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Closed formulas for the total Roman domination number of lexicographic product graphs

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Abstract

Let G be a graph with no isolated vertex and $f : V(G) \rightarrow \{0,1,2\}$ a function. Let $V_i = \{x \in V(G) : f(x) = i\}$ for every $i \in \{0,1,2\}$. We say that f is a total Roman dominating function on G if every vertex in V_0 is adjacent to at least one vertex in V_2 and the subgraph induced by $V_1 \cup V_2$ has no isolated vertex. The weight of f is $\omega(f) = \sum_{v \in V(G)} f(v)$. The minimum weight among all total Roman dominating functions on G is the total Roman domination number of G, denoted by $\gamma_{tR}(G)$. It is known that the general problem of computing $\gamma_{tR}(G)$ is NP-hard. In this paper, we show that if G is a graph with no isolated vertex and H is a nontrivial graph, then the total Roman domination number of the lexicographic product graph $G \circ H$ is given by

$$\gamma_{tR}(G \circ H) = \begin{cases} 2\gamma_t(G) & \text{if } \gamma(H) \geq 2, \\ \xi(G) & \text{if } \gamma(H) = 1, \end{cases}$$

where $\gamma(H)$ is the domination number of H, $\gamma_t(G)$ is the total domination number of G and $\xi(G)$ is a domination parameter defined on G.

Keywords: Total Roman domination, total domination, lexicographic product graph.

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1 Introduction

Let G be a graph with no isolated vertex and $f : V(G) \rightarrow \{0,1,2\}$ a function. Let $V_i = \{x \in V(G) : f(x) = i\}$ for every $i \in \{0,1,2\}$. We will identify f with the partition of $V(G)$ induced by f and write $f(V_0, V_1, V_2)$. The weight of f is defined to be

$$\omega(f) = f(V(G)) = \sum_{v \in V(G)} f(v) = |V_1| + 2|V_2|.$$

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A function $f(V_0, V_1, V_2)$ is said to be *total Roman dominating function* on G if every vertex in V_0 is adjacent to at least one vertex in V_2 and the subgraph induced by $V_1 \cup V_2$ has no isolated vertex [15]. The minimum weight among all total Roman dominating functions on G is the *total Roman domination number* of G, denoted by $\gamma_{tR}(G)$. In this article, we continue the study initiated in [5] on the total Roman domination number of lexicographic product graphs. In particular, we give closed formulas for the total Roman domination number of lexicographic product graphs.

Let G and H be two graphs. The *lexicographic product* of G and H is the graph $G \circ H$ whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and $(u, v)(x, y) \in E(G \circ H)$ if and only if $ux \in E(G)$ or $u = x$ and $vy \in E(H)$. Notice that for any $u \in V(G)$ the subgraph of $G \circ H$ induced by $\{u\} \times V(H)$ is isomorphic to H. For simplicity, we will denote this subgraph by H_u.

For a basic introduction to the lexicographic product of two graphs we suggest the books [7, 12]. One of the main problems in the study of $G \circ H$ consists of finding exact values or tight bounds for specific parameters of these graphs and express them in terms of known invariants of G and H. In particular, we cite the following works on domination theory of lexicographic product graphs: (total) domination [16, 17, 18, 21], Roman domination [18], weak Roman domination [20], rainbow domination [19], super domination [6], doubly connected domination [2], secure domination [13], double domination [4] and total Roman domination [5].

We assume that the reader is familiar with the basic concepts and terminology of domination in graph. If this is not the case, we suggest the textbooks [8, 9, 11]. In particular, we use the standard notation $\gamma(G)$ and $\gamma_t(G)$ for the domination number and the total domination number of a graph G, respectively. Throughout the paper, $N(v)$ will denote the set of neighbours or *open neighbourhood* of v in G. The *closed neighbourhood* of v, denoted by $N[v]$, equals $N(v) \cup \{v\}$. A vertex $v \in V(G)$ such that $N[v] = V(G)$ is said to be a *universal vertex*. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

2 The case where $\gamma(H) \geq 2$

The next theorem merges two results obtained in [18] and [21].

Theorem 2.1 ([18] and [21]). *For any graph G with no isolated vertex and any nontrivial graph H,*

$$\gamma(G \circ H) = \begin{cases} \gamma(G), & \text{if } \gamma(H) = 1, \\ \gamma_t(G), & \text{if } \gamma(H) \geq 2. \end{cases}$$

Below we present two theorems that complete the tools we need to deduce our first result.

Theorem 2.2. [1] *For any graph G with no isolated vertex,*

$$2\gamma(G) \leq \gamma_{tR}(G) \leq \min\{2\gamma_t(G), 3\gamma(G)\}.$$

Theorem 2.3. [3] *For any graph G with no isolated vertex and any nontrivial graph H,*

$$\gamma_t(G \circ H) = \gamma_t(G).$$

From the results above we deduce the following main theorem.

Theorem 2.4. *For any graph G with no isolated vertex and any graph H with $\gamma(H) \geq 2$,*

$$\gamma_{tR}(G \circ H) = 2\gamma_t(G).$$

Proof. The result immediately follows by applying Theorems 2.1, 2.3 and 2.2, in this order, i.e., $2\gamma_t(G) = 2\gamma(G \circ H) \leq \gamma_{tR}(G \circ H) \leq 2\gamma_t(G \circ H) = 2\gamma_t(G)$. □

Notice that, since the general optimization problem of finding the total domination number of a graph is NP-hard [14], by Theorem 2.4 we can conclude that the problem of finding the total Roman domination number is NP-hard. Even so, we would like to point out that there are several families of graphs for which the total domination number can be found in polynomial time [10].

3 The case where $\gamma(H) = 1$

The following two lemmas are the main tools in this section.

Lemma 3.1. *Let G be a graph with no isolated vertex. For any nontrivial graph H with $\gamma(H) = 1$, there exists a $\gamma_{tR}(G \circ H)$-function f satisfying the following two conditions.*

(i) *$f(V(H_u)) \leq 2$ for every $u \in V(G)$.*

(ii) *If $f(V(H_u)) = 2$, then $f(u, v) = 2$ for some universal vertex v of H.*

Proof. Given a TRDF f on $G \circ H$, we define the set $R_f = \{x \in V(G) : f(V(H_x)) \geq 3\}$. Let f be a $\gamma_{tR}(G \circ H)$-function such that $|R_f|$ is minimum among all $\gamma_{tR}(G \circ H)$-functions. Let $v \in V(H)$ be a universal vertex and suppose that there exists $u \in R_f$. We differentiate the following two cases.

Case 1. There exists $u' \in N(u)$ such that $f(V(H_{u'})) \geq 1$. Let f' be the function defined by $f'(V(H_u)) = f'(u, v) = 2$ and $f'(x, y) = f(x, y)$ for every $x \in V(G) \setminus \{u\}$. It is readily seen that f' is a $\gamma_{tR}(G \circ H)$-function with $|R_{f'}| < |R_f|$, which is a contradiction.

Case 2. $f(N(u) \times V(H)) = 0$. In this case, we choose a vertex $u' \in N(u)$ and define a function f' as $f'(V(H_{u'})) = f'(u', v) = 1$, $f'(V(H_u)) = f'(u, v) = 2$ and $f'(x, y) = f(x, y)$ for every $x \in V(G) \setminus \{u, u'\}$. As in Case 1, f' is a $\gamma_{tR}(G \circ H)$-function with $|R_{f'}| < |R_f|$, which is a contradiction.

According to the two cases above, (i) follows. Now, for any $\gamma_{tR}(G \circ H)$-function $f(V_0, V_1, V_2)$ satisfying (i), we define $R'_f = \{x \in V(G) : f(V(H_x)) = 2 \text{ and } V(H_x) \cap V_2 = \emptyset\}$. Let $g(V'_0, V'_1, V'_2)$ be a $\gamma_{tR}(G \circ H)$-function such that $|R'_g|$ is minimum among all $\gamma_{tR}(G \circ H)$-functions satisfying (i). Suppose that there exists $u \in R'_g$. If there exists $u' \in N(u)$ such that, $g(V(H_{u'})) = 2$, then the function g' defined by $g'(V(H_u)) = g'(u, v) = 1$ and $g'(x, y) = g(x, y)$ for every $x \in V(G) \setminus \{u\}$, is a TRDF on $G \circ H$ of weight $\omega(g') < \omega(g) = \gamma_{tR}(G \circ H)$, which is a contradiction. Hence, $g(N(u) \times V(H)) \leq 1$ and we can differentiate the following two cases.

Case 1'. There exists $u' \in N(u)$ such that $g(V(H_{u'})) = 1$. In this case, we define a function g' by $g'(V(H_u)) = g'(u, v) = 2$ and $g'(x, y) = g(x, y)$ for every $x \in V(G) \setminus \{u\}$. Notice that g' is a $\gamma_{tR}(G \circ H)$-function satisfying (i) and $|R'_{g'}| < |R'_g|$, which is a contradiction.

Case 2'. $g(N(u) \times V(H)) = 0$. We fix $u' \in N(u)$. Notice that there exists $u'' \in N(u') \setminus \{u\}$, with $V(H_{u''}) \cap V'_2 \neq \emptyset$. Hence, we can define a function g' as $g'(V(H_{u'})) =$

$g'(u',v) = g'(V(H_u)) = g'(u,v) = 1$ and $g'(x,y) = g(x,y)$ for every $x \in V(G) \setminus \{u,u'\}$. As in Case 1', g' is a $\gamma_{tR}(G \circ H)$-function satisfying (i) and $|R'_{g'}| < |R'_g|$, which is a contradiction.

According to the two cases above, $R'_g = \emptyset$, and so there exists a $\gamma_{tR}(G \circ H)$-function h defined as $h(V(H_u)) = h(u,v) = 2$ whenever $g(V(H_u)) = 2$ and $h(V(H_u)) = g(V(H_u))$ otherwise. Therefore, h satisfies (i) and (ii). □

Lemma 3.2. *Let G be a graph with no isolated vertex and H a nontrivial graph with $\gamma(H) = 1$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{tR}(G \circ H)$-function, $A = \{x \in V(G) : V(H_x) \cap V_1 \neq \emptyset\}$ and $B = \{x \in V(G) : V(H_x) \cap V_2 \neq \emptyset\}$. If f satisfies Lemma 3.1, then B is a dominating set and $A \cup B$ is a total dominating set of G.*

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_{tR}(G \circ H)$-function which satisfies Lemma 3.1. Let $C = V(G) \setminus (A \cup B)$. Obviously, if $x \in C$, then $V(H_x) \subseteq V_0$, which implies that x is adjacent to some vertex in B and, since H is a nontrivial graph and f satisfies Lemma 3.1, if $x \in A$, then there exists $y \in V(H)$ such that $(x,y) \in V_0$, and so x is adjacent to some vertex in B. Hence, B is a dominating set of G. Now, since the subgraph of $G \circ H$ induced by $V_1 \cup V_2$ does not have isolated vertices, the subgraph of G induced by $A \cup B$ does not have isolated vertices, which implies that $A \cup B$ is total dominating set of G. □

For any graph G, let $\mathcal{D}(G)$ be the set of dominating sets of G, and $\mathcal{D}_t(G)$ the set of total dominating sets of G. We now proceed to introduce our main tool, which is the following domination parameter.

$$\xi(G) = \min\{|A| + 2|B| : \ B \in \mathcal{D}(G) \text{ and } A \cup B \in \mathcal{D}_t(G)\}.$$

We say that an ordered pair (A,B) of subsets of $V(G)$ is a $\xi(G)$-pair if $B \in \mathcal{D}(G)$, $A \cup B \in \mathcal{D}_t(G)$ and $\xi(G) = |A| + 2|B|$.

Theorem 3.3. *For any graph G with no isolated vertex and any nontrivial graph H with $\gamma(H) = 1$,*

$$\gamma_{tR}(G \circ H) = \xi(G).$$

Proof. Let v be a universal vertex of H. From any $\xi(G)$-pair (A,B) we define the function $f(V_0, V_1, V_2)$ as $V_2 = B \times \{v\}$, $V_1 = A \times \{v\}$ and $V_0 = V(G \circ H) \setminus (V_1 \cup V_2)$. Since V_2 is a dominating set of $G \circ H$ and $V_1 \cup V_2$ is a total dominating set of $G \circ H$, we can conclude that f is a TRDF on $G \circ H$. Therefore, $\gamma_{tR}(G \circ H) \leq \omega(f) = |V_1| + 2|V_2| = |A| + 2|B| = \xi(G)$.

Now, let $f'(V'_0, V'_1, V'_2)$ be a $\gamma_{tR}(G \circ H)$-function which satisfies Lemma 3.1. Let $A = \{x \in V(G) : f'(V(H_x)) = 1\}$ and $B = \{x \in V(G) : f'(V(H_x)) = 2\}$. By lemma 3.2, B is a dominating set of G and $A \cup B$ is a total dominating set, which implies that $\xi(G) \leq |A| + 2|B| = |V'_1| + 2|V'_2| = \gamma_{tR}(G \circ H)$. Therefore, the result follows. □

Let G be the graph shown in Figure 1 and H a nontrivial graph with $\gamma(H) = 1$. Notice that $\gamma_{tR}(G \circ H) = \xi(G) = \gamma_{tR}(G) = 8$, where $f_1(V_0, V_1, V_2)$ and $f_2(W_0, W_1, W_2)$ are $\gamma_{tR}(G)$-functions for $V_1 = \{b, e\}$, $V_2 = \{a, d, f\}$, $W_1 = \emptyset$, $W_2 = \{a, b, e, f\}$. Furthermore, both (V_1, V_2) and (W_1, W_2) are $\xi(G)$-pairs, where V_2 is a $\gamma(G)$-set and $|V_1| + |V_2| > \gamma_t(G)$, while W_2 is a $\gamma_t(G)$-set which does not contain any $\gamma(G)$-set.

The following bounds were given in [5]. In fact the lower bound was stated for any connected non-trivial graph G, although it also holds for any graph G with no isolated vertex.

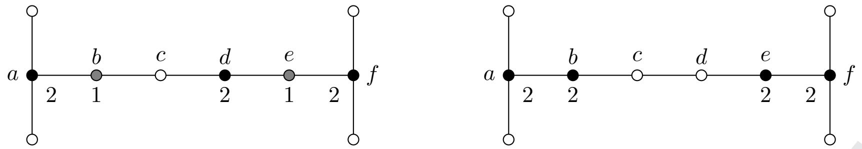

Figure 1: The labels correspond to two different $\gamma_{tR}(G)$-functions $f_1(V_0, V_1, V_2)$, on the left, and $f_2(W_0, W_1, W_2)$, on the right. In this case, $\gamma_{tR}(G) = 2\gamma_t(G) = 8$, $V_2 = \{a, d, f\}$ is a $\gamma(G)$-set and $W_2 = \{a, b, e, f\}$ is the only $\gamma_t(G)$-set.

Theorem 3.4. [5] *For any graph H and any graph G with no isolated vertex,*

$$\gamma_{tR}(G) \leq \gamma_{tR}(G \circ H) \leq 2\gamma_t(G).$$

Furthermore, if $\gamma(H) = 1$, then

$$\gamma_{tR}(G \circ H) \leq 3\gamma(G).$$

In order to improve some of these bounds, we need to introduce some additional terminology. Given a set $S \subseteq V(G)$, we define

$$\psi(S) = \min\{|S'| : S' \subseteq V(G) \setminus S \text{ and } S \subseteq N(S' \cup S)\}.$$

We also define the following parameter associated to G.

$$\mu(G) = \min\{\psi(S) : S \text{ is a } \gamma(G)\text{-set}\}.$$

It is readily seen that $0 \leq \mu(G) \leq \gamma(G)$. Furthermore, $\mu(G) = 0$ if and only if $\gamma_t(G) = \gamma(G)$, while $\mu(G) = \gamma(G)$ if and only if for every $\gamma(G)$-set S and every pair of different vertices $x, y \in S$ we have that $N[x] \cap N[y] = \emptyset$, i.e., if and only if every $\gamma(G)$-set is a 2-packing of G.

With the notation above in mind, we state the following theorem.

Theorem 3.5. *Let G and H be two graphs with no isolated vertex. If $\gamma(H) = 1$, then*

$$\max\{\gamma_{tR}(G), \gamma_t(G) + \gamma(G)\} \leq \gamma_{tR}(G \circ H) \leq \min\{2\gamma(G) + \mu(G), 2\gamma_t(G)\}.$$

Proof. Our main tool is Theorem 3.3. For any $\xi(G)$-pair (A, B) we have that $\gamma_{tR}(G \circ H) = \xi(G) = 2|B| + |A| \geq |(A \cup B)| + |B| \geq \gamma_t(G) + \gamma(G)$.

Now, let S be a $\gamma(G)$-set with $\mu(G) = \psi(S)$ and $S' \subseteq V(G) \setminus S$ a set of minimum cardinality among the subsets of $V(G) \setminus S$ satisfying that $S \subseteq N(S' \cup S)$. Since $S \cup S'$ is a total dominating set, $\gamma_{tR}(G \circ H) = \xi(G) \leq |S \cup S'| + |S| = 2|S| + |S'| = 2\gamma(G) + \mu(G)$.

Finally, by Theorem 3.4, $\gamma_{tR}(G) \leq \gamma_{tR}(G \circ H) \leq 2\gamma_t(G)$, which completes the proof. □

Since $\mu(G) \leq \gamma(G)$, we can conclude that the bound $\gamma_{tR}(G \circ H) \leq 2\gamma(G) + \mu(G)$ is never worse than the known bound $\gamma_{tR}(G \circ H) \leq 3\gamma(G)$. In order to see that the upper bounds given by Theorem 3.5 are tight, we take the graph G shown in Figure 1 and any nontrivial graph H with $\gamma(H) = 1$. In this case, $\gamma_{tR}(G \circ H) = 2\gamma_t(G) = 2\gamma(G) + \mu(G) = 8$.

We would point out the following result which is a direct consequence of Theorems 2.2 and 3.5.

Theorem 3.6. *If G is a graph with $\gamma_t(G) = \gamma(G)$ and H is a nontrivial graph with $\gamma(H) = 1$, then*

$$\gamma_{tR}(G \circ H) = \gamma_{tR}(G) = 2\gamma(G).$$

We now proceed to characterize the graphs achieving the lower bounds given by Theorem 3.5.

Theorem 3.7. *Let G and H be two graphs with no isolated vertex. If $\gamma(H) = 1$, then the following statements are equivalent.*

(i) $\gamma_{tR}(G \circ H) = \gamma_{tR}(G)$.

(ii) *There exists a $\gamma_{tR}(G)$-function $f(V_0, V_1, V_2)$ such that V_2 is dominating set of G.*

Proof. If there exists a $\gamma_{tR}(G)$-function $f(V_0, V_1, V_2)$ such that V_2 is dominating set of G, then $\gamma_{tR}(G \circ H) = \xi(G) \leq |V_1 \cup V_2| + |V_2| = |V_1| + 2|V_2| = \gamma_{tR}(G)$. Since $\gamma_{tR}(G) \leq \gamma_{tR}(G \circ H)$, we conclude that $\gamma_{tR}(G \circ H) = \gamma_{tR}(G)$.

Conversely, assume that $\gamma_{tR}(G \circ H) = \gamma_{tR}(G)$. Let $g(V_0', V_1', V_2')$ be a $\gamma_{tR}(G \circ H)$-function satisfying Lemma 3.1. Let $A = \{x \in V(G) : g(V(H_x)) = 1\}$ and $B = \{x \in V(G) : g(V(H_x)) = 2\}$. By Lemma 3.2, B is a dominating set of G and $A \cup B$ is a total dominating set. Hence, we can define a TRDF $h(V_0'', V_1'', V_2'')$ from $V_1'' = A$ and $V_2'' = B$. Since $\omega(h) = |A| + 2|B| = |V_1'| + 2|V_2'| = \gamma_{tR}(G \circ H) = \gamma_{tR}(G)$, we conclude that h is a $\gamma_{tR}(G)$-function where V_2'' is a dominating set, as desired. □

The next result gives a characterization for the case $\gamma_{tR}(G \circ H) = \gamma_t(G) + \gamma(G)$ whenever $\gamma(H) = 1$.

Theorem 3.8. *Let G and H be two graphs with no isolated vertex. If $\gamma(H) = 1$, then the following statement are equivalent.*

(i) $\gamma_{tR}(G \circ H) = \gamma_t(G) + \gamma(G)$.

(ii) *There exists a $\gamma_t(G)$-set that contains some $\gamma(G)$-set.*

Proof. If there exists a $\gamma_t(G)$-set X which contains a $\gamma(G)$-set B, then $\gamma_{tR}(G \circ H) = \xi(G) \leq |X \setminus B| + 2|B| = |X| + |B| = \gamma_t(G) + \gamma(G)$, and by (i) we conclude that $\gamma_{tR}(G \circ H) = \gamma_t(G) + \gamma(G)$.

Conversely, assume that $\gamma_{tR}(G \circ H) = \gamma_t(G) + \gamma(G)$ and let (A, B) be a $\xi(G)$-pair. If the total dominating set $A \cup B$ is a $\gamma_t(G)$-set, then we are done, as B is a dominating set and from $\gamma_t(G) + \gamma(G) = \gamma_{tR}(G \circ H) = \xi(G) = |A| + 2|B| = |A \cup B| + |B| = \gamma_t(G) + |B|$ we deduce that B is a $\gamma(G)$-set. Suppose to the contrary, that $|A \cup B| > \gamma_t(G)$. In such a case, $\gamma_t(G) + \gamma(G) = \xi(G) = |A| + 2|B| \geq |A \cup B| + |B| > \gamma_t(G) + \gamma(G)$, which is a contradiction. Therefore, the result follows. □

Figure 2 shows a graph G such that $\gamma_{tR}(G \circ H) = \gamma_t(G) + \gamma(G) = 7 > 6 = \gamma_{tR}(G)$ for every nontrivial graph H with $\gamma(H) = 1$.

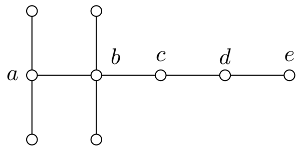

Figure 2: The $\gamma_t(G)$-set $D = \{a, b, d, e\}$ contains the $\gamma(G)$-set $S = \{a, b, d\}$.

4 Small values of $\gamma_{tR}(G \circ H)$

In this short section we characterize the graphs G and H for which $\gamma_{tR}(G \circ H) \in \{3, 4\}$.

Theorem 4.1. *For any graph G and H with no isolated vertex, the following statements are equivalent.*

(i) $\gamma_{tR}(G \circ H) = 3$.

(ii) $\gamma(G) = \gamma(H) = 1$.

Proof. If $\gamma_{tR}(G \circ H) = 3$, then by Theorem 2.4 we deduce that $\gamma(H) = 1$. Moreover, by Theorem 3.5 we have that $3 = \gamma_{tR}(G \circ H) \geq \gamma_t(G) + \gamma(G) \geq 3$. Hence, $\gamma(G) = 1$, as required. Conversely, if $\gamma(G) = \gamma(H) = 1$, then by Theorem 3.8 we deduce that $\gamma_{tR}(G \circ H) = 3$. □

Theorem 4.2. *For any graph G and H with no isolated vertex, $\gamma_{tR}(G \circ H) = 4$ if and only if one of the following conditions are satisfied.*

(i) $\gamma_t(G) = 2$ *and* $\gamma(H) \geq 2$.

(ii) $\gamma_t(G) = \gamma(G) = 2$ *and* $\gamma(H) = 1$.

Proof. We first notice that if conditions (i) or (ii) holds, then by Theorem 2.4 or by Theorem 3.5, respectively, it follows that $\gamma_{tR}(G \circ H) = 4$.

Conversely, assume that $\gamma_{tR}(G \circ H) = 4$. If $\gamma(H) \geq 2$, then Theorem 2.4 leads to $\gamma_t(G) = 2$. From now on, we assume that $\gamma(H) = 1$. By Theorem 3.8, we have that $4 = \gamma_{tR}(G \circ H) \geq \gamma_t(G) + \gamma(G)$. Hence, $1 \leq \gamma(G) \leq 2$. If $\gamma(G) = 1$, then by Theorem 4.1 we obtain that $\gamma_{tR}(G \circ H) = 3$, which is a contradiction. Hence, $\gamma(G) = 2$ and so $\gamma_t(G) = 2$. Therefore, the result follows. □

5 Open problems

By Theorem 3.3 we learned that, if we want to know the behaviour of $\gamma_{tR}(G \circ H)$ when $\gamma(H) = 1$, then it is crucial to obtain the exact value or derive tight bounds on $\xi(G)$. In this sense, the study of $\xi(G)$ is an interesting challenge.

In particular, Theorem 3.5 states that

$$\max\{\gamma_{tR}(G), \gamma_t(G) + \gamma(G)\} \leq \xi(G) \leq \min\{2\gamma(G) + \mu(G), 2\gamma_t(G)\}.$$

The graphs achieving the equalities $\xi(G) = \gamma_{tR}(G)$ and $\xi(G) = \gamma_t(G) + \gamma(G)$ were characterized in Theorems 3.7 and 3.8, respectively. Therefore, the problems of characterizing the graphs achieving the equalities $\xi(G) = 2\gamma_t(G)$ and $\xi(G) = 2\gamma(G) + \mu(G) = 3\gamma(G)$ remain open.

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