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Cospectrality of multipartite graphs*

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Abstract

Let G be a graph on n vertices and consider the adjacency spectrum of G as the ordered n-tuple whose entries are eigenvalues of G written decreasingly. Let G and H be two non-isomorphic graphs on n vertices with spectra S and T, respectively. Define the distance between the spectra of G and H as the distance of S and T to a norm N of the n-dimensional vector space over real numbers. Define the cospectrality of G as the minimum of distances between the spectrum of G and spectra of all other non-isomorphic n vertices graphs to the norm N. In this paper we investigate copsectralities of the cocktail party graph and the complete tripartite graph with parts of the same size to the Euclidean or Manhattan norms.

Keywords: Spectra of graphs, cospectrality of graphs, adjacency matrix of a graph, Euclidean norm, Manhattan norm.

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1 Introduction and results

All graphs considered here are simple, that is finite and undirected without loops and multiple edges. Let G be a graph with vertex set $\{v_1, \dots, v_n\}$. The adjacency matrix of G is an $n \times n$ matrix $A(G) = [a_{ij}]$ such that $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. By the eigenvalues of G, we mean those of its adjacency matrix. We denote by $Spec(G)$ the multiset of the eigenvalues of the graph G.

Richard Brualdi proposed in [24] the following problem:

Problem ([24, Problem AWGS.4])**.** Let G_n and G'_n be two non-isomorphic graphs on n vertices with spectra

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad \text{and} \quad \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n,$$

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respectively. Define the distance between the spectra of G_n and G'_n as

$$\lambda(G_n, G'_n) = \sum_{i=1}^{n} (\lambda_i - \lambda'_i)^2 \quad \Big(\text{or use } \sum_{i=1}^{n} |\lambda_i - \lambda'_i|\Big).$$

Define the cospectrality of G_n by

$$\mathrm{cs}(G_n) = \min\{\lambda(G_n, G'_n) \colon G'_n \text{ not isomorphic to } G_n\}.$$

Let

$$\mathrm{cs}_n = \max\{\mathrm{cs}(G_n) \colon G_n \text{ a graph on } n \text{ vertices}\}.$$

This function measures how far apart the spectrum of a graph with n vertices can be from the spectrum of any other graph with n vertices.

Problem A. Investigate $\mathrm{cs}(G_n)$ for special classes of graphs.

Problem B. Find a good upper bound on cs_n.

In [15], Jovanović et al. studied the spectral distance between certain graphs to the ℓ^1-norm i.e. $\sigma(G_n, G'_n) = \sum_{i=1}^{n} |\lambda_i - \lambda'_i|$. In [1], Abdollahi et al. completely answered Problem B to any ℓ^p-norm by proving that $\mathrm{cs}_n = 2$ for all $n \geq 2$, whenever $1 \leq p < \infty$ and $\mathrm{cs}_n = 1$ to the ℓ^∞-norm. In [2, 20], the authors studied Problem A to the Euclidean norm (the ℓ^2-norm) and determined the cospectralities of classes of complete graphs and complete bipartite graphs. In [3] we compute the cospectralities to the ℓ^1-norm of complete graphs and complete bipartite graphs with parts of the same size. In [4, 10, 11, 13, 14, 16, 17, 18], Problems A or B are studied based on different matrix representations. To find some applications of the cospectrality of graphs, we refer to [6, 25, 27].

In this paper we study Problem A and investigate the cospectralities of CP_n and $K_{n,n,n}$, $(n \geq 3)$, to the ℓ^1- and ℓ^2-norms i.e. $\sigma(G_n, G'_n) = \sum_{i=1}^{n} |\lambda_i - \lambda'_i|$ and $\lambda(G_n, G'_n) = \sum_{i=1}^{n} (\lambda_i - \lambda'_i)^2$, respectively. We find some conditions for the eigenvalues of a graph H such that $\mathrm{cs}(G) = \sigma(G, H)$ and G is isomorphic to CP_n or $K_{n,n,n}$. Also we give some computational results and conjectures to find $\mathrm{cs}(CP_n)$ and $\mathrm{cs}(K_{n,n,n})$.

In the last section we consider cospectralities of null graphs, complete graphs and complete bipartite graphs using the ℓ^p-norm for $p > 2$ and we see that similar known conclusions using with ℓ^1 and ℓ^2-norms (see [2, 3, 11, 20]) hold more or less valid.

Let us first introduce some notations. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively; By the order of G we mean the number of vertices; Denote by $\overline{G}$ the complement of G. The degree of a vertex of a graph is the number of edges that are incident with the vertex and Δ is the maximum degree of the vertices. An r-regular graph is a graph where all vertices have degree r.

For two graphs G and H with disjoint vertex sets, $G + H$ denotes the graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$, i.e. the disjoint union of two graphs G and H. The complete product (join) $G \nabla H$ of graphs G and H is the graph obtained from $G + H$ by joining every vertex of G with every vertex of H. In particular, nG denotes $\underbrace{G + \cdots + G}_{n}$ and $\nabla_n G$ denotes $\underbrace{G \nabla \cdots \nabla G}_{n}$. The coalescence $G \cdot H$ is obtained by the disjoint union of two graphs G and H by identifying a vertex u of G with a vertex v of H.

For positive integers $n_1, \ldots, n_\ell$, $K_{n_1,\ldots,n_\ell}$ denotes the complete multipartite graph with ℓ parts of sizes $n_1, \ldots, n_\ell$. Let K_n denote the complete graph on n vertices, $nK_1 = \overline{K_n}$ denote the null graph on n vertices and P_n denote the path with n vertices. The cocktail party graph CP_n has $2n$ vertices and it is a complement of nK_2. So for $n = 1$, $CP_1 = K_{1,1}$ and for $n \geq 2$ we have $CP_n = K_{\underbrace{2,\ldots,2}_{n}}$.

Since CP_n and $K_{n,n,n}$ are regular graphs, by Propositions 3 and 6 of [9], CP_n and $K_{n,n,n}$ are determined by their spectrum. So we can compute the values of $\mathrm{cs}(CP_n)$ and $\mathrm{cs}(K_{n,n,n})$.

Our main results are as follows.

Theorem 1.1. *If $n \geq 2$ and $cs(CP_n) = \sigma(CP_n, H)$ for some graph H with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{2n}$, then*

(1) *If H is a connected graph, then $2n-3 \leq \lambda_1 < 2n-1$. Otherwise $2n-3 \leq \lambda_1 < 2n-2$ and H has two connected components such that one of them is K_1.*

(2) $0 \leq \lambda_2 \leq 1$,

(3) $-1 \leq \lambda_i \leq \frac{1}{2}$, *for any integer i, $3 \leq i \leq n+1$, and if $n \geq 13$, then $0 \leq \lambda_3 \leq \frac{1}{2}$,*

(4) $-3 \leq \lambda_{n+2} \leq -1$,

(5) $-3 \leq \lambda_i \leq \frac{-3}{2}$, *for any integer i, $n+3 \leq i \leq 2n$.*

Theorem 1.2. *Let $n \geq 4$ and $cs(K_{n,n,n}) = \sigma(K_{n,n,n}, H)$ for some graph H with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{3n}$. For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have*

(1) $2n - \frac{\sqrt{3}}{3} - \frac{\varepsilon}{2} < \lambda_1 < 2n + \frac{\sqrt{3}}{3} + \frac{\varepsilon}{2}$,

(2) $\sqrt{2} - 1 < \lambda_2 < \frac{\sqrt{3}}{3} + \frac{\varepsilon}{2}$ *or $\lambda_2 = 0$ and $H \cong tK_1 + K_{p,q,r}$ for some positive integers p, q and r such that at least one of them is greater than 1,*

(3) $0 \leq \lambda_3 < \frac{\sqrt{3}}{6} + \frac{\varepsilon}{4}$,

(4) $-\frac{\sqrt{3}}{3} - \frac{\varepsilon}{2} < \lambda_i < \frac{\sqrt{3}}{6} + \frac{\varepsilon}{4}$, *for any integer i, $4 \leq i \leq 3n-2$,*

(5) $-n - \frac{\sqrt{3}}{3} - \frac{\varepsilon}{2} < \lambda_{3n-1} < -n + \frac{\sqrt{3}}{3} + \frac{\varepsilon}{2}$,

(6) $-n - \frac{\sqrt{3}}{3} - \frac{\varepsilon}{2} < \lambda_{3n} < -n + \frac{\sqrt{3}}{6} + \frac{\varepsilon}{4}$.

2 Cospectrality of cocktail party graphs

In this section $\mathrm{cs}(CP_n)$ is investigated to the ℓ^1- and ℓ^2-norms. We need the following results in the sequel. The proofs of next two results are similar to those of Lemma 2.2 and Corollary 2.3 of [18]. We give them here for the reader's convenience.

Lemma 2.1. *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be two sequences with $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 0$. If there exist some $1 \leq j \leq n$ and a real positive number α such that $|a_j - b_j| > \alpha$, then $\sum_{i=1}^n |a_i - b_i| > 2\alpha$.*

Proof. Without loss of generality, we may assume that $a_j - b_j > \alpha$. Suppose that $a_{i_1} \geq b_{i_1}, \ldots, a_{i_s} \geq b_{i_s}$ and $a_{i_{s+1}} \leq b_{i_{s+1}}, \ldots, a_{i_n} \leq b_{i_n}$, then

$$\begin{aligned}\sum_{i=1}^{n} |a_i - b_i| &= \sum_{t=1}^{s} (a_{i_t} - b_{i_t}) + \sum_{t=s+1}^{n} (b_{i_t} - a_{i_t}) \\ &= 2\sum_{t=1}^{s} (a_{i_t} - b_{i_t}) \\ &\geq 2(a_j - b_j) \\ &> 2\alpha.\end{aligned}$$

□

Corollary 2.2. *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be two sequences with $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 0$. If there exist $1 \leq j_1 \neq j_2 \leq n$ and a real positive number α such that $a_{j_1} - b_{j_1} + a_{j_2} - b_{j_2} > \alpha$, then $\sum_{i=1}^{n} |a_i - b_i| > 2\alpha$.*

Proof. If either $a_{j_1} - b_{j_1} > \alpha$ or $a_{j_2} - b_{j_2} > \alpha$, then by Lemma 2.1, the result holds. So we may assume that $0 < a_{j_1} - b_{j_1} \leq \alpha$ and $0 < a_{j_2} - b_{j_2} \leq \alpha$. Let $a'_{j_1} = a_{j_1} + a_{j_2}$, $b'_{j_1} = b_{j_1} + b_{j_2}$, $a'_i = a_i$ and $b'_i = b_i$ for $i \neq j_1, j_2$. So $\sum_{i=1, i\neq j_2}^{n} a'_i = \sum_{i=1, i\neq j_2}^{n} b'_i = 0$ and $a'_{j_1} - b'_{j_1} > \alpha$. Thus the result follows from Lemma 2.1. □

Theorem 2.3. *Let G be a graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. If $cs(G) = \sigma(G, H)$ for some graph H with eigenvalues $\lambda'_1 \geq \cdots \geq \lambda'_n$, then for all integers i and j, $1 \leq j < i \leq n$,*

(1) $|\lambda_i - \lambda'_i| \leq 1$,

(2) $\lambda_i - \lambda'_j \leq \frac{1}{2}$.

Proof. By Theorem 1.1 of [1], $\mathrm{cs}_n = 2$ for all $n \geq 2$, so $\mathrm{cs}(G) \leq 2$. Now the result follows from Lemma 2.1 and Corollary 2.2. □

Theorem 2.4 ([5, Theorem 1])**.** *Let G be a simple graph of order n without isolated vertices. If $\lambda_2(G)$ is the second largest eigenvalue of G, then*

(1) *$\lambda_2(G) = -1$ if and only if G is a complete graph with at least two vertices,*

(2) *$\lambda_2(G) = 0$ if and only if G is a complete k-partite graph with $2 \leq k \leq n-1$,*

(3) *there exists no graph G such that $-1 < \lambda_2(G) < 0$.*

Theorem 2.5 ([21, Theorem 3.8])**.** *Let G be a graph of order n. If $\lambda_3(G) < 0$, then G has at least $n - 12$ eigenvalues -1.*

Theorem 2.6 ([7, Theorem 3.2.1])**.** *Let λ_1 be the greatest eigenvalue of the graph G, and let $\overline{d}$ and Δ be its average degree and maximum degree, respectively. Then*

$$\overline{d} \leq \lambda_1 \leq \Delta.$$

Moreover, $\overline{d} = \lambda_1$ if and only if G is regular. For a connected graph G, $\lambda_1 = \Delta$ if and only if G is regular.

Proof of Theorem 1.1. Since

$$Spec(CP_n) = \{2n-2, \underbrace{0,\dots,0}_{n}, \underbrace{-2,\dots,-2}_{n-1}\},$$

we have

$$\sigma(CP_n, H) = |2n-2-\lambda_1| + \sum_{i=2}^{n+1} |\lambda_i| + \sum_{i=n+2}^{2n} |2+\lambda_i|.$$

If $\mathrm{cs}(CP_n) = \sigma(CP_n, H)$, then by Theorem 1.1 of [1], $\mathrm{cs}(CP_n) \le 2$. By Theorems 2.3, 2.4, 2.5 and Corollary 2.2, we obtain (2) – (5) and $2n-3 \le \lambda_1 \le 2n-1$.

If H is a connected graph and $\lambda_1 = 2n-1$, then by Theorem 2.6, $H \cong K_{2n}$, a contradiction. So $2n-3 \le \lambda_1 < 2n-1$. Now suppose that H is not connected. Let $H_1, \dots, H_k$ be the connected components of H. There exists an unique i, $1 \le i \le k$, such that $\lambda_1(H) = \lambda_1(H_i)$. We can assume that $\lambda_1(H) = \lambda_1(H_1)$. Thus $\lambda_1(H_j) \le \lambda_2(H) \le 1$, for every j, $2 \le j \le k$. So $\lambda_1(H_j) = 0$ or $\lambda_1(H_j) = 1$, $2 \le j \le k$. Since $-1 \le \lambda_3(H) \le \frac{1}{2}$, there exists at most one connected component with $\lambda_1(H_j) = 1$, $2 \le j \le k$. Therefore $H \cong H_1 + tK_1$ or $H \cong H_1 + K_2 + sK_1$, for some integers $t > 0$ and $s \ge 0$. By Theorem 2.6, $2n-3 \le \lambda_1(H) = \lambda_1(H_1) \le \Delta \le 2n-1$, where Δ is the maximum degree of the vertices of H. If $\Delta = 2n-1$, then, by Theorem 2.6, $H_1 \cong K_{2n}$, a contradiction. Let $\Delta = 2n-3$. Therefore by Theorem 2.6, $H_1 \cong K_{2n-2}$, a contradiction. Now suppose that $\Delta = 2n-2$. If $\lambda_1(H_1) = 2n-2$, then by Theorem 2.6, $H_1 \cong K_{2n-1}$, a contradiction. Hence we can assume that $H \cong H_1 + K_1$ and $2n-3 \le \lambda_1(H) < 2n-2$. This completes the proof. □

Remark 2.7. Let H be a connected graph with m edges. If $\mathrm{cs}(CP_n) = \sigma(CP_n, H)$, then, by Theorem 1.1 and Theorem 1 in [26], it is not hard to see that $2n^2 - 5n + 4 \le m < 2n^2 - n$.

Now we find $\sigma(CP_n, (CP_{n-1} \bigtriangledown K_1) \cdot K_2)$ and $\lambda(CP_n, CP_n \setminus e)$ and propose two conjectures. We need the following results.

Theorem 2.8 ([7, Theorem 2.1.8])**.** *If G_1 is r_1-regular with n_1 vertices, and G_2 is r_2-regular with n_2 vertices, then the characteristic polynomial of the join $G_1 \bigtriangledown G_2$ is given by*

$$P_{G_1 \bigtriangledown G_2}(x) = \frac{P_{G_1}(x)P_{G_2}(x)}{(x-r_1)(x-r_2)}((x-r_1)(x-r_2)-n_1n_2).$$

Theorem 2.9 ([7, Theorem 2.2.3])**.** *Let $G \cdot H$ be the coalescence in which the vertex u of G is identified with the vertex v of H. Then*

$$P_{G \cdot H}(x) = P_G(x)P_{H-v}(x) + P_{G-u}(x)P_H(x) - xP_{G-u}(x)P_{H-v}(x).$$

Lemma 2.10. *If $(CP_{n-1} \bigtriangledown K_1) \cdot K_2$ is the coalescence of K_2 with $CP_{n-1} \bigtriangledown K_1$ with its vertex of maximum degree as distinguished vertex, then for $n \ge 3$,*

$$Spec((CP_{n-1} \bigtriangledown K_1) \cdot K_2) = \{x_1, x_2, \underbrace{0,\dots,0}_{n-1}, x_3, \underbrace{-2,\cdots,-2}_{n-2}\},$$

such that $x_1 > x_2 > 0 > x_3$ are the roots of the polynomial $x^3 + (4-2n)x^2 + (1-2n)x + 2n - 4$.

Proof. Since $P_{CP_{n-1}}(x) = x^{n-1}(x+2)^{n-2}(x-2n+4)$ and $P_{K_1}(x) = x$, Theorem 2.8 implies that

$$P_{CP_{n-1}\triangledown K_1}(x) = x^{n-1}(x+2)^{n-2}(x^2+(4-2n)x+2-2n).$$

Since $P_{K_2}(x) = x^2-1$, it follows from Theorem 2.9,

$$P_{(CP_{n-1}\triangledown K_1)\cdot K_2}(x) = x^{n-1}(x+2)^{n-2}(x^3+(4-2n)x^2+(1-2n)x+2n-4).$$

Thus $(CP_{n-1} \triangledown K_1) \cdot K_2$ has $n-1$ and $n-2$ eigenvalues 0 and -2, respectively. The remaining eigenvalues are the roots of the polynomial $x^3+(4-2n)x^2+(1-2n)x+2n-4$. If

$$\begin{aligned}
a &= \Big(8n^3-30n^2+24n+8+3(-60n^4+312n^3-648n^2+606n-237)^{\frac{1}{2}}\Big)^{\frac{1}{3}},\\
b &= -\frac{4}{9}n^2+\frac{10}{9}n-\frac{13}{9},\\
r &= \big((8n^3-30n^2+24n+8)^2+540n^4-2808n^3+5832n^2-5454n+2133\big)^{\frac{1}{6}},\\
\theta &= \frac{1}{3}\arctan\left(\frac{3(60n^4-312n^3+648n^2-606n+237)^{\frac{1}{2}}}{8n^3-30n^2+24n+8}\right).
\end{aligned}$$

Then

$$\begin{aligned}
x_1 &= \frac{2n}{3}-\frac{4}{3}+\frac{a}{3}-\frac{3b}{a},\\
x_2 &= \frac{2n}{3}-\frac{4}{3}+(\frac{3b}{2r}-\frac{r}{6})\cos\theta-\sqrt{3}(\frac{3b}{2r}-\frac{r}{6})\sin\theta,\\
x_3 &= \frac{2n}{3}-\frac{4}{3}+(\frac{3b}{2r}-\frac{r}{6})\cos\theta+\sqrt{3}(\frac{3b}{2r}-\frac{r}{6})\sin\theta.
\end{aligned}$$

This completes the proof. □

Lemma 2.11. $\lim_{n\longrightarrow\infty}\sigma\big(CP_n, (CP_{n-1} \triangledown K_1)\cdot K_2\big) = 2$, *whenever* $(CP_{n-1} \triangledown K_1)\cdot K_2$ *is the coalescence of* K_2 *with* $CP_{n-1} \triangledown K_1$ *with its vertex of maximum degree as distinguished vertex.*

Proof. By Lemma 2.10 and using the symbolic computational software Maple [19] (see https://data.amc-journal.eu/cospectrality/maplecode1.mw), the result follows. □

Theorem 2.12 ([7, Theorem 2.1.5])**.** *Let G, H be graphs with n_1, n_2 vertices respectively. The characteristic polynomial of the join $G \triangledown H$ is given by the relation*

$$\begin{aligned}
P_{G\triangledown H}(x) = (-1)^{n_2}P_G(x)P_{\overline{H}}(-x-1) + (-1)^{n_1}P_H(x)P_{\overline{G}}(-x-1)\\
-(-1)^{n_1+n_2}P_{\overline{G}}(-x-1)P_{\overline{H}}(-x-1).
\end{aligned}$$

Lemma 2.13. *For $n \geq 3$ and any edge e,*

$$Spec(CP_n \setminus e) = \Bigg\{x_1, \frac{\sqrt{5}-1}{2}, \underbrace{0,\ldots,0}_{n-2}, x_2, -\frac{\sqrt{5}+1}{2}, \underbrace{-2,\ldots,-2}_{n-3}, x_3\Bigg\},$$

where $x_1 > 0 > x_2 > x_3$ are the roots of the polynomial $x^3-(2n-5)x^2-(6n-9)x-2n+2$.

Proof. For any edge e, $CP_n \setminus e = P_4 \bigtriangledown CP_{n-2}$. Let $G = P_4$ and $H = CP_{n-2}$. Thus $\overline{G} = G$ and $\overline{H} = (n-2)K_2$. We have

$$P_G(x) = P_{\overline{G}}(x) = x^4 - 3x^2 + 1,$$
$$P_H(x) = (x - 2n + 6)x^{n-2}(x + 2)^{n-3},$$
$$P_{\overline{H}}(x) = (x^2 - 1)^{n-2}.$$

Therefore

$$P_{CP_n\setminus e} = P_{G\bigtriangledown H}(x) = x^{n-2}(x+2)^{n-3}(x^2+x-1)(x^3-(2n-5)x^2-(6n-9)x-2n+2).$$

It follows $CP_n \setminus e$ has $n-2$ and $n-3$ eigenvalues 0 and -2, respectively. The remaining eigenvalues are $\frac{\sqrt{5}-1}{2}$, $-\frac{\sqrt{5}+1}{2}$ and the roots of $x^3 - (2n-5)x^2 - (6n-9)x - 2n + 2$. If

$$a = \big(64n^3 - 48n^2 - 312n + 404 + 12(-240n^4 + 528n^3 + 396n^2 - 1740n + 1137)^{\frac{1}{2}}\big)^{\frac{1}{3}},$$
$$b = -\frac{4}{9}n^2 + \frac{2}{9}(n+1),$$
$$r = \big((64n^3 - 48n^2 - 312n + 404)^2 + 34560n^4 - 76032n^3 - 57024n^2 + 250560n - 163728\big)^{\frac{1}{6}},$$
$$\theta = \frac{1}{3}\arctan\left(\frac{12(240n^4 - 528n^3 - 396n^2 + 1740n - 1137)^{\frac{1}{2}}}{64n^3 - 48n^2 - 312n + 404}\right).$$

Then

$$x_1 = \frac{2n}{3} - \frac{5}{3} + \frac{a}{6} - \frac{6b}{a},$$
$$x_2 = \frac{2n}{3} - \frac{5}{3} + (\frac{3b}{r} - \frac{r}{12})\cos\theta - \sqrt{3}(\frac{3b}{r} - \frac{r}{12})\sin\theta,$$
$$x_3 = \frac{2n}{3} - \frac{5}{3} + (\frac{3b}{r} - \frac{r}{12})\cos\theta + \sqrt{3}(\frac{3b}{r} - \frac{r}{12})\sin\theta,$$

and we are done. □

Lemma 2.14. $\lim_{n\longrightarrow\infty} \lambda(CP_n, CP_n \setminus e) = 10 - 4\sqrt{5}$.

Proof. By Lemma 2.13 and using the symbolic computational software Maple [19] (see https://data.amc-journal.eu/cospectrality/maplecode2.mw), the result follows. □

We have the following conjectures:

Conjecture 2.15. *For every integer $n \geq 2$, cs$(CP_n) = \sigma(CP_n, H)$ for some graph H if and only if $H \cong (CP_{n-1} \bigtriangledown K_1) \cdot K_2$, whenever $(CP_{n-1} \bigtriangledown K_1) \cdot K_2$ is the coalescence of K_2 with $CP_{n-1} \bigtriangledown K_1$ with its vertex of maximum degree as distinguished vertex.*

Conjecture 2.16. *For every integer $n \geq 4$, cs$(CP_n) = \lambda(CP_n, H)$ for some graph H if and only if $H \cong CP_n \setminus e$, for any edge e.*

For $n = 2$ and $n = 3$, cs$(CP_n) = \lambda(CP_n, H)$ if and only if $H \cong (CP_{n-1} \bigtriangledown K_1) \cdot K_2$. Our computational results confirm Conjectures 2.15 and 2.16 for all graphs of order at most 10.

3 Cospectrality of complete tripartite graphs

In this section we investigate $\mathrm{cs}(K_{n,n,n})$, for $n \geq 3$, to the ℓ^1- and ℓ^2-norms. First we need the following results.

Theorem 3.1 ([12, Theorem 9.1.1]). *Let G be a graph of order n and H be an induced subgraph of G with order m. Suppose that $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$ and $\lambda_1(H) \geq \cdots \geq \lambda_m(H)$ are the eigenvalues of G and H, respectively. Then for every i, $1 \leq i \leq m$, $\lambda_i(G) \geq \lambda_i(H) \geq \lambda_{n-m+i}(G)$.*

Theorem 3.2 (See [23] and also [8, Theorem 6.7]). *A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

Lemma 3.3 ([22, Lemma 7]). *$\lambda_2((K_1 + K_{r,s})\nabla\overline{K_q}) \leq \sqrt{2} - 1$ $(r \leq s)$ if and only if one of the conditions $1 - 10$ holds:*

(1) $r > 1$, $s \geq r$, $q = 1$;

(2) $r = 1$, $s \geq 1$, $q \geq 2$;

(3) $r = 2$, $s \geq 2$, $q = 2$;

(4) $r = 2$, $2 \leq s \leq 3$, $q \geq 3$;

(5) $r = 2$, $s = 4$, $3 \leq q \leq 7$;

(6) $r = 2$, $s = 5$, $3 \leq q \leq 4$;

(7) $r = 2$, $6 \leq s \leq 8$, $q = 3$;

(8) $r = 3$, $s = 3$, $2 \leq q \leq 4$;

(9) $r = 3$, $4 \leq s \leq 7$, $q = 2$;

(10) $r = 4$, $s = 4$, $q = 2$.

Lemma 3.4 ([22, Lemma 8]). *$\lambda_2((K_1 + K_{r,s})\nabla K_{p,q}) \leq \sqrt{2} - 1$ $(r \leq s,\ p \leq q)$ if and only if one of the conditions $1 - 5$ holds:*

(1) $r = 1$, $s = 1$, $p \geq 1$, $q \geq p$;

(2) $r = 1$, $s = 2$, $1 \leq p \leq 2$, $q \leq p$;

(3) $r = 1$, $s = 2$, $p = 3$, $3 \leq q \leq 7$;

(4) $r = 1$, $s = 2$, $p = 4$, $q = 4$;

(5) $r = 1$, $s = 3$, $p = 1$, $q = 1$.

Theorem 3.5 ([22, Theorem]). *Let G be a graph without isolated vertices and let $\lambda_2(G)$ be the second largest eigenvalue of G. Then $0 < \lambda_2(G) \leq \sqrt{2} - 1$ if and only if one of the following holds:*

(1) $G \cong (\nabla_t(K_1 + K_2))\nabla K_{n_1,\ldots,n_m}$,

(2) $G \cong (K_1 + K_{r,s})\nabla\overline{K_q}$, *and parameters q, r and s satisfy one of the conditions* (1) – (10) *from* Lemma 3.3,

(3) $G \cong (K_1 + K_{r,s})\nabla K_{p,q}$, *and parameters p, q, r and s satisfy one of the conditions* (1) – (5) *from* Lemma 3.4.

Lemma 3.6. *Let $n \geq 3$ and $x_1 > 0 > x_2 > x_3$ be the roots of the polynomial $x^3 - (3n^2 - 1)x - 2n^3 + 2n$. Then*

$$Spec(K_{n-1,n,n+1}) = \{x_1, \underbrace{0, \ldots, 0}_{3n-3}, x_2, x_3\}.$$

Proof. Since $P_{K_{n_1,\ldots,n_k}}(x) = x^{\sum_{i=1}^{k} n_i - k}\left(1 - \sum_{i=1}^{k} \frac{n_i}{x+n_i}\right) \prod_{i=1}^{k}(x+n_i)$,

$$P_{K_{n-1,n,n+1}}(x) = x^{3n-3}(x^3 - (3n^2-1)x - 2n^3 + 2n).$$

Thus $K_{n-1,n,n+1}$ has $3n-3$ eigenvalues 0 and 3 eigenvalues

$$\begin{aligned} x_1 &= \frac{a^2 + 9n^2 - 3}{3a}, \\ x_2 &= \left(\frac{-r}{6} + \frac{1-3n^2}{2r}\right)\cos\theta - \sqrt{3}\left(\frac{-r}{6} + \frac{1-3n^2}{2r}\right)\sin\theta, \\ x_3 &= \left(\frac{-r}{6} + \frac{1-3n^2}{2r}\right)\cos\theta + \sqrt{3}\left(\frac{-r}{6} + \frac{1-3n^2}{2r}\right)\sin\theta, \end{aligned}$$

where

$$\begin{aligned} a &= \left(27n^3 - 27n + 3(-81n^4 + 54n^2 + 3)^{\frac{1}{2}}\right)^{\frac{1}{3}}, \\ r &= \left((27n^3 - 27n)^2 + 729n^4 - 486n^2 - 27\right)^{\frac{1}{6}}, \\ \theta &= \frac{1}{3}\arctan\left(\frac{(81n^4 - 54n^2 - 3)^{\frac{1}{2}}}{9n^3 - 9n}\right). \end{aligned}$$ □

Lemma 3.7. $\lim_{n\longrightarrow\infty} \sigma(K_{n,n,n}, K_{n-1,n,n+1}) = \frac{2\sqrt{3}}{3}$.

Proof. Since $Spec(K_{n,n,n}) = \{2n, \underbrace{0,\ldots,0}_{3n-3}, -n, -n\}$, by Lemma 3.6 and using the computational software Maple [19] (see https://data.amc-journal.eu/cospectrality/maplecode3.mw), the result follows. □

Proof of Theorem 1.2. Note that

$$\sigma(K_{n,n,n}, H) = |2n - \lambda_1| + \sum_{i=2}^{3n-2} |\lambda_i| + |n + \lambda_{3n-1}| + |n + \lambda_{3n}|.$$

By Lemma 3.7, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\mathrm{cs}(K_{n,n,n}) < \frac{2\sqrt{3}}{3} + \varepsilon$. By Lemma 2.1, Corollary 2.2, Theorems 2.4 and 2.5, we obtain (1), (3) – (6) and $0 \leq \lambda_2 < \frac{\sqrt{3}}{3} + \frac{\varepsilon}{2}$. Suppose that $0 < \lambda_2 \leq \sqrt{2} - 1$. Hence Theorem 3.5 can be applied.
Case 1: $H \cong (\nabla_t(K_1 + K_2))\nabla K_{n_1,\ldots,n_m}$. If $t \geq 2$, then $(K_1 + K_2)\nabla(K_1 + K_2)$ is an induced subgraph of H. Since

$$Spec((K_1 + K_2)\nabla(K_1 + K_2)) = \{3.73205, .41421, .26795, -1, -1, -2.41421\},$$

by Theorem 3.1, $\lambda_{3n-2} \leq -1$, a contradiction. Now, suppose that $t = 1$. If $m = 1$, then $H \cong (K_1 + K_2)\nabla\overline{K_{3n-3}}$. We have $P_H(x) = x^{3n-4}f(x)$, whenever $f(x) =$

$x^4-(9n-8)x^2-(6n-6)x+3n-3$. So the non-zero eigenvalues of H are the roots of $f(x)=0$. By computing the roots, it implies that $\lambda_{3n-1}=-1$, a contradiction. Therefore $m\geq 2$. If $n_1=\cdots=n_m=1$, then $H\cong(K_1+K_2)\nabla K_{3n-3}$. So $(K_1+K_2)\nabla K_2$ is an induced subgraph of H. Since

$$Spec((K_1+K_2)\nabla K_2)=\{3.32340, .35793, -1, -1, -1.68133\},$$

by Theorem 3.1, $\lambda_{3n-2}\leq -1$, a contradiction. Now, we can assume that $n_i\geq 2$, for some $1\leq i\leq m$. Thus $(K_1+K_2)\nabla K_{1,2}$ is an induced subgraph of H. Since

$$Spec((K_1+K_2)\nabla K_{1,2})=\{4.06779, .36162, 0, -1, -1.24464, -2.18477\},$$

by Theorem 3.1, $\lambda_{3n-2}\leq -1$, a contradiction.

Case 2: $H\cong(K_1+K_{r,s})\nabla\overline{K_q}$ and parameters q,r and s satisfy conditions 1–10 from Lemma 3.3. We have $P_H(x)=x^{3n-4}f(x)$ whenever $f(x)=x^4-(q+qr+qs+rs)x^2-2qrsx+qrs$. The non-zero eigenvalues of H are determined by equation $f(x)=0$. By computing the roots, we have $\lambda_1=-\lambda_{3n}$ and $\lambda_2=-\lambda_{3n-1}$, a contradiction.

Case 3: $H\cong(K_1+K_{r,s})\nabla K_{p,q}$, and parameters p,q,r and s satisfy conditions 1–5 from Lemma 3.4. In this case, H can be isomorphic to one of these graphs: $(K_1+K_{1,2})\nabla K_{3,5}$, $(K_1+K_{1,2})\nabla K_{4,4}$ and $(K_1+K_{1,1})\nabla K_{p,q}$ whenever $q\geq p\geq 1$ and $p+q=3n-3$. All of these graphs have $(K_1+K_{1,1})\nabla K_{1,2}$ as an induced subgraph. Since

$$Spec((K_1+K_{1,1})\nabla K_{1,2})=\{4.06779, .36162, 0, -1, -1.24464, -2.18477\},$$

by Theorem 3.1, $\lambda_{3n-2}\leq -1$, a contradiction.

So $\sqrt{2}-1<\lambda_2<\frac{\sqrt{3}}{3}+\frac{\varepsilon}{2}$ or $\lambda_2=0$. If $\lambda_2=0$, then, by Theorem 3.2, there are some positive integers $k,n_1,\ldots,n_k$ and an integer $t\geq 0$ such that $H\cong tK_1+K_{n_1,\ldots,n_k}$. If $k=1$, then $H\cong\overline{K_{3n}}$, a contradiction. If $k=2$, then $H\cong tK_1+K_{r,s}$. Since

$$Spec(H)=\{\sqrt{rs},\underbrace{0,\ldots,0}_{3n-2},-\sqrt{rs}\},$$

$\lambda_{3n-1}=0$, a contradiction. Thus $k\geq 3$. Suppose that $k\geq 4$. If $n_1=\cdots=n_k=1$, then $H\cong tK_1+K_{3n-t}$. We have

$$Spec(H)=\{3n-t-1,\underbrace{0,\ldots,0}_{t},\underbrace{-1,\ldots,-1}_{3n-t-1}\}.$$

Hence $\lambda_{3n}=-1$, a contradiction. If there exists an unique i, $1\leq i\leq k$, such that $n_i\geq 2$, then $K_{1,1,1,2}$ is an induced subgraph of H. Since

$$Spec(K_{1,1,1,2})=\{3.64575, 0, -1, -1, -1.64575\},$$

by Theorem 3.1, $\lambda_{3n-2}\leq -1$, a contradiction. Thus there exist i and j such that $n_i,n_j\geq 2$. Hence H has $K_{1,1,2,2}$ as an induced subgraph. We have

$$Spec(K_{1,1,2,2})=\{4.37228, 0, 0, -1, -1.37228, -2\}.$$

So by Theorem 3.1, $\lambda_{3n-2}\leq -1$, a contradiction. Therefore we can assume that $k=3$ and $H\cong tK_1+K_{p,q,r}$, for some positive integers p,q and r. If $p=q=r=1$, then, by similar argument given in $k\geq 4$, we have $\lambda_{3n}=-1$, a contradiction. So $H\cong tK_1+K_{p,q,r}$ such that at least one of p,q and r is greater than 1. This completes the proof. □

Lemma 3.8. $\lim_{n\longrightarrow\infty} \lambda(K_{n,n,n}, K_{n-1,n,n+1}) = \frac{2}{3}$.

Proof. By Lemma 3.6 and using the symbolic computational software Maple [19] (see https://data.amc-journal.eu/cospectrality/maplecode4.mw), the result follows. □

The graph H in Figure 1 is the only unique graph such that $\sigma(K_{3,3,3}, H)$ and $\lambda(K_{3,3,3}, H)$ have the minimum possible values. For $n \geq 4$, we have the following conjectures:

Conjecture 3.9. *For every integer $n \geq 4$, cs$(K_{n,n,n}) = \sigma(K_{n,n,n}, H)$ for some graph H if and only if $H \cong K_{n-1,n,n+1}$.*

Conjecture 3.10. *For every integer $n \geq 4$, cs$(K_{n,n,n}) = \lambda(K_{n,n,n}, H)$ for some graph H if and only if $H \cong K_{n-1,n,n+1}$.*

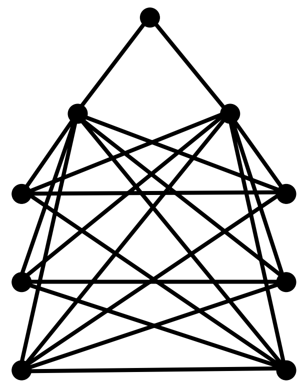

Figure 1: The graph which is closest to $K_{3,3,3}$ both in the ℓ^1- and ℓ^2-norm.

4 Cospectrality of some families of graphs using ℓ^p-norm for $p > 2$

Let $p > 2$ be an arbitrary positive integer. First we determine the cospectrality of the null graphs on n vertices.

Theorem 4.1. *For every integer $n \geq 2$, cs$(nK_1) = 2$. Moreover, cs$(nK_1) = \lambda^{(p)}(nK_1, H)$ for some graph H if and only if $H \cong K_2 + (n-2)K_1$.*

Proof. It is not hard to see that $\lambda^{(p)}(nK_1, K_2 + (n-2)K_1) = 2$. Let H be a simple graph of order n. Thus cs$(nK_1) = \lambda^{(p)}(nK_1, H) \leq 2$. So $|\lambda_1(H)| \leq \sqrt[p]{2}$, where $\lambda_1(H)$ is the greatest eigenvalue of H. Since the greatest eigenvalue of a graph is always non-negative and $H \ncong nK_1$, we have $0 < \lambda_1(H) \leq \sqrt[p]{2}$. Moreover, there is no graph whose greatest eigenvalue lies in the intervals $(0, 1)$ and $(1, \sqrt{2})$. Hence $\lambda_1(H) = 1$. Thus $H \cong K_2 + (n-2)K_1$. □

In the following we show that the minimum value of $\lambda^{(p)}(K_n, H)$ occurs whenever $H \cong K_n \setminus e$, where $K_n \setminus e$ is the graph obtaining from K_n by deletion one edge e. First we need the following results.

Lemma 4.2. *$\lambda^{(p)}(K_2, K_2 \setminus e) = 2$ and for every integer $n \geq 3$ and every edge e of K_n, $\lambda^{(p)}(K_n, K_n \setminus e) < 2$.*

Proof. It is easy to see that $\lambda^{(p)}(K_2, K_2 \setminus e) = 2$. By Corollary 3.4 and Lemma 3.6 in [2], one can obtain the result. □

Theorem 4.3. *For every integer $n \geq 2$, $cs(K_n) = \lambda^{(p)}(K_n, H)$ for some graph H if and only if $H \cong K_n \setminus e$ for any edge e, where $K_n \setminus e$ is the graph obtaining from K_n by deletion one edge e.*

Proof. For $n = 2$ and $n = 3$, It is easy to see that $\text{cs}(K_n) = \lambda^{(p)}(K_n, K_n \setminus e)$. Let $n \geq 4$. We show that if H is not isomorphic to K_n and $K_n \setminus e$, then $\lambda^{(p)}(K_n, H) \geq 2$.

Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of H. Therefore

$$\lambda^{(p)}(K_n, H) = |\lambda_1 - n + 1|^p + \sum_{i=2}^{n} |\lambda_i + 1|^p.$$

One can obtain this if one of the following cases holds, then $\lambda^{(p)}(K_n, H) \geq 2$.

Case 1: $\lambda_1 - n + 1 \leq -\sqrt[3]{2}$.

Case 2: $\lambda_2 + 1 \geq \sqrt[3]{2}$.

Case 3: $\lambda_3 \geq 0$.

Now suppose that none of the above cases occurs. Thus we can assume that $\lambda_1 > n - 1 - \sqrt[3]{2}$, $\lambda_2 < \sqrt[3]{2} - 1$ and $\lambda_3 < 0$. If $\lambda_2 \leq 0$, then, by Lemma 3.9 in [2], $H \cong K_{n-1} + K_1$ and $\lambda^{(p)}(K_n, H) = 2$.

Now suppose that $\lambda_2 > 0$. Since $0 < \lambda_2 < \sqrt[3]{2} - 1 < \frac{1}{3}$, by Theorem 2 in [5], there exists an integer t such that $H \cong tK_1 + (K_1 + K_2)\nabla\overline{K_{n-3-t}}$ where $0 \leq t \leq n - 4$.

If $n - 3 - t > 1$, then $(K_1 + K_2)\nabla\overline{K_2}$ is an induced subgraph of H. Since

$$Spec((K_1 + K_2)\nabla\overline{K_2}) = \{2.85577, 0.32164, 0, -1, -2.17741\},$$

by Theorem 3.1, $\lambda_3 \geq 0$, a contradiction. If $n - 3 - t = 1$, then $H \cong (n - 4)K_1 + (K_1 + K_2)\nabla\overline{K_1}$. Since

$$Spec(H) = \{2.17009, 0.31111, \underbrace{0, \ldots, 0}_{n-4}, -1, -1.48119\},$$

$\lambda^{(p)}(K_n, H) > 2$. Therefore by Lemma 4.2, $\text{cs}(K_n) = \lambda^{(p)}(K_n, K_n \setminus e)$. This completes the proof. □

In the following, we investigate the cospectrality of complete bipartite graphs. The proofs of Lemmas 2.5 and 2.7 and Theorem 2.8 in [20] are also working for $p > 2$, an arbitrary positive integer. First we need the following results, the "$\ell^p-version$" of Lemmas 2.5 and 2.7 in [20].

Lemma 4.4. *Let m and n be two positive integers and G be a graph of order $m + n$. If G has $K_{1,1,2}$ or $(K_1 + K_2)\nabla K_1$ as an induced subgraph, then $\lambda^{(p)}(G, K_{m,n}) \geq 1$.*

Lemma 4.5. *Let m and n be two positive integers and G be a graph of order $m+n$. Suppose that there are no positive integers r, s and a non-negative integer t such that $G \cong K_{r,s} + tK_1$. If $\lambda_2(G) \leq \sqrt{2}-1$, then $\lambda^{(p)}(G, K_{m,n}) \geq 1$.*

Theorem 4.6. *Let m and n be two positive integers such that $(m, n) \neq (1, 1)$. Then*

$$cs(K_{m,n}) = \lambda^{(p)}(K_{m,n}, K_{r,s} + tK_1),$$

for some integers $r, s \geq 1$ and $t \geq 0$ such that $r+s+t = m+n$ and $r, s \neq m, n$. Moreover, if $cs(K_{m,n}) = \lambda^{(p)}(K_{m,n}, H)$ for some graph H, then $H \cong K_{i,j} + hK_1$, where $i, j \geq 1$ and $h \geq 0$ are some integers so that $i+j+h = m+n$.

Proof. It is easy to see that $\text{cs}(K_{1,2}) = \lambda^{(p)}(K_{1,2}, K_{1,1} + K_1)$. So we can assume that $m+n \geq 4$. Let $i, j \geq 1$ and $h \geq 0$ be some integers such that $i+j+h = m+n$. Thus $\lambda^{(p)}(K_{m,n}, K_{i,j} + hK_1) = 2|\sqrt{mn} - \sqrt{ij}|^p$. By Lemma 2.4 in [20], there are some positive integers r and s such that $r+s \leq m+n$ and $\{r, s\} \neq \{m, n\}$ so that $|\sqrt{mn} - \sqrt{rs}|^p < (\frac{\sqrt{2}-1}{\sqrt{2}})^p$. Let $t = m+n-r-s$. Hence we obtain $\lambda^{(p)}(K_{m,n}, K_{r,s} + tK_1) < (\sqrt{2}-1)^p$. Therefore $\text{cs}(K_{m,n}) < (\sqrt{2}-1)^p < 1$. Now suppose that H is a graph such that $\text{cs}(K_{m,n}) = \lambda^{(p)}(K_{m,n}, H)$. Thus $\lambda^{(p)}(K_{m,n}, H) < (\sqrt{2}-1)^p$. Let $\lambda_2(H)$ be the second largest eigenvalue of H. So we have $|\lambda_2(H)| < \sqrt{2}-1$. Since $\lambda^{(p)}(K_{m,n}, H) < 1$, by Lemma 4.5, there are some integers $r, s \geq 1$ and $t \geq 0$ such that $H \cong K_{r,s} + tK_1$. This completes the proof. □

Theorem 4.7. *Let $n \geq 1$ be an integer. Then, the following hold:*

(1) $cs(K_{1,1}) = \lambda^{(p)}(K_{1,1}, 2K_1) = 2$,

(2) $cs(K_{1,2}) = \lambda^{(p)}(K_{1,2}, K_{1,1} + K_1) = 2|\sqrt{2}-1|^p$,

(3) *If $n \geq 3$ is a prime number, then*

$$cs(K_{1,n}) = \lambda^{(p)}(K_{1,n}, K_{2,\frac{n+1}{2}} + \frac{n-3}{2}K_1) = 2|\sqrt{n+1} - \sqrt{n}|^p,$$

(4) *If $n \geq 3$ is not a prime number, then*

$$cs(K_{1,n}) = \lambda^{(p)}(K_{1,n}, K_{r,s} + (n+1-r-s)K_1) = 0,$$

where r and s are some positive integers such that $r, s < n$ and $n = rs$.

Proof. The method is similar to that of Theorem 2.10 in [20]. □

By Theorem 4.6, one can easily obtain the following results.

Theorem 4.8. *For every integer $n \geq 2$, $cs(K_{n,n}) = 2|n - \sqrt{n^2-1}|^p$. Moreover, $cs(K_{n,n}) = \lambda^{(p)}(K_{n,n}, H)$ for some graph H if and only if $H \cong K_{n-1,n+1}$.*

Theorem 4.9. *For every integer $n \geq 2$, $cs(K_{n,n+1}) = 2|\sqrt{n^2+n} - \sqrt{n^2+n-2}|^p$. Moreover, $cs(K_{n,n+1}) = \lambda^{(p)}(K_{n,n+1}, H)$ for some graph H if and only if $H \cong K_{n-1,n+2}$.*

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