R}(H)N/N$ acts regularly on $\bar{H}_0$. Then $|\mathcal{R}(H)/(\mathcal{R}(H) \cap N)| = |\mathcal{R}(H)N/N| = |H_0/N|$, and so $|N| = |\mathcal{R}(H) \cap N|$, forcing $N \leq \mathcal{R}(H)$, a contradiction. Thus, $\mathcal{R}(H)_\Delta \nleq \langle\mathcal{R}(a)\rangle$, and so $\langle\mathcal{R}(a)\rangle\mathcal{R}(H)_\Delta = \mathcal{R}(H)$. This implies that $\langle\mathcal{R}(a), N\rangle/N$ is transitive and so regular on $\bar{H}_0$. Similarly, $\langle\mathcal{R}(a), N\rangle/N$ is also regular on $\bar{H}_1$. Thus, Γ_N is a trivalent 2-type bi-Cayley graph over $\langle\mathcal{R}(a), N\rangle/N$. By [24, Lemma 5.3], $\bar{H}_0$ and $\bar{H}_0$ are blocks of imprimitivity of G/N, and so H_0 and H_1 are blocks of imprimitivity of G, a contradiction.

So far, we have completed the proof of (2). Then N does not fix H_0 setwise, and then $N\mathcal{R}(H)$ is transitive on $V(\Gamma)$. The minimality of G gives that $G = N\mathcal{R}(H)$. Let P and P_1 be Sylow 2-subgroups of G and $\mathcal{R}(H)$, respectively, such that $P_1 \leq P$. Then $N \leq P$ and $P = NP_1$.

If n is even, then by a similar argument to the second paragraph, a contradiction occurs.

Thus, n is odd. As $H \cong D_{2n}$, $P_1 \cong \mathbb{Z}_2$ and P_1 is non-normal in $\mathcal{R}(H)$. So $N \cap \mathcal{R}(H) = 1$. Clearly, $|V(\Gamma)| = 4n$. If N is semiregular on $V(\Gamma)$, then $N \cong \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, and then $|G| = |\mathcal{R}(H)||N| = 2|\mathcal{R}(H)|$ or $4|\mathcal{R}(H)|$. Since $\mathrm{Core}_G(\mathcal{R}(H)) = 1$, we must have $|G : \mathcal{R}(H)| = 4$ and $G \lesssim \mathrm{Sym}(4)$. Since n is odd, one has $n = 3$ and $H \cong \mathrm{Sym}(3)$. So $G \cong \mathrm{Sym}(4)$ and hence $G_{1_0} \cong \mathbb{Z}_2$. Then all involutions of $G(\cong \mathrm{Sym}(4))$ not contained in N are conjugate. Take $1 \neq g \in G_{1_0}$. Then g is an involution which is not contained in N because N is semiregular on $V(\Gamma)$. Since $\mathcal{R}(H) \cap N = 1$, every involution in $\mathcal{R}(H)$ would be conjugate to g. This is clearly impossible because $\mathcal{R}(H)$ is semiregular on $V(\Gamma)$. Thus, N is not semiregular on $V(\Gamma)$. (3) is proved.

Since n is odd, we have $|V(\Gamma_N)| > 2$. Since N is not semiregular on $V(\Gamma)$, Γ_N has valency 2 and $\Gamma[\Delta]$ has valency 0. This implies that the subgraph induced by any two adjacent two orbits of N is either a union of several cycles or a perfect matching. Thus, Γ_N has even order. As Γ has order $4n$ with n odd, every orbit of N has length 2. It is easy to see that Γ is isomorphic to the cross ladder graph CL_{4n}. □

The following is the main result of this section.

Theorem 7.6. *Suppose that H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$. Then $\Gamma = \mathrm{BiCay}(H, R, L, S)$ is vertex-transitive non-Cayley if and only if one of the followings occurs:*

(1) $(R, L, S) \equiv (\{b, ba\}, \{b, ba^{2m}\}, \{1\})$, *where* $n = 2(2m + 1)$, $m \not\equiv 1 \pmod 3$, *and the corresponding graph is isomorphic the multi-cross ladder graph* $\mathrm{MCL}_{4m,2}$*;*

(2) $(R, L, S) \equiv (\{b, ba\}, \{ba^{24\ell}, ba^{12\ell-1}\}, \{1\})$, *where* $n = 48\ell$ *and* $\ell \geq 1$.

Proof. The sufficiency can be obtained from Theorem 1.2 and Lemma 5.1. We shall prove the necessity in the following subsection by a series of lemmas. □

7.3 Proof of the necessity of Theorem 7.6

The purpose of this subsection is to prove the necessity of Theorem 7.6. Throughout this subsection, we shall always assume that H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$ and that $\Gamma = \mathrm{BiCay}(H, R, L, S)$ is vertex-transitive non-Cayley. In this subsection, we shall always use the following notation.

Assumption II. Let $N = \mathrm{Core}_G(\mathcal{R}(H))$.

Our first lemma gives some properties of the group N.

Lemma 7.7. $1 < N < \langle \mathcal{R}(a) \rangle$, $|\langle \mathcal{R}(a) \rangle : N| = n/|N|$ *is odd and the quotient graph* Γ_N *of* Γ *relative to* N *is isomorphic to the cross ladder graph* $\mathrm{CL}_{4n/|N|}$.

Proof. If $N = 1$, then from Lemma 7.5 it follows that $\Gamma \cong \mathrm{CL}_{4n}$ which is a Cayley graph by Theorem 1.1, a contradiction. Thus, $N > 1$. Since H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$, one has $N < \mathcal{R}(H)$.

Consider the quotient graph Γ_N. Clearly, N fixes H_0 setwise. Recall that H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$ and that Γ is non-Cayley. Applying Lemma 7.4, we see that Γ_N is a trivalent 2-type bi-Cayley graph over $\mathcal{R}(H)/N$. This implies that $|\mathcal{R}(H) : N| > 2$, and since H is a dihedral group, one has $N < \langle \mathcal{R}(a) \rangle$.

Again, by Lemma 7.4, $\mathcal{R}(H)/N$ acts semiregularly on $V(\Gamma_N)$ with two orbits, $\bar{H}_0$ and $\bar{H}_1$, where $\bar{H}_i$ is the set of orbits of N contained in H_i with $i = 1, 0$. Furthermore,

N is just the kernel of G acting on $V(\Gamma_N)$ and N acts semiregularly on $V(\Gamma)$. Then G/N is also a minimal vertex-transitive automorphism group of Γ_N containing $\mathcal{R}(H)/N$. If $\bar{H}_0$ and $\bar{H}_1$ are blocks of imprimitivity of G/N on $V(\Gamma_N)$, then H_0 and H_1 will be blocks of imprimitivity of G on $V(\Gamma)$, which is impossible by our assumption. Thus, $\bar{H}_0$ and $\bar{H}_1$ are not blocks of imprimitivity of G/N on $V(\Gamma_N)$. Since $N = \mathrm{Core}_G(\mathcal{R}(H))$, $\mathrm{Core}_{G/N}(\mathcal{R}(H)/N)$ is trivial. Then from Lemma 7.5 it follows that $\Gamma_N \cong \mathrm{CL}_{\frac{4n}{|N|}}$, where $\frac{n}{|N|}$ is odd. □

Next, we introduce another notation which will be used in the proof.

Assumption III. Take M/N to be a minimal normal subgroup of G/N.

We shall first consider some basic properties of the quotient graph Γ_M of Γ relative to M.

Lemma 7.8. *The quotient graph Γ_M of Γ relative to M is a cycle of length $n/|N|$. Furthermore, every orbit of M on $V(\Gamma)$ is a union of an orbit of N on H_0 and an orbit of N on H_1, and these two orbits of N are non-adjacent.*

Proof. Applying Lemma 7.5 to Γ_N and G/N, we obtain the following facts:

(a) M/N is an elementary abelian 2-group which is not regular on $V(\Gamma_N)$,

(b) M/N does not fix $\bar{H}_0$ setwise,

(c) every orbit of M/N on $V(\Gamma_N)$ consists of two non-adjacent vertices of Γ_N.

From (b) and (c) it follows that every orbit of M on $V(\Gamma)$ is just a union of an orbit of N on H_0 and an orbit of N on H_1, and these two orbits are non-adjacent. Since every orbit of N on $V(\Gamma)$ is an independent subset of $V(\Gamma)$, each orbit of M on $V(\Gamma)$ is also an independent subset.

Recall that $\Gamma_N \cong \mathrm{CL}_{4m}$ where $m = \frac{n}{|N|}$ is odd. The quotient graph of Γ_N relative to M/N is just a cycle of length m, and so the quotient graph Γ_M of Γ relative to M is also a cycle of length m. □

By Lemma 7.8, each orbit of M on $V(\Gamma)$ is an independent subset. It follows that the subgraph induced by any two adjacent orbits of M is either a perfect matching or a union of several cycles. For convenience of the statement, the following notations will be used in the remainder of the proof:

Assumption IV.

(1) Let Δ and Δ' be two adjacent orbits of M on $V(\Gamma)$ such that $\Gamma[\Delta \cup \Delta']$ is a union of several cycles.

(2) Let $\Delta = \Delta_0 \cup \Delta_1$ and $\Delta' = \Delta'_0 \cup \Delta'_1$, where $\Delta_0, \Delta'_0 \subseteq H_0$ and $\Delta_1, \Delta'_1 \subseteq H_1$ are four orbits of N on $V(\Gamma)$.

(3) $1_0 \in \Delta_0$.

Since $\Gamma[\Delta]$ and $\Gamma[\Delta']$ are both null graphs and since $\Gamma[\Delta \cup \Delta']$ is a union of several cycles, we have the following easy observation.

Lemma 7.9. $\Gamma[\Delta_i \cup \Delta'_j]$ *is a perfect matching for any* $0 \leq i, j \leq 1$.

The following lemma tells us the possibility of R (Recall that we assume that $\Gamma = \mathrm{BiCay}(H, R, L, \{1\})$).

Lemma 7.10. *Up to graph isomorphism, we may assume that $R = \{b, ba^i\}$ with $i \in \mathbb{Z}_n \setminus \{0\}$ and that $b_0 \in \Delta'_0$. Furthermore, we have*

$$\Delta_0 = \{h_0 \mid \mathcal{R}(h) \in N\}, \Delta'_0 = \{(bh)_0 \mid \mathcal{R}(h) \in N\},$$
$$\Delta'_1 = \{h_1 \mid \mathcal{R}(h) \in N\}, \Delta_1 = \{(bh)_1 \mid \mathcal{R}(h) \in N\},$$

and 1_1 is adjacent to $(ba^l)_1 \in \Delta_1$ for some $\mathcal{R}(a^l) \in N$.

Proof. Recall that N is a proper subgroup of $\langle \mathcal{R}(a) \rangle$ and that $n/|N|$ is odd. Since n is even by Proposition 7.3, it follows that N is of even order, and so the unique involution $\mathcal{R}(a^{n/2})$ of $\langle \mathcal{R}(a) \rangle$ is contained in N. As $1_0 \in \Delta_0$ and $N \leq \langle \mathcal{R}(a) \rangle$ acts on H_0 by right multiplication, one has $\Delta_0 = \{h_0 \mid h \in N\}$. Since $\Gamma[\Delta_0]$ is an empty graph, one has $a^{n/2} \notin R$. By Proposition 2.3 (1), we have $\langle R \cup L \rangle = H$, and since R and L are both self-inverse, either $R \subseteq b\langle a \rangle$ or $L \subseteq b\langle a \rangle$. By Proposition 2.3 (4), we may assume that $R \subseteq b\langle a \rangle$.

Recall that $\Gamma[\Delta_i \cup \Delta'_j]$ is a perfect matching for any $0 \leq i, j \leq 1$. Then 1_0 is adjacent to $r_0 \in \Delta'_0$ for some $r \in R$. Since $R \subseteq b\langle a \rangle$ and $\mathrm{Aut}(H)$ is transitive on $b\langle a \rangle$, by Proposition 2.3 (3), we may assume that $r = b$. So 1_0 is adjacent to $b_0 \in \Delta'_0$. Since $N \leq \langle \mathcal{R}(a) \rangle$ acts on H_i with $i = 0$ or 1 by right multiplication, we see that the two orbits Δ_0, Δ'_0 of N are just the form as given in the lemma. Since $S = \{1\}$, the edges between H_0 and H_1 form a perfect matching. This enables us to obtain another two orbits Δ_1, Δ'_1 of N which have the form as given in the lemma.

By Lemma 7.9, $\Gamma[\Delta_1 \cup \Delta'_1]$ is a perfect matching. So we may assume that 1_1 is adjacent to $(ba^l)_1 \in \Delta_1$ for some $\mathcal{R}(a^l) \in N$. □

Now we shall introduce some new notations which will be used in the following.

Assumption V.

(1) Let $T = \langle \mathcal{R}(a^l) \rangle$ be of order t, where a^l is given in the above lemma.

(2) Let

$$\Omega_0 = \{(a^{i\frac{n}{t}})_0 \mid 0 \leq i \leq t-1\}, \quad \Omega_1 = \{(ba^{i\frac{n}{t}})_1 \mid 0 \leq i \leq t-1\},$$
$$\Omega'_0 = \{(ba^{i\frac{n}{t}})_0 \mid 0 \leq i \leq t-1\}, \quad \Omega'_1 = \{(a^{i\frac{n}{t}})_1 \mid 0 \leq i \leq t-1\}.$$

(3) $\mathcal{B} = \{B^{\mathcal{R}(h)} \mid h \in H\}$, where $B = \Omega_0 \cup \Omega_1$.

(4) Let $B' = \Omega'_0 \cup \Omega'_1$. Then $B' = B^{\mathcal{R}(b)}$.

Lemma 7.11. *The followings hold.*

(1) $T \leq N$.

(2) $\Omega_0, \Omega_1, \Omega'_0, \Omega'_1$ *are four orbits of* T.

(3) $\Gamma[\Omega_0 \cup \Omega_1 \cup \Omega'_0 \cup \Omega'_1]$ *is a cycle of length* $4t$.

(4) $\mathcal{B}$ *is a* G*-invariant partition of* $V(\Gamma)$.

Proof. By Lemma 7.10, we see that $\mathcal{R}(a^l) \in N$, and so $T \leq N$. (1) holds. Since $T = \langle \mathcal{R}(a^l) \rangle$ is assumed to be of order t, one has $T = \langle \mathcal{R}(a^{n/t}) \rangle$, and then one can obtain (2). By the adjacency rule of bi-Cayley graph, we can obtain (3).

Set $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_0' \cup \Omega_1'$ and $B = \Omega_0 \cup \Omega_1$. By Lemma 7.8, $\Gamma[\Delta]$ is a null graph, and so $B = \Delta \cap \Omega$. Since Γ has valency 3, it follows that $\Delta \cup \Delta'$ is a block of imprimitivity of G on $V(\Gamma)$, and hence Ω is also a block of imprimitivity of G on $V(\Gamma)$ since $\Gamma[\Omega]$ is a component of $\Gamma[\Delta \cup \Delta']$. Since Δ is also a block of imprimitivity of G on $V(\Gamma)$, $B(= \Delta \cap \Omega)$ is a block of imprimitivity of G on $V(\Gamma)$. Then $\mathcal{B} = \{B^{\mathcal{R}(h)} \mid h \in H\}$ is a G-invariant partition of $V(\Gamma)$. □

Lemma 7.12. *$T < N$ and the quotient graph $\Gamma_{\mathcal{B}}$ of Γ relative to $\mathcal{B}$ is isomorphic to the cross ladder graph $\mathrm{CL}_{\frac{4n}{2t}}$. Moreover, T is the kernel of G acting on $\mathcal{B}$.*

Proof. Let $K_{\mathcal{B}}$ be the kernel of G acting on $\mathcal{B}$. Clearly, $T \leq K_{\mathcal{B}}$. Let $B' = \Omega_0' \cup \Omega_1'$. Then $B' = B^{\mathcal{R}(b)} \in \mathcal{B}$. Let $B^{\mathcal{R}(h)} \in \mathcal{B}$ be adjacent to B and $B^{\mathcal{R}(h)} \neq B'$.

Suppose that $\Gamma[B \cup B^{\mathcal{R}(h)}]$ is a perfect matching. Since G is transitive on $\mathcal{B}$, $\Gamma_{\mathcal{B}}$ is a cycle of length $\frac{2n}{t}$. Clearly, $G/K_{\mathcal{B}}$ is vertex-transitive but not edge-transitive on $\Gamma_{\mathcal{B}}$, so $G/K_{\mathcal{B}} \cong D_{2n/t}$. If $t = 1$, then it is easy to see that $\Gamma \cong \mathrm{CL}_{4n}$ which is a Cayley graph by Theorem 1.1, a contradiction. If $t > 1$, then since $\Gamma[\Omega] = \Gamma[B \cup B']$ is a cycle of length $4t$, $K_{\mathcal{B}}$ acts faithfully on B, and so $K_{\mathcal{B}} \leq \mathrm{Aut}\,(\Gamma[B \cup B']) \cong D_{8t}$. Since $K_{\mathcal{B}}$ fixes B, one has $|K_{\mathcal{B}}| \mid 4t$, implying that $|G| = |K_{\mathcal{B}}| \cdot \frac{2n}{t} \mid 8n$. As $|R(H)| = 2n$ and $\mathcal{R}(H)$ is non-normal in G, one has $|K_{\mathcal{B}}| = 4t$ due to $T \leq K_{\mathcal{B}}$. In view of the fact that $K_{\mathcal{B}} \lesssim D_{8t}$, $K_{\mathcal{B}}$ has a characteristic cyclic subgroup, say J, of order $2t$. Then we have $J \trianglelefteq G$ because $K_{\mathcal{B}} \trianglelefteq G$. Clearly, J is regular on B and $J \cap N = T$, so $J\mathcal{R}(H)$ is regular on $V(\Gamma)$. It follows from Proposition 2.1 that Γ is a Cayley graph, a contradiction.

Therefore, $\Gamma[B \cup B^{\mathcal{R}(h)}]$ is not a perfect matching. If $N = T$, then $B = \Delta$ and $B' = \Delta'$ are orbits of M, and then $\Gamma[B \cup B^{\mathcal{R}(h)}]$ will be a perfect matching, a contradiction. Thus, $N > T$.

Now we are going to prove that $\Gamma_{\mathcal{B}} \cong \mathrm{CL}_{\frac{n}{2t}}$. Since B is adjacent to $B^{\mathcal{R}(h)}$, Ω_i is adjacent to $\Omega_j^{\mathcal{R}(h)}$ for some $i, j \in \{0, 1\}$. Then because Ω_i and $\Omega_j^{\mathcal{R}(h)}$ are orbits of T, $\Gamma[\Omega_i \cup \Omega_j^{\mathcal{R}(h)}]$ is a perfect matching. This implies that $\Gamma_{\mathcal{B}}$ is of valency 3, and so $K_{\mathcal{B}}$ is intransitive on B. As every $B^h \in \mathcal{B}$ is a union of two orbits of T on $V(\Gamma)$, $K_{\mathcal{B}}$ fixes every orbit of T. Since N is cyclic, the normality of N in G implies that $T \trianglelefteq G$. Clearly, Ω_0 is adjacent to three pair-wise different orbits of T, so the quotient graph Γ_T of Γ relative to T is of valency 3. Consequently, the kernel of G acting on $V(\Gamma_T)$ is T. Then $K_{\mathcal{B}} = T$. Now $\mathcal{R}(H)/T \cong D_{2n/t}$ is regular on $\mathcal{B}$, and so $\Gamma_{\mathcal{B}}$ is a Cayley graph over $\mathcal{R}(H)/T$. Furthermore, G/T is not arc-transitive on $\Gamma_{\mathcal{B}}$. Since $\mathcal{R}(H)/T$ is non-normal in G/T, $\Gamma_{\mathcal{B}}$ is a non-normal Cayley graph over $\mathcal{R}(H)/T$. If $\Gamma_{\mathcal{B}}$ is arc-transitive, then by [13, Theorem 1], either $|\mathrm{Aut}\,(\Gamma_{\mathcal{B}})| = 3k|\mathcal{R}(H)/T|$ with $k \leq 2$, or $\Gamma_{\mathcal{B}}$ has order $2 \cdot p$ with $p = 3$ or 7. For the former, since G/T is not arc-transitive on $\Gamma_{\mathcal{B}}$, one has $|G/T : \mathcal{R}(H)/T| \leq 2$, implying $\mathcal{R}(H) \trianglelefteq G$, a contradiction. For the latter, we have $\frac{2n}{t} = 6$ or 14, implying $\frac{n}{t} = 3$ or 7. It follows that T is a maximal subgroup of $\langle \mathcal{R}(a) \rangle$, and so $T = N$, a contradiction. Therefore, $\Gamma_{\mathcal{B}}$ is not arc-transitive. Since $\mathcal{R}(H)/T$ is non-normal in G/T, by Theorem 1.1, one has $\Gamma_{\mathcal{B}} \cong \mathrm{CL}_{\frac{4n}{2t}}$, as required. □

Proof of Theorem 7.6. By Lemma 7.12, we have $\Gamma_{\mathcal{B}} \cong \mathrm{CL}_{\frac{4n}{2t}}$. By the definition of $\mathrm{CL}_{\frac{4n}{2t}}$, we may partition the vertex set of $\Gamma_{\mathcal{B}}$ in the following way:

$$V(\Gamma_{\mathcal{B}}) = V_0 \cup V_1 \cup \cdots V_{\frac{2n}{2t}-2} \cup V_{\frac{2n}{2t}-1}, \text{ where } V_i = \{B_i^0, B_i^1\}, i \in \mathbb{Z}_{\frac{2n}{2t}}$$

and

$$E(\Gamma_{\mathcal{B}}) = \{\{B_{2i}^r, B_{2i+1}^r\}, \{B_{2i+1}^r, B_{2i+2}^s\} \mid i \in \mathbb{Z}_{\frac{n}{2t}}, r, s \in \mathbb{Z}_2\}.$$

Assume that $B_0^0 = B$ and $B_1^0 = B'$. Recall that $B = \Omega_0 \cup \Omega_1$ and $B' = \Omega_0' \cup \Omega_1' = B^{\mathcal{R}(b)}$. Moreover, $\Omega_0, \Omega_1, \Omega_0'$ and Ω_1' are four orbits of T. Then every $B_i^j \in \mathcal{B}$ is just a union of two orbits of T. For convenience, we may let

$$B_i^j = \Omega_{i0}^j \cup \Omega_{i1}^j, i \in \mathbb{Z}_{\frac{2n}{2t}}, j \in \mathbb{Z}_2,$$

where $\Omega_{i0}^j, \Omega_{i1}^j$ are two orbits of T. For $B = B_0^0$, we let $\Omega_0 = \Omega_{00}^0$ and $\Omega_1 = \Omega_{01}^0$, and for $B' = B_1^0$, we let $\Omega_0' = \Omega_{10}^0$ and $\Omega_1' = \Omega_{11}^0$.

For convenience, in the remainder of the proof, we shall use C_{4t} to denote a cycle of length $4t$, and we also call C_{4t} a $4t$-cycle. Recall that $\Gamma[B \cup B'] = \Gamma[B_0^0 \cup B_1^0] \cong \mathrm{C}_{4t}$, and that the edges between $\Omega_{0i}^0 (= \Omega_i)$ and $\Omega_{1j}^0 (= \Omega_j')$ form a perfect matching for all $i, j \in \mathbb{Z}_2$. Since $T \trianglelefteq G$, the quotient graph Γ_T of Γ relative to T has valency 3. So the edges between any two adjacent orbits of T form a perfect matching.

From the construction of $\Gamma_{\mathcal{B}}$, one may see that there exists $g \in G$ such that $\{V_0, V_1\}^g = \{V_{2i}, V_{2i+1}\}$ for each $i \in \mathbb{Z}_{\frac{n}{2t}}$. So for each $i \in \mathbb{Z}_{\frac{n}{2t}}, r \in \mathbb{Z}_2$, we may assume that $\Gamma[B_{2i}^r \cup B_{2i+1}^r] \cong \mathrm{C}_{4t}$, and $\Omega_{(2i)s}^r \sim \Omega_{(2i+1)t}^r$ for all $s, t \in \mathbb{Z}_2$. (Here $\Omega_{(2i)s}^r \sim \Omega_{(2i+1)t}^r$ means that $\Omega_{(2i)s}^r$ and $\Omega_{(2i+1)t}^r$ are adjacent in $\Gamma_{\mathcal{B}}$.) Again, from the construction of $\Gamma_{\mathcal{B}}$, we may assume that

$$\Omega_{(2i+2)0}^0 \sim \Omega_{(2i+1)0}^0, \Omega_{(2i+2)1}^0 \sim \Omega_{(2i+1)0}^1, \Omega_{(2i+2)1}^1 \sim \Omega_{(2i+1)0}^1, \Omega_{(2i+2)1}^1 \sim \Omega_{(2i+1)0}^0,$$

for each $i \in \mathbb{Z}_{\frac{n}{2t}}$. We draw a local subgraph of $\Gamma_{\mathcal{B}}$ in Figure 3. Observing that every

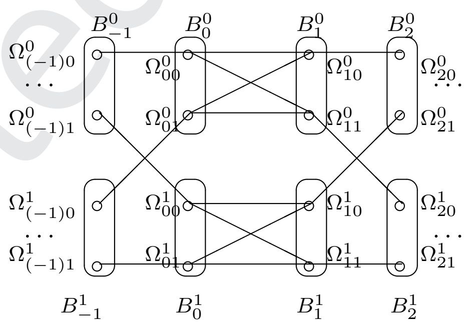

Figure 3: The sketch graph of $\Gamma_{\mathcal{B}}$

$V_i = \{B_i^0, B_i^1\}$ with $i \in \mathbb{Z}_{\frac{2n}{2t}}$ is a block of imprimitivity of $G/K_{\mathcal{B}}$ acting on $V(\Gamma_{\mathcal{B}})$. So every $B_i^0 \cup B_i^1$ with $i \in \mathbb{Z}_{\frac{2n}{2t}}$ is a block of imprimitivity of G acting on $V(\Gamma)$. Let E be the kernel of G acting on the block system $\Lambda = \{B_i^0 \cup B_i^1 \mid i \in \mathbb{Z}_{\frac{2n}{2t}}\}$. Then $G/E \cong D_{\frac{n}{t}}$ acts regularly on Λ. Clearly, $\mathcal{R}(H)$ is also transitive on Ω, so $G/E = \mathcal{R}(H)E/E$. By Lemma 7.12, T is a the kernel of G acting on $\mathcal{B}$. So E/T is an elementary 2-group. From $\mathcal{R}(H)/(\mathcal{R}(H) \cap E) \cong D_{\frac{n}{t}}$ it follows that $\mathcal{R}(H) \cap E = \langle \mathcal{R}(a^{\frac{n}{2t}}) \rangle \cong \mathbb{Z}_{2t}$, and so $(\mathcal{R}(H) \cap E)/T$ is a normal subgroup of G/T of order 2. This implies that $B_i^1 = (B_i^0)^{\mathcal{R}(a^{\frac{n}{2t}})}$ for $i \in \mathbb{Z}_{\frac{2n}{2t}}$. We may further assume that $\Omega_{01}^1 = (\Omega_{00}^0)^{\mathcal{R}(a^{\frac{n}{2t}})} \subseteq B_0^1$. So $\Omega_{00}^0 \cup \Omega_{01}^1$ is just the orbit of $\langle \mathcal{R}(a^{\frac{n}{2t}})\rangle$ containing 1_0.

Observing that $\Omega_{10}^0 \sim \Omega_{20}^0$ and the edges between them are of the form $\{g_0, (ba^i g)_0\}$ with $g_0 \in \Omega_{10}^0$, one has $\Omega_{20}^0 = ba^i\Omega_{10}^0 = ba^i(\Omega_{00}^0)^{\mathcal{R}(b)} = (\Omega_{00}^0)^{\mathcal{R}(a^{-i})}$. So $\Omega_{20}^1 \subseteq (B_0^1)^{\mathcal{R}(a^{-i})}$.

Since $B_1^0 = B' = B^{\mathcal{R}(b)} = (B_0^0)^{\mathcal{R}(b)}$, one has $B_1^1 = (B_0^1)^{\mathcal{R}(b)}$. Recall that $1_1 \in \Omega_{11}^0 = \Omega_1'$ and 1_1 is adjacent to $1_0 \in \Omega_{00}^0 = \Omega_0$ and $(ba^l)_1 \in \Omega_{01}^0 = \Omega_1$. As we assume that $\Omega_{11}^0 \sim \Omega_{20}^1$, 1_1 is adjacent to some vertex in Ω_{20}^1. So $\Omega_{20}^1 \subseteq H_1$ and hence

$$\Omega_{20}^1 = (\Omega_{00}^1)^{\mathcal{R}(a^{-i})} = (\Omega_{01}^0)^{\mathcal{R}(a^{\frac{n}{2t}})\mathcal{R}(a^{-i})} = (\Omega_{01}^0)^{\mathcal{R}(a^{\frac{n}{2t}-i})} = \{(ba^{k\frac{n}{t}})_1 \mid 0 \le k \le t-1\}^{\mathcal{R}(a^{\frac{n}{2t}-i})}.$$

So we have the following claim.

Claim 1 $L = \{ba^l, ba^{k\frac{n}{t}+\frac{n}{2t}-i}\}$ and $R = \{ba^i, b\}$, where $|\mathcal{R}(a^l)| = t$, $i \in \mathbb{Z}_n$ and $0 \le k \le t-1$.

Let $G_{1_0}^*$ be the kernel of G_{1_0} acting on the neighborhood of 1_0 in Γ. Then $G_{1_0}^* \le E_{1_0}$. Recall that for each $i \in \mathbb{Z}_{\frac{n}{2t}}, r \in \mathbb{Z}_2$, $\Gamma[B_{2i}^r \cup B_{2i+1}^r] \cong \mathrm{C}_{4t}$ and the edges between $B_{2i+1}^0 \cup B_{2i+1}^1$ and $B_{2i+2}^0 \cup B_{2i+2}^1$ form a perfect matching. It follows that E acts faithfully on each $B_i^0 \cup B_i^1$. Clearly, $G_{1_0}^* \le E_{1_0}$, so $G_{1_0}^*$ acts faithfully on each $B_i^0 \cup B_i^1$.

Claim 2 If $t > 2$ then $G_{1_0}^* = 1$, and if $t = 2$ then $G_{1_0}^* \le \mathbb{Z}_2$ and $3 \mid n$.

Assume that $t \ge 2$. Since $\Gamma[B_0^0 \cup B_1^0] \cong \mathrm{C}_{4t}$, $G_{1_0}^*$ fixes every vertex in B_0^0, and so fixes every vertex in $\Omega_{(-1)0}^0$ since $\Omega_{(-1)0}^0 \sim \Omega_{00}^0$ (see Figure 3). This implies that $G_{1_0}^*$ fixes $\Omega_{(-1)1}^0$ setwise, and so fixes Ω_{00}^1 setwise since $\Omega_{(-1)1}^0 \sim \Omega_{00}^1$. Consequently, $G_{1_0}^*$ also fixes Ω_{01}^1 setwise. Similarly, by considering the edges between $B_1^0 \cup B_1^1$ and $B_2^0 \cup B_2^1$, we see that $G_{1_0}^*$ fixes both Ω_{10}^1 and Ω_{11}^1 setwise. Recall that the edges between Ω_{0i}^1 and Ω_{1j}^1 form a perfect matching for $i, j \in \mathbb{Z}_2$. As $\Gamma[B_1^0 \cup B_1^1] \cong \mathrm{C}_{4t}$, $G_{1_0}^*$ acts faithfully on Ω_{00}^1 (or Ω_{01}^1), and so $G_{1_0}^* \le \mathbb{Z}_2$.

If $t > 2$, then since $\Gamma[B_{-2}^0 \cup B_{-1}^0] \cong \mathrm{C}_{4t}$, $G_{1_0}^*$ will fix every vertex in this cycle, and in particular, $G_{1_0}^*$ will fix every vertex in $\Omega_{(-1)1}^0$. As $\Omega_{(-1)1}^0 \sim \Omega_{00}^1$, $G_{1_0}^*$ will fix every vertex in Ω_{00}^1. Since $G_{1_0}^*$ acts faithfully on Ω_{00}^1, one has $G_{1_0}^* = 1$.

Let $t = 2$. We shall show that $3 \mid n$. Then $T = \langle \mathcal{R}(a^{\frac{n}{2}})\rangle$. Recall that $(\mathcal{R}(H) \cap E)/T$ is a normal subgroup of G/T of order 2. Let $M = \mathcal{R}(H) \cap E$. Then M is a normal subgroup of G of order 4. Since $\mathcal{R}(H)$ is dihedral, one has $M = \langle \mathcal{R}(a^{\frac{n}{4}})\rangle$. Let $C = C_G(M)$. Then $\mathcal{R}(a) \in C$ and $\mathcal{R}(b) \notin C$. It follows that C is a proper subgroup of G. Since G/E acts regularly on Λ, C_{1_0} fixes every element in Λ. Since C_{1_0} centralizes M, C_{1_0} fixes every vertex in the orbit $\Omega_{00}^0 \cup \Omega_{01}^1$ of M containing 1_0. Clearly, $C_{1_0} \le G_{1_0}$, so $C_{1_0}/(C_{1_0} \cap G_{1_0}^*) \le \mathbb{Z}_2$. As we have shown that $G_{1_0}^*$ acts faithfully on Ω_{01}^1, it follows that $C_{1_0} \cap G_{1_0}^* = 1$ since C_{1_0} fixes Ω_{01}^1 pointwise, and hence $C_{1_0} \le \mathbb{Z}_2$. On the other hand, as $G_{1_0}^* \le \mathbb{Z}_2$, one has $|G| \mid 4 \cdot 4n = 16n$. Since $C < G$ and $\mathcal{R}(a) \in C$, one has $|C| = kn$ with $k \mid 8$.

Suppose that $3 \nmid n$. For any odd prime divisor p of n, let P be a Sylow p-subgroup of $\langle \mathcal{R}(a)\rangle$. Then P is also a Sylow p-subgroup of C. If P is not normal in C, then by Sylow's theorem, we have $|C : N_C(P)| = k'p + 1 \mid 8$ for some integer k'. Since $p \ne 3$, one has $p = 7$ and $k' = 1$. This implies that $|C| = 8|N_C(P)|$, and so $|C| = 8n$ due to $\mathcal{R}(a) \in C$ and $C < G$. Since $C_{1_0} \le \mathbb{Z}_2$, one has $|C : C_{1_0}| \ge 4n$, and so C is transitive on $V(\Gamma)$. Moreover, we have $C_C(P) = N_C(P) = \langle \mathcal{R}(a)\rangle$. By Burnside theorem, C has a normal subgroup M such that $C = M \rtimes P$. Then the quotient graph Γ_M of Γ relative to M would be a cycle of length $|P|$, and the subgraph induced by each orbit of M is just a perfect matching. This implies that M is just the kernel of G acting on $V(\Gamma_M)$. Furthermore, C/M is a vertex-transitive subgroup of Aut (Γ_M). Since Γ_M is a cycle, C/M must contain a subgroup, say B/M acting regularly on $V(\Gamma_M)$. Then B will be regular on $V(\Gamma)$, and so by Proposition 2.1, Γ is a Cayley graph, a contradiction. Therefore, $P \trianglelefteq C$, and since $C \trianglelefteq G$, one has $P \trianglelefteq G$, implying $P \leq N$. By the arbitrariness of P, $n/|N|$ must be even, contrary to Lemma 7.7. Thus, $3 \mid n$, as claimed.

The following claim shows that $t = 1$ or 2.

Claim 3 $t \leq 2$.

By way of contradiction, suppose that $t > 2$. Let $C = C_G(T)$. Then $\langle \mathcal{R}(a)\rangle \leq C$ and $\mathcal{R}(H) \nleq C$ since $|T| = t > 2$. Clearly, $C_{1_0} \leq E_{1_0}$. As C_{1_0} centralizes T, C_{1_0} will fixes every vertex in Ω_{00}^0 since Ω_{00}^0 is an orbit of T containing 1_0. Since $\Gamma[B_0^0 \cup B_1^0] \cong \mathrm{C}_{4t}$, C_{1_0} fixes every vertex in this $4t$-cycle, and so $C_{1_0} \leq G_{1_0}^* = 1$ (by Claim 2). Thus, C acts semiregularly on $V(\Gamma)$. If $C = \langle \mathcal{R}(a)\rangle$, then by N/C-theorem, we have $G/\langle \mathcal{R}(a)\rangle = G/C \leq \mathrm{Aut}\,(T)$. Since $T \leq N \leq \langle \mathcal{R}(a)\rangle$ is cyclic, Aut (T) is abelian. It then follows that $\mathcal{R}(H)/C \trianglelefteq G/C$, and hence $\mathcal{R}(H) \trianglelefteq G$, a contradiction. If $C > \langle \mathcal{R}(a)\rangle$, then $|C| = 2n$ because Γ is non-Cayley. Since H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$, C does not fix H_0 setwise, and so $\mathcal{R}(H)C$ is transitive on $V(\Gamma)$. Clearly, $\mathcal{R}(H) \cap C = \langle \mathcal{R}(a)\rangle$, so $|\mathcal{R}(H)C| = |\mathcal{R}(H)||C|/|\langle \mathcal{R}(a)\rangle| = 4n$. It follows that $\mathcal{R}(H)C$ is regular on $V(\Gamma)$, contradicting that Γ is non-Cayley.

By Claim 3, we only need to consider the following two cases:

Case 1 $t = 1$.

In this case, by Claim 1, we have $R = \{b, ba^i\}$ and $L = \{b, ba^{\frac{n}{2}-i}\}$. For convenience, we let $n = 2\ell$. Then $R = \{b, ba^i\}$ and $L = \{b, ba^{\ell-i}\}$.

By Proposition 2.3 (1), the connectedness of Γ implies that $\langle a^i, a^\ell\rangle = \langle a\rangle$. Then either $(i, 2\ell) = 1$, or $i = 2k$ with $(k, 2\ell) = 1$ and ℓ is odd. Recall that $H = \langle a, b \mid a^{2\ell} = b^2 = 1, bab = a^{-1}\rangle$. For any $\lambda \in \mathbb{Z}_{2\ell}^*$, let α_λ be the automorphism of H induced by the map

$$a^\lambda \mapsto a,\ b \mapsto b.$$

So if $(i, 2\ell) = 1$, then we have

$$(R, L)^{\alpha_i} = (\{b, ba\}, \{b, ba^{\ell-1}\}),$$

and if $i = 2k$ with $(k, 2\ell) = 2$ and ℓ is odd, then we have

$$(R, L)^{\alpha_k} = (\{b, ba^2\}, \{b, ba^{\ell-2}\}).$$

So by Proposition 2.3 (3), we have

$$(R, L, S) \equiv (\{b, ba\}, \{b, ba^{\ell-1}\}, \{1\}) \text{ or } (\{b, ba^2\}, \{b, ba^{\ell-2}\}, \{1\})(\ell \text{ is odd}).$$

Suppose that ℓ is even. Then $(R, L, S) \equiv (\{b, ba\}, \{b, ba^{\ell-1}\}, \{1\})$. Since ℓ is even, one has $(2\ell, \ell+1) = 1$ and $(\ell+1)^2 \equiv 1 \pmod{2\ell}$. Then it is easy to check that $\alpha_{\ell+1}$ is an automorphism of H of order 2 that swaps $\{b, ba\}$ and $\{b, ba^{\ell-1}\}$. By Proposition 2.4, we have $\delta_{\alpha_{\ell+1},1,1} \in I$, and then $\Gamma \cong \mathrm{BiCay}(H, \{b, ba\}, \{b, ba^{\ell+1}\}, \{1\})$ is a Cayley graph, a contradiction.

Now we assume that $n = 2\ell$ with $\ell = 2m + 1$ for some integer m. Let

$$\Gamma_1 = \mathrm{BiCay}(H, \{b, ba\}, \{b, ba^{2m}\}, \{1\}), \Gamma_2 = \mathrm{BiCay}(H, \{b, ba^2\}, \{b, ba^{2m-1}\}, \{1\}).$$

Direct calculation shows that $(n, 2m-1) = 1$, and $2m(2m-1) \equiv 2 \pmod n$. Then the automorphism $\alpha_{2m-1} : a \mapsto a^{2m-1},\ b \mapsto b$ maps the pair of two subsets $(\{b, ba\}, \{b, ba^{2m}\})$ to $(\{b, ba^{2m-1}\}, \{b, ba^2\})$. So, we have $(R, L, S) \equiv (\{b, ba\}, \{b, ba^{2m}\}, \{1\})$. By Lemma 4.1 and Theorem 1.2, $\Gamma \cong \mathrm{MCL}(4m, 2)$ and Γ is non-Cayley if and only if $3 \nmid (2m+1)$. Note that $3 \nmid (2m+1)$ is equivalent to $m \not\equiv 1 \pmod 3$. So we obtain the first family of graphs in Theorem 7.6.

Case 2 $t = 2$.

In this case, by Claim 1, we have $R = \{b, ba^i\}$ and $L = \{ba^{\frac{n}{2}}, ba^{\frac{3n}{4}-i}\}$ or $\{ba^{\frac{n}{2}}, ba^{\frac{n}{4}-i}\}$. We still use the following notation: For any $\lambda \in \mathbb{Z}_{2\ell}^*$, let α_λ be the automorphism of H induced by the map

$$a^\lambda \mapsto a,\ b \mapsto b.$$

Note that

$$(\{b, ba^i\}, \{ba^{\frac{n}{2}}, ba^{\frac{3n}{4}-i}\})^{\alpha_{-1}} = (\{b, ba^{-i}\}, \{ba^{\frac{n}{2}}, ba^{\frac{n}{4}-(-i)}\}).$$

By replacing $-i$ by i, we may always assume that

$$(R, L) = (\{b, ba^i\}, \{ba^{\frac{n}{2}}, ba^{\frac{n}{4}-i}\}).$$

By Claim 2, we have $3 \mid n$. So we may assume that $n = 12m$ for some integer m. Then we have

$$(R, L) = (\{b, ba^i\}, \{ba^{6m}, ba^{3m-i}\}).$$

Since Γ is connected, by Proposition 2.3, we have $\langle a^i, a^{3m}\rangle = \langle a\rangle$. If m is odd, by Lemma 6.1, Γ will be a Cayley graph which is impossible. Thus, m is even. It then follows that $\langle a^i\rangle \cap \langle a^{3m}\rangle > 1$ since $\langle a^i, a^{3m}\rangle = \langle a^i\rangle\langle a^{3m}\rangle = \langle a\rangle$. Since $\langle a^{3m}\rangle \cong \mathbb{Z}_4$, one has $|\langle a^i\rangle \cap \langle a^{3m}\rangle| = 2$ or 4. For the former, we would have $|\langle a^i\rangle| = 6m$, and since m is even, one has $4 \mid |\langle a^i\rangle|$, and hence $a^{3m} \in \langle a^i\rangle$, a contradiction. Thus, we have $|\langle a^i\rangle \cap \langle a^{3m}\rangle| = 4$, that is, $\langle a^i\rangle = \langle a\rangle$. So $(i, 12m) = 1$, and then $\alpha_i \in \mathrm{Aut}\,(H)$ which maps $(\{b, ba^i\}, \{ba^{6m}, ba^{3m-i}\})$ to $(\{b, ba\}, \{ba^{6m}, ba^{3m-1}\})$ or $(\{b, ba\}, \{ba^{6m}, ba^{-3m-1}\})$. Then

$$(R, L, S) \equiv (\{b, ba\}, \{ba^{6m}, ba^{3m-1}\}, \{1\}) \text{ or } (\{b, ba\}, \{ba^{6m}, ba^{-3m-1}\}), \{1\}.$$

If $m \equiv 2 \pmod 4$, then by Lemma 6.2, we see that Γ will be a Cayley graph, a contradiction. Thus, $m \equiv 0 \pmod 4$. Clearly, $(3m-1, 12m) = 1$, and hence the map $a \mapsto a^{3m-1}, b \mapsto ba^{6m}$ induces an automorphism, say β of H. It is easy to check that

$$(\{b, ba\}, \{ba^{6m}, ba^{3m-1}\})^\beta = (\{ba^{6m}, ba^{-3m-1}\}, \{b, ba\}).$$

Thus,

$$(R, L, S) \equiv (\{b, ba\}, \{ba^{6m}, ba^{3m-1}\}, \{1\}).$$

By Proposition 5.1, Γ is a non-Cayley graph. Let $m = 4\ell$ for some integer ℓ. Then $n = 48\ell$ and then we get the second family of graphs in Theorem 7.6. This completes the proof of Theorem 7.6. □

7.4 Proof of Theorem 1.3

By [20, Theorem 1.2], if Γ is 0- or 1-type, then Γ is a Cayley graph. Let Γ be of 2-type. Suppose that Γ is a non-Cayley graph. Let $G \leq \mathrm{Aut}(\Gamma)$ be minimal subject to that $\mathcal{R}(H) \leq G$ and G is transitive on $V(\Gamma)$. If H_0 and H_1 are blocks of imprimitivity of G on $V(\Gamma)$, then by Proposition 7.2, we obtain the first two families of graphs of Theorem 1.3. Otherwise, H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$, by Theorem 7.6, we obtain the last two families of graphs of Theorem 1.3. □

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