On Hermitian varieties in $\text{PG}(6, q^2)$

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Abstract

In this paper we characterize the non-singular Hermitian variety $\mathcal{H}(6, q^2)$ of $\text{PG}(6, q^2)$, $q \neq 2$ among the irreducible hypersurfaces of degree $q + 1$ in $\text{PG}(6, q^2)$ not containing solids by the number of its points and the existence of a solid $S$ meeting it in $q^4 + q^2 + 1$ points.

Keywords: Unital, Hermitian variety, algebraic hypersurface.


1 Introduction

The set of all absolute points of a non-degenerate unitary polarity in $\text{PG}(r, q^2)$ determines the Hermitian variety $\mathcal{H}(r, q^2)$. This is a non-singular algebraic hypersurface of degree $q + 1$ in $\text{PG}(r, q^2)$ with a number of remarkable properties, both from the geometrical and the combinatorial point of view; see [5, 16]. In particular, $\mathcal{H}(r, q^2)$ is a 2-character set with respect to the hyperplanes of $\text{PG}(r, q^2)$ and 3-character blocking set with respect to the...
lines of $\text{PG}(r, q^2)$ for $r > 2$. An interesting and widely investigated problem is to provide combinatorial descriptions of $\mathcal{H}(r, q^2)$ among all hypersurfaces of the same degree.

First, we observe that a condition on the number of points and the intersection numbers with hyperplanes is not in general sufficient to characterize Hermitian varieties; see [1],[2]. On the other hand, it is enough to consider in addition the intersection numbers with codimension 2 subspaces in order to get a complete description; see [7].

In general, a hypersurface $\mathcal{H}$ of $\text{PG}(r, q)$ is viewed as a hypersurface over the algebraic closure of $\text{GF}(q)$ and a point of $\text{PG}(r, q^l)$ in $\mathcal{H}$ is called a $\text{GF}(q^l)$-point. A $\text{GF}(q)$-point of $\mathcal{H}$ is also said to be a rational point of $\mathcal{H}$. Throughout this paper, the number of $\text{GF}(q^l)$-points of $\mathcal{H}$ will be denoted by $N_{q^l}(\mathcal{H})$. For simplicity, we shall also use the convention $|\mathcal{H}| = N_q(\mathcal{H})$.

In the present paper, we shall investigate a combinatorial characterization of the Hermitian hypersurface $\mathcal{H}(6, q^2)$ in $\text{PG}(6, q^2)$ among all hypersurfaces of the same degree having also the same number of $\text{GF}(q^2)$-rational points.

More in detail, in [12, 13] it has been proved that if $\mathcal{X}$ is a hypersurface of degree $q+1$ in $\text{PG}(r, q^2)$, $r \geq 3$ odd, with $|\mathcal{X}| = |\mathcal{H}(r, q^2)| = (q^{r+1} + (1)^r)(q^r - (1)^r)/(q^2 - 1)$ $\text{GF}(q^2)$-rational points, not containing linear subspaces of dimension greater than $r^{-1}$, then $\mathcal{X}$ is a non-singular Hermitian variety of $\text{PG}(r, q^2)$. This result generalizes the characterization of [8] for the Hermitian curve of $\text{PG}(2, q^2)$, $q \neq 2$.

The case where $r > 4$ is even is, in general, currently open. A starting point for a characterization in arbitrary even dimension can be found in [3] where the case of a hypersurface $\mathcal{X}$ of degree $q+1$ in $\text{PG}(4, q^2)$, $q > 3$ is considered. There, it is shown that when $\mathcal{X}$ has the same number of rational points as $\mathcal{H}(4, q^2)$, does not contain any subspaces of dimension greater than 1 and meets at least one plane $\pi$ in $q^2 + 1$ $\text{GF}(q^2)$-rational points, then $\mathcal{X}$ is a Hermitian variety.

In this article we deal with hypersurfaces of degree $q+1$ in $\text{PG}(6, q^2)$ and we prove that a characterization similar to that of [3] holds also in dimension 6. We conjecture that this can be extended to arbitrary even dimension.

**Theorem 1.1.** Let $S$ be a hypersurface of $\text{PG}(6, q^2)$, $q > 2$, defined over $\text{GF}(q^2)$, not containing solids. If the degree of $S$ is $q+1$ and the number of its rational points is $q^{11} + q^9 + q^7 + q^4 + q^2 + 1$, then every solid of $\text{PG}(6, q^2)$ meets $S$ at least $q^4 + q^2 + 1$ rational points. If there is at least a solid $\Sigma_3$ such that $|\Sigma_3 \cap S| = q^4 + q^2 + 1$, then $S$ is a non-singular Hermitian variety of $\text{PG}(6, q^2)$.

Furthermore, we also extend the result obtained in [3] to the case $q = 3$.

### 2 Preliminaries and notation

In this section we collect some useful information and results that will be crucial to our proof.

A Hermitian variety in $\text{PG}(r, q^2)$ is the algebraic variety of $\text{PG}(r, q^2)$ whose points $\langle v \rangle$ satisfy the equation $\eta(v, v) = 0$ where $\eta$ is a sesquilinear form $\text{GF}(q^2)^{r+1} \times \text{GF}(q^2)^{r+1} \rightarrow \text{GF}(q^2)$. The radical of the form $\eta$ is the vector subspace of $\text{GF}(q^2)^{r+1}$ given by

$$\text{Rad}(\eta) := \{ w \in \text{GF}(q^2)^{r+1} : \forall v \in \text{GF}(q^2)^{r+1}, \eta(v, w) = 0 \}.$$  

The form $\eta$ is non-degenerate if $\text{Rad}(\eta) = \{0\}$. If the form $\eta$ is non-degenerate, then the corresponding Hermitian variety is denoted by $\mathcal{H}(r, q^2)$ and it is non-singular, of degree
$q + 1$ and contains

$$(q^r+1+(-1)^r)(q^r-(-1)^r)/(q^2-1)$$

$\text{GF}(q^2)$-rational points. When $\eta$ is degenerate we shall call vertex $R_t$ of the degenerate Hermitian variety associated to $\eta$ the projective subspace $R_t := \text{PG}(\text{Rad}(\eta)) := \{\langle w \rangle : w \in \text{Rad}(\eta)\}$ of $\text{PG}(r, q^2)$. A degenerate Hermitian variety can always be described as a cone of vertex $R_t$ and basis a non-degenerate Hermitian variety $H(r-t, q^2)$ disjoint from $R_t$ where $t = \dim(\text{Rad}(\eta))$ is the vector dimension of the radical of $\eta$. In this case we shall write the corresponding variety as $R_t H(r-t, q^2)$. Indeed,

$$R_t H(r-t, q^2) := \{X \in \langle P, Q \rangle : P \in R_t, Q \in H(r-t, q^2)\}.$$

Any line of $\text{PG}(r, q^2)$ meets a Hermitian variety (either degenerate or not) in either $1, q + 1$ or $q^2 + 1$ points (the latter value only for $r > 2$). The maximal dimension of projective subspaces contained in the non-degenerate Hermitian variety $H(r, q^2)$ is $(r - 2)/2$, if $r$ is even, or $(r - 1)/2$, if $r$ is odd. These subspaces of maximal dimension are called generators of $H(r, q^2)$ and the generators of $H(r, q^2)$ through a point $P$ of $H(r, q^2)$ span a hyperplane $P^\perp$ of $\text{PG}(r, q^2)$, the tangent hyperplane at $P$.

It is well known that this hyperplane meets $H(r, q^2)$ in a degenerate Hermitian variety $PH(r-2, q^2)$, that is in a Hermitian cone having as vertex the point $P$ and as base a non-singular Hermitian variety of $\Theta \cong \text{PG}(r-2, q^2)$ contained in $P^\perp$ with $P \not\in \Theta$.

Every hyperplane of $\text{PG}(r, q^2)$, which is not tangent, meets $H(r, q^2)$ in a non-singular Hermitian variety $H(r-1, q^2)$, and is called a secant hyperplane of $H(r, q^2)$. In particular, a tangent hyperplane contains

$$1 + q^2(q^{r-1} + (-1)^r)(q^{r-2} - (-1)^r)/(q^2 - 1)$$

$\text{GF}(q^2)$-rational points of $H(r, q^2)$, whereas a secant hyperplane contains

$$(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})/(q^2 - 1)$$

$\text{GF}(q^2)$-rational points of $H(r, q^2)$.

We now recall several results which shall be used in the course of this paper.

**Lemma 2.1** ([15]). Let $d$ be an integer with $1 \leq d \leq q + 1$ and let $C$ be a curve of degree $d$ in $\text{PG}(2, q)$ defined over $\text{GF}(q)$, which may have $\text{GF}(q)$-linear components. Then the number of its rational points is at most $dq + 1$ and $N_q(C) = dq + 1$ if and only if $C$ is a pencil of $d$ lines of $\text{PG}(2, q)$.

**Lemma 2.2** ([10]). Let $d$ be an integer with $2 \leq d \leq q + 2$, and $C$ a curve of degree $d$ in $\text{PG}(2, q)$ defined over $\text{GF}(q)$ without any $\text{GF}(q)$-linear components. Then $N_q(C) \leq (d-1)q + 1$, except for a class of plane curves of degree 4 over $\text{GF}(4)$ having 14 rational points.

**Lemma 2.3** ([11]). Let $S$ be a surface of degree $d$ in $\text{PG}(3, q)$ over $\text{GF}(q)$. Then

$$N_q(S) \leq dq^2 + q + 1$$

**Lemma 2.4** ([8]). Suppose $q \neq 2$. Let $C$ be a plane curve over $\text{GF}(q^2)$ of degree $q + 1$ without $\text{GF}(q^2)$-linear components. If $C$ has $q^3 + 1$ rational points, then $C$ is a Hermitian curve.
Lemma 2.5 ([7]). A subset of points of $\text{PG}(r, q^2)$ having the same intersection numbers with respect to hyperplanes and spaces of codimension 2 as non-singular Hermitian varieties, is a non-singular Hermitian variety of $\text{PG}(r, q^2)$.

From [9, Th 23.5.1, Th 23.5.3] we have the following.

Lemma 2.6. If $\mathcal{W}$ is a set of $q^7 + q^4 + q^2 + 1$ points of $\text{PG}(4, q^2)$, $q > 2$, such that every line of $\text{PG}(4, q^2)$ meets $\mathcal{W}$ in $1, q + 1$ or $q^2 + 1$ points, then $\mathcal{W}$ is a Hermitian cone with vertex a line and base a unital.

Finally, we recall that a blocking set with respect to lines of $\text{PG}(r, q^2)$ is a point set which blocks all the lines, i.e., intersects each line of $\text{PG}(r, q^2)$ in at least one point.

3 Proof of Theorem 1.1

We first provide an estimate on the number of points of a curve of degree $q + 1$ in $\text{PG}(2, q^2)$, where $q$ is any prime power.

Lemma 3.1. Let $\mathcal{C}$ be a plane curve over $\text{GF}(q^2)$, without $\text{GF}(q^2)$-lines as components and of degree $q + 1$. If the number of $\text{GF}(q^2)$-rational points of $\mathcal{C}$ is $N < q^3 + 1$, then

$$N \leq \begin{cases} q^3 - (q^2 - 2) & \text{if } q > 3 \\ 24 & \text{if } q = 3 \\ 8 & \text{if } q = 2. \end{cases}$$

(3.1)

Proof. We distinguish the following three cases:

(a) $\mathcal{C}$ has two or more $\text{GF}(q^2)$-components;

(b) $\mathcal{C}$ is irreducible over $\text{GF}(q^2)$, but not absolutely irreducible;

(c) $\mathcal{C}$ is absolutely irreducible.

Suppose first $q \neq 2$.

Case (a) Suppose $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Let $d_i$ be the degree of $\mathcal{C}_i$, for each $i = 1, 2$. Hence $d_1 + d_2 = q + 1$. By Lemma 2.2,

$$N \leq N_{q^2}(\mathcal{C}_1) + N_{q^2}(\mathcal{C}_2) \leq [(q + 1) - 2]q^2 + 2 = q^3 - (q^2 - 2)$$

Case (b) Let $\mathcal{C}'$ be an irreducible component of $\mathcal{C}$ over the algebraic closure of $\text{GF}(q^2)$. Let $\text{GF}(q^{2t})$ be the minimum defining field of $\mathcal{C}'$ and $\sigma$ be the Frobenius morphism of $\text{GF}(q^{2t})$ over $\text{GF}(q^2)$. Then

$$\mathcal{C} = \mathcal{C}' \cup \mathcal{C}'^\sigma \cup \mathcal{C}'^{\sigma^2} \cup \ldots \cup \mathcal{C}'^{\sigma^{t-1}},$$

and the degree of $\mathcal{C}'$, say $e$, satisfies $q + 1 = te$ with $e > 1$. Hence any $\text{GF}(q^2)$-rational point of $\mathcal{C}$ is contained in $\cap_{i=0}^{t-1} \mathcal{C}'^{\sigma^i}$. In particular, $N \leq e^2 \leq (q + 1)^2$ by Bezout’s Theorem and $(q + 1)^2 < q^3 - (q^2 - 2)$.

Case (c) Let $\mathcal{C}$ be an absolutely irreducible curve over $\text{GF}(q^2)$ of degree $q + 1$. Either $\mathcal{C}$ has a singular point or not.

In general, an absolutely irreducible plane curve $\mathcal{M}$ over $\text{GF}(q^2)$ is $q^2$-Frobenius non-classical if for a general point $P(x_0, x_1, x_2)$ of $\mathcal{M}$ the point $Pq^2 = Pq^2(x_0^q, x_1^q, x_2^q)$ is
on the tangent line to \(M\) at the point \(P\). Otherwise, the curve \(M\) is said to be Frobenius classical. A lower bound on the number of \(\mathbb{GF}(q^2)\)-points for \(q^2\)-Frobenius non-classical curves is given by [6, Corollary 1.4]: for a \(q^2\)-Frobenius non-classical curve \(C\) of degree \(d\), we have \(N_{q^2}(C') \geq d(q^2 - d + 2)\). In particular, if \(d = q + 1\), the lower bound is just \(q^3 + 1\).

Going back to our original curve \(C\), we know \(C\) is Frobenius classical because \(N < q^3 + 1\). Let \(F(x, y, z) = 0\) be an equation of \(C\) over \(\mathbb{GF}(q^2)\). We consider the curve \(D\) defined by \(\frac{\partial F}{\partial x}x + \frac{\partial F}{\partial y}y + \frac{\partial F}{\partial z}z = 0\). Then \(C\) is not a component of \(D\) because \(C\) is Frobenius classical. Furthermore, any \(\mathbb{GF}(q^2)\)-point \(P\) lies on \(C \cap D\) and the intersection multiplicity of \(C\) and \(D\) at \(P\) is at least 2 by Euler’s theorem for homogeneous polynomials. Hence by Bézout’s theorem, \(2N \leq (q + 1)(q^2 + q)\). Hence

\[
N \leq \frac{1}{2} q(q + 1)^2.
\]

This argument is due to Stöhr and Voloch [18, Theorem 1.1]. This Stöhr and Voloch’s bound is lower than the estimate for \(N\) in case (a) for \(q > 4\) and it is the same for \(q = 4\). When \(q = 3\) the bound in case (a) is smaller than the Stöhr and Voloch’s bound.

Finally, we consider the case \(q = 2\). Under this assumption, \(C\) is a cubic curve and neither case (a) nor case (b) might occur. For a degree 3 curve over \(\mathbb{GF}(q^2)\) the Stöhr and Voloch’s bound is loose, thus we need to change our argument. If \(C\) has a singular point, then \(C\) is a rational curve with a unique singular point. Since the degree of \(C\) is 3, singular points are either cusps or ordinary double points. Hence \(N \in \{4, 5, 6\}\). If \(C\) is nonsingular, then it is an elliptic curve and, by the Hasse-Weil bound, see [19], \(N \in I\) where \(I = \{1, 2, \ldots, 9\}\) and for each number \(N\) belonging to \(I\) there is an elliptic curve over \(\mathbb{GF}(4)\) with \(N\) points, from [14, Theorem 4.2]. This completes the proof. 

Henceforth, we shall always suppose \(q > 2\) and we denote by \(S\) an algebraic hypersurface of \(\mathbb{PG}(6, q^2)\) satisfying the following hypotheses of Theorem 1.1:

(S1) \(S\) is an algebraic hypersurface of degree \(q + 1\) defined over \(\mathbb{GF}(q^2)\);
(S2) \(|S| = q^{11} + q^9 + q^7 + q^4 + q^2 + 1\);
(S3) \(S\) does not contain projective 3-spaces (solids);
(S4) there exists a solid \(\Sigma_3\) such that \(|S \cap \Sigma_3| = q^4 + q^2 + 1\).

We first consider the behavior of \(S\) with respect to the lines.

**Lemma 3.2.** An algebraic hypersurface \(T\) of degree \(q + 1\) in \(\mathbb{PG}(r, q^2)\), \(q \neq 2\), with \(|T| = |\mathbb{H}(r, q^2)|\) is a blocking set with respect to lines of \(\mathbb{PG}(r, q^2)\).

**Proof.** Suppose on the contrary that there is a line \(\ell\) of \(\mathbb{PG}(r, q^2)\) which is disjoint from \(T\). Let \(\alpha\) be a plane containing \(\ell\). The algebraic plane curve \(C = \alpha \cap T\) of degree \(q + 1\) cannot have \(\mathbb{GF}(q^2)\)-linear components and hence it has at most \(q^3 + 1\) points because of Lemma 2.2. If \(C\) had \(q^3 + 1\) rational points, then from Lemma 2.4, \(C\) would be a Hermitian curve with an external line, a contradiction since Hermitian curves are blocking sets. Thus \(N_{q^2}(C) \leq q^3\). Since \(q > 2\), by Lemma 3.1, \(N_{q^2}(C) < q^3 - 1\) and hence every plane through \(r\) meets \(T\) in at most \(q^3 - 1\) rational points. Consequently, by considering all planes through \(r\), we can bound the number of rational points of \(T\) by \(N_{q^2}(T) \leq (q^3 - 1) \frac{q^{r - 4} - 1}{q^2 - 1}\).
\[ q^{2r-3} + \cdots < |\mathcal{H}(r, q^2)|, \] which is a contradiction. Therefore there are no external lines to \( T \) and so \( T \) is a blocking set w.r.t. lines of \( \PG(r, q^2) \).

\[ \square \]

**Remark 3.3.** The proof of [3, Lemma 3.1] would work perfectly well here under the assumption \( q > 3 \). The alternative argument of Lemma 3.2 is simpler and also holds for \( q = 3 \).

By the previous Lemma and assumptions (S1) and (S2), \( S \) is a blocking set for the lines of \( \PG(6, q^2) \). In particular, the intersection of \( S \) with any 3-dimensional subspace \( \Sigma \) of \( \PG(6, q^2) \) is also a blocking set with respect to lines of \( \Sigma \) and hence it contains at least \( q^4 + q^2 + 1 \) GF\((q^2)\)-rational points; see [4].

**Lemma 3.4.** Let \( \Sigma_3 \) be a solid of \( \PG(6, q^2) \) satisfying condition (S4), that is \( \Sigma_3 \) meets \( S \) in exactly \( q^4 + q^2 + 1 \) points. Then, \( \Pi := S \cap \Sigma_3 \) is a plane.

**Proof.** \( S \cap \Sigma_3 \) must be a blocking set for the lines of \( \PG(3, q^2) \); also it has size \( q^4 + q^2 + 1 \). It follows from [4] that \( \Pi := S \cap \Sigma_3 \) is a plane. \[ \square \]

**Lemma 3.5.** Let \( \Sigma_3 \) be a solid of satisfying condition (S4). Then, any 4-dimensional projective space \( \Sigma_4 \) through \( \Sigma_3 \) meets \( S \) in a Hermitian cone with vertex a line and basis a Hermitian curve.

**Proof.** Consider all of the \( q^6 + q^4 + q^2 + 1 \) subspaces \( \Sigma_3 \) of dimension 3 in \( \PG(6, q^2) \) containing \( \Pi \).

From Lemma 2.3 and condition (S3) we have \( |\Sigma_3 \cap S| \leq q^5 + q^4 + q^2 + 1 \). Hence,

\[ |S| = (q^7 + 1)(q^4 + q^2 + 1) \leq (q^6 + q^4 + q^2)q^5 + q^4 + q^2 + 1 = |S| \]

Consequently, \( |\Sigma_3 \cap S| = q^5 + q^4 + q^2 + 1 \) for all \( \Sigma_3 \neq \Sigma_3 \) such that \( \Pi \subset \Sigma_3 \).

Let \( C := \Sigma_4 \cap S \). Counting the number of rational points of \( C \) by considering the intersections with the \( q^2 + 1 \) subspaces \( \Sigma_3' \) of dimension 3 in \( \Sigma_4 \) containing the plane \( \Pi \) we get

\[ |C| = q^2 \cdot q^5 + q^4 + q^2 + 1 = q^7 + q^4 + q^2 + 1. \]

In particular, \( C \cap \Sigma_3' \) is a maximal surface of degree \( q + 1 \); so it must split in \( q + 1 \) distinct planes through a line of \( \Pi \); see [17]. So \( C \) consists of \( q^3 + 1 \) distinct planes belonging to distinct \( q^2 \) pencils, all containing \( \Pi \); denote by \( \mathcal{L} \) the family of these planes. Also for each \( \Sigma_3' \neq \Sigma_3 \), there is a line \( \ell' \) such that all the planes of \( \mathcal{L} \) in \( \Sigma_3' \) pass through \( \ell' \). It is now straightforward to see that any line contained in \( C \) must necessarily belong to one of the planes of \( \mathcal{L} \) and no plane not in \( \mathcal{L} \) is contained in \( C \).

In order to get the result it is now enough to show that a line of \( \Sigma_4 \) meets \( C \) in either 1, \( q + 1 \) or \( q^2 + 1 \) points. To this purpose, let \( \ell \) be a line of \( \Sigma_4 \) and suppose \( \ell \not\subset C \). Then, by Bezout’s theorem,

\[ 1 \leq |\ell \cap C| \leq q + 1. \]

Assume \( |\ell \cap C| > 1 \). Then we can distinguish two cases:

1. \( \ell \cap \Pi \neq \emptyset \). If \( \ell \) and \( \Pi \) are incident, then we can consider the 3-dimensional subspace \( \Sigma_3' := \langle \ell, \Pi \rangle \). Then \( \ell \) must meet each plane of \( \mathcal{L} \) in \( \Sigma_3' \) in different points (otherwise \( \ell \) passes through the intersection of these planes and then \( |\ell \cap C| = 1 \)). As there are \( q + 1 \) planes of \( \mathcal{L} \) in \( \Sigma_3' \), we have \( |\ell \cap C| = q + 1 \).
2. $\ell \cap \Pi = \emptyset$. Consider the plane $\Lambda$ generated by a point $P \in \Pi$ and $\ell$. Clearly $\Lambda \not\in \mathcal{L}$. The curve $\Lambda \cap S$ has degree $q + 1$ by construction, does not contain lines (for otherwise $\Lambda \in \mathcal{L}$) and has $q^3 + 1$ GF($q^2$)-rational points (by a counting argument).

So from Lemma 2.4 it is a Hermitian curve. It follows that $\ell$ is a $q + 1$ secant.

We can now apply Lemma 2.6 to see that $C_1$ is a Hermitian cone with vertex a line. \hfill $\square$

**Lemma 3.6.** Let $\Sigma_3$ be a space satisfying condition (S4) and take $\Sigma_5$ to be a 5-dimensional projective space with $\Sigma_3 \subseteq \Sigma_5$. Then $\Sigma \cap \Sigma_5$ is a Hermitian cone with vertex a point and basis a Hermitian hypersurface $\mathcal{H}(4, q^2)$.

**Proof.** Let

$$\Sigma_4 := \Sigma_1^4, \Sigma_2^4, \ldots, \Sigma^{q^2+1}_4$$

be the 4-spaces through $\Sigma_3$ contained in $\Sigma_5$. Put $C_i := \Sigma_i^4 \cap S$, for all $i \in \{1, \ldots, q^2 + 1\}$ and $\Pi := \Sigma_3 \cap C_1$. From Lemma 3.5 $C_i$ is a Hermitian cone with vertex a line, say $\ell_i$. Furthermore $\Pi \subseteq \Sigma_3 \subseteq \Sigma_4$ where $\Pi$ is a plane. Choose a plane $\Pi' \subseteq \Sigma_3^4$ such that $m := \Pi' \cap C_1$ is a line $m$ incident with $\Pi$ but not contained in it. Let $P_1 := m \cap \Pi$. It is straightforward to see that in $\Sigma_4^4$ there is exactly 1 plane through $m$ which is a $(q^4 + q^2 + 1)$-secant, $q^2$ planes which are $(q^3 + q^2 + 1)$-secant and $q^2$ planes which are $(q^2 + 1)$-secant.

Also $P_1$ belongs to the line $\ell_1$. There are now two cases to consider:

(a) There is a plane $\Pi'' \neq \Pi'$ not contained in $\Sigma_4^4$ for all $i = 1, \ldots, q^2 + 1$ with $m \subseteq \Pi'' \subseteq S \cap \Sigma_5$.

We first show that the vertices of the cones $C_i$ are all concurrent. Consider $m_i := \Pi'' \cap \Sigma_i^4$. Then $\{m_i : i = 1, \ldots, q^2 + 1\}$ consists of $q^2 + 1$ lines (including $m$) all through $P_1$. Observe that for all $i$, the line $m_i$ meets the vertex $\ell_i$ of the cone $C_i$ in $P_i \in \Pi$. This forces $P_1 = P_2 = \cdots = P_{q^2+1}$. So $P_1 \in \ell_1, \ldots, \ell_{q^2+1}$.

Now let $\Sigma_4$ be a 4-dimensional space in $\Sigma_5$ with $P_1 \not\in \Sigma_4$; in particular $\Pi \not\subseteq \Sigma_4$. Put also $\Sigma_3 := \Sigma_1^3 \cap \Sigma_4$. Clearly, $r := \Sigma_3 \cap \Pi$ is a line and $P_1 \not\in r$. So $\Sigma_3 \cap \Pi$ cannot be the union of $q + 1$ planes, since if this were to be the case, these planes would have to pass through the vertex $\ell_1$. It follows that $\Sigma_3 \cap S$ must be a Hermitian cone with vertex a point and basis a Hermitian curve. Let $\mathcal{W} := \Sigma_4 \cap S$. The intersection $\mathcal{W} \cap \Sigma_4^4$ as $i$ varies is a Hermitian cone with basis a Hermitian curve, so, the points of $\mathcal{W}$ are

$$|\mathcal{W}| = (q^2 + 1)q^5 + q^2 + 1 = (q^2 + 1)(q^5 + 1);$$

in particular, $\mathcal{W}$ is a hypersurface of $\Sigma_4$ of degree $q + 1$ such that there exists a plane of $\Sigma_4$ meeting $\mathcal{W}$ in just one line (such planes exist in $\Sigma_3$). Also suppose $\mathcal{W}$ to contain planes and let $\Pi''' \subseteq \mathcal{W}$ be such a plane. Since $\Sigma_4^4 \cap \mathcal{W}$ does not contain planes, all $\Sigma_4^4$ meet $\Pi'''$ in a line $t_i$. Also $\Pi'''$ must be contained in $\bigcup_{i=1}^{q^2+1} t_i$. This implies that the set $\{t_i\}_{i=1,...,q^2+1}$ consists of $q^2 + 1$ lines through a point $P \in \Pi \setminus \{P_1\}$.

Furthermore each line $t_i$ passing through $P$ must meet the radical line $\ell_i$ of the Hermitian cone $S \cap \Sigma_3$ and this forces $P$ to coincide with $P_1$, a contradiction. It follows that $\mathcal{W}$ does not contain planes.

So by the characterization of $\mathcal{H}(4, q^2)$ of [3] we have that $\mathcal{W}$ is a Hermitian variety $\mathcal{H}(4, q^2)$. 


We also have that $|S \cap \Sigma_5| = |P_1 \mathcal{H}(4, q^2)|$. Let now $r$ be any line of $\mathcal{H}(4, q^2) = S \cap \Sigma_4$ and let $\Theta$ be the plane $\langle r, P_1 \rangle$. The plane $\Theta$ meets $\Sigma_4'$ in a line $q_i \subseteq S$ for each $i = 1, \ldots, q^2 + 1$ and these lines are concurrent in $P_1$. It follows that all the points of $\Theta$ are in $S$. This completes the proof for the current case and shows that $S \cap \Sigma_5$ is a Hermitian cone $P_1 \mathcal{H}(4, q^2)$.

(b) All planes $\Pi^\prime\prime$ with $m \subseteq \Pi^\prime\prime \subseteq S \cap \Sigma_5$ are contained in $\Sigma_4'$ for some $i = 1, \ldots, q^2 + 1$. We claim that this case cannot happen. We can suppose without loss of generality $m \cap \ell_i = P_1$ and $P_1 \not\in \ell_i$ for all $i = 2, \ldots, q^2 + 1$. Since the intersection of the subspaces $\Sigma_4'$ is $\Sigma_3$, there is exactly one plane through $m$ in $\Sigma_5$ which is $(q^2 + q^2 + 1)$-secant, namely the plane $\langle \ell_1, m \rangle$. Furthermore, in $\Sigma_4'$ there are $q^4$ planes through $m$ which are $(q^3 + q^2 + 1)$-secant and $q^2$ planes which are $(q^2 + 1)$-secant. We can provide an upper bound to the points of $S \cap \Sigma_5$ by counting the number of points of $S \cap \Sigma_5$ on planes in $\Sigma_5$ through $m$ and observing that a plane through $m$ not in $\Sigma_5$ and not contained in $S$ has at most $q^3 + q^2 + 1$ points in common with $S \cap \Sigma_5$. So

$$|S \cap \Sigma_5| \leq q^6 \cdot q^3 + q^7 + q^4 + q^2 + 1.$$ 

As $|S \cap \Sigma_5| = q^9 + q^7 + q^4 + q^2 + 1$, all planes through $m$ which are neither $(q^4 + q^2 + 1)$-secant nor $(q^2 + 1)$-secant are $(q^3 + q^2 + 1)$-secant. That is to say that all of these planes meet $S$ in a curve of degree $q + 1$ which must split into $q + 1$ lines through a point because of Lemma 2.1.

Take now $P_2 \in \Sigma_4^2 \cap S$ and consider the plane $\Xi := \langle m, P_2 \rangle$. The line $\langle P_1, P_2 \rangle$ is contained in $\Sigma_4^2$; so it must be a $(q + 1)$-secant, as it does not meet the vertex line $\ell_2$ of $C_2$ in $\Sigma_4^2$. Now, $\Xi$ meets every of $\Sigma_4^i$ for $i = 2, \ldots, q^2 + 1$ in a line through $P_1$ which is either a $1$-secant or a $q + 1$-secant; so

$$|S \cap \Xi| \leq q^2(q) + q^2 + 1 = q^3 + q^2 + 1.$$ 

It follows $|S \cap \Xi| = q^3 + q^2 + 1$ and $S \cap \Xi$ is a set of $q + 1$ lines all through the point $P_1$. This contradicts our previous construction.

□

**Lemma 3.7.** Every hyperplane of $\text{PG}(6, q^2)$ meets $S$ either in a non-singular Hermitian variety $\mathcal{H}(5, q^2)$ or in a cone over a Hermitian hypersurface $\mathcal{H}(4, q^2)$.

**Proof.** Let $\Sigma_3$ be a solid satisfying condition (S4). Denote by $\Lambda$ a hyperplane of $\text{PG}(6, q^2)$. If $\Lambda$ contains $\Sigma_3$ then, from Lemma 3.6 it follows that $\Lambda \cap S$ is a Hermitian cone $P \mathcal{H}(4, q^2)$.

Now assume that $\Lambda$ does not contain $\Sigma_3$. Denote by $S^j_3$, with $j = 1, \ldots, q^2 + 1$ the $q^2 + 1$ hyperplanes through $\Sigma_4^i$, where as before, $\Sigma_4^i$ is a 4-space containing $\Sigma_3$. By Lemma 3.6 again we get that $S^j_3 \cap S = P \mathcal{H}(4, q^2)$. We count the number of rational points of $\Lambda \cap S$ by studying the intersections of $S^j_3 \cap S$ with $\Lambda$ for all $j \in \{1, \ldots, q^2 + 1\}$. Setting $W_j := S^j_3 \cap S \cap \Lambda$, $\Omega := \Sigma_4^i \cap S \cap \Lambda$ then

$$|S \cap \Lambda| = \sum_j |W_j \setminus \Omega| + |\Omega|.$$ 

If $\Pi$ is a plane of $\Lambda$ then $\Omega$ consists of $q + 1$ planes of a pencil. Otherwise let $m$ be the line in which $\Lambda$ meets the plane $\Pi$. Then $\Omega$ is either a Hermitian cone $P_0 \mathcal{H}(2, q^2)$, or $q + 1$
planes of a pencil, according as the vertex $P^j \in \Pi$ is an external point with respect to $m$ or not.

In the former case $\mathcal{W}_j$ is a non singular Hermitian variety $\mathcal{H}(4, q^2)$ and thus $|S \cap \Lambda| = (q^2 + 1)(q^7) + q^5 + q^2 + 1 = q^9 + q^7 + q^5 + q^2 + 1$.

In the case in which $\Omega$ consists of $q + 1$ planes of a pencil then $\mathcal{W}_j$ is either a $P_0 \mathcal{H}(3, q^2)$ or a Hermitian cone with vertex a line and basis a Hermitian curve $\mathcal{H}(2, q^2)$.

If there is at least one index $j$ such that $\mathcal{W}_j = \ell_1 \mathcal{H}(2, q^2)$ then, there must be a 3-dimensional space $\Sigma_j'$ of $S_j' \cap \Lambda$ meeting $S$ in a generator. Hence, from Lemma 3.6 we get that $S \cap \Lambda$ is a Hermitian cone $P' \mathcal{H}(4, q^2)$.

Assume that for all $j \in \{1, \ldots, q^2 + 1\}$, $\mathcal{W}_j$ is a $P_0 \mathcal{H}(3, q^2)$. In this case $|S \cap \Lambda| = (q^2 + 1)q^7 + (q + 1)q^4 + q^2 + 1 = q^9 + q^7 + q^5 + q^4 + q^2 + 1 = |\mathcal{H}(5, q^2)|$.

We are going to prove that the intersection numbers of $S$ with hyperplanes are only two that is $q^9 + q^7 + q^5 + q^4 + q^2 + 1$ or $q^9 + q^7 + q^4 + q^2 + 1$.

Denote by $x_i$ the number of hyperplanes meeting $S$ in $i$ rational points with $i \in \{q^9 + q^7 + q^5 + q^4 + q^2 + 1, q^9 + q^7 + q^5 + q^4 + q^2 + 1\}$. Double counting arguments give the following equations for the integers $x_i$:

$$
\begin{align*}
\sum_i x_i &= q^{12} + q^{10} + q^8 + q^6 + q^4 + q^2 + 1 \\
\sum_i i x_i &= |S|(q^{10} + q^8 + q^6 + q^4 + q^2 + 1) \\
\sum_{i=1}^n i(i-1)x_i &= |S|(|S| - 1)(q^8 + q^6 + q^4 + q^2 + 1).
\end{align*}
$$

Solving (3.2) we obtain $x_{q^9+q^7+q^5+q^2+1} = 0$. In the case in which $|S \cap \Lambda| = |\mathcal{H}(5, q^2)|$, since $S \cap \Lambda$ is an algebraic hypersurface of degree $q + 1$ not containing 3-spaces, from [19, Theorem 4.1] we get that $S \cap \Lambda$ is a Hermitian variety $\mathcal{H}(5, q^2)$ and this completes the proof.

**Proof of Theorem 1.1.** The first part of Theorem 1.1 follows from Lemma 3.4. From Lemma 3.7, $S$ has the same intersection numbers with respect to hyperplanes and 4-spaces as a non-singular Hermitian variety of PG($6, q^2$), hence Lemma 2.5 applies and $S$ turns out to be a $\mathcal{H}(6, q^2)$.

**Remark 3.8.** The characterization of the non-singular Hermitian variety $\mathcal{H}(4, q^2)$ given in [3] is based on the property that a given hypersurface is a blocking set with respect to lines of PG($4, q^2$), see [3, Lemma 3.1]. This lemma holds when $q > 3$. Since Lemma 3.2 extends the same property to the case $q = 3$ it follows that the result stated in [3] is also valid in PG($4, 3^2$).

4 Conjecture

We propose a conjecture for the general $2n$-dimensional case.

Let $S$ be a hypersurface of PG($2d, q^2$), $q > 2$, defined over GF($q^2$), not containing $d$-dimensional projective subspaces. If the degree of $S$ is $q + 1$ and the number of its rational points is $|\mathcal{H}(2d, q^2)|$, then every $d$-dimensional subspace of PG($2d, q^2$) meets $S$ in at least $\theta_{q^2}(d - 1) := (q^{2d-2} - 1)/(q^2 - 1)$ rational points. If there is at least a $d$-dimensional
subspace $\Sigma_d$ such that $|\Sigma_d \cap S| = |\text{PG}(d-1, q^2)|$, then $S$ is a non-singular Hermitian variety of $\text{PG}(2d, q^2)$.

Lemma 3.1 and Lemma 3.2 can be a starting point for the proof of this conjecture since from them we get that $S$ is a blocking set with respect to lines of $\text{PG}(2d, q^2)$.

References