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On the Smith normal form of the Varchenko matrix*

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Abstract

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{A}$ isomorphic to $\mathbb{R}^n$. Let V_q be the q-Varchenko matrix for the arrangement $\mathcal{A}$ with all hyperplane parameters equal to q. In this paper, we consider three interesting cases of q-Varchenko matrices associated to hyperplane arrangements. We show that they have a Smith normal form over $\mathbb{Z}[q]$.

Keywords: Hyperplane arrangement, Smith normal form, Varchenko matrix.

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1 Introduction

Let M be an $n \times n$ matrix over a commutative unital ring R. We say that M has a Smith normal form (SNF for short) over R if there are matrices $P, Q \in R^{n\times n}$ such that $\det(P)$ and $\det(Q)$ are units in R and PMQ is a diagonal matrix $\operatorname{diag}(d_1, d_2, \ldots, d_n)$ where d_i divides d_j in R for all $i < j$.

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Recently, there is an interest in SNF in combinatorics. A survey of this topic was given by Stanley in [11]. The SNF of a matrix of a differential operator was considered by Stanley and the first author in [2], where they proved a special case of a conjecture given by Miller and Reiner [7]. In [13], interesting results concerning the SNF of random integer matrix were found.

It is well known that M has an SNF if R is a principal ideal domain (PID), but not much is known for general rings. In this paper we are interested in the integer polynomial ring $\mathbb{Z}[q]$. Some matrices in $\mathbb{Z}[q]^{n\times n}$ do not have an SNF over R. For example, it is not hard to show that $\left[\begin{smallmatrix} 2 & 0 \\ 0 & q \end{smallmatrix}\right]$ does not have an SNF over $\mathbb{Z}[q]$. However, lots of matrices in $\mathbb{Z}[q]^{n\times n}$ do have SNF over $\mathbb{Z}[q]$. For example, it is asked whether every matrix of the form $A = (q^{a_{ij}})$, where a_{ij} are nonnegative integers, has an SNF over $\mathbb{Z}[q]$. There is not a general solution to this question. But we could give a positive answer which arises from some special cases of geometrical structures. The matrices we are interested in are called Varchenko matrices (see [12]). These matrices are associated to a hyperplane arrangement (see Definition 1.2). The Varchenko matrix was studied in the papers of Varchenko [12], Schechtman and Varchenko [8], and Brylawski and Varchenko [1]. These matrices describe the analogue of Serre's relations for quantum Kac-Moody Lie algebras and are relevant to the study of hypergeometric functions and the representation theory of quantum groups [6]. Entries appearing in the diagonal of a Smith normal form of a matrix are called invariant factors. Applications of invariant factors of a q-matrix can be found in [3, 4, 9]. We are going to prove that Varchenko matrices associated to some hyperplane arrangements do have an SNF.

We use the notation and terminology on hyperplane arrangements in [10]. A finite (real) hyperplane arrangement $\mathcal{A}$ is a finite set of affine hyperplanes in some affine space $\mathbb{A}$ isomorphic to $\mathbb{R}^n$.

For a hyperplane H in $\mathcal{A}$, let

$$\mathcal{A}^H = \{H \cap H' : H' \in \mathcal{A} \text{ such that } H' \cap H \neq \emptyset \text{ and } H' \neq H\}.$$

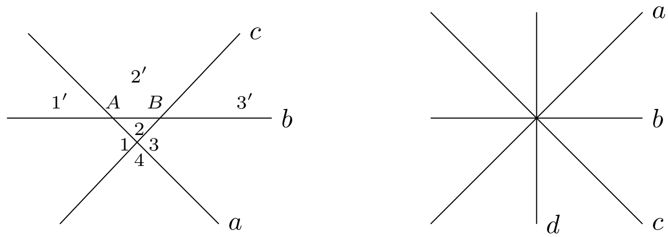
This is a hyperplane arrangement in the affine space H. We also write $\mathcal{A} - \{H\}$ for the arrangement from $\mathcal{A}$ with H removed.

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{A}$. Then $\mathbb{A}$ is divided into some regions by these hyperplanes. Explicitly, a region is a connected component of $\mathbb{A} - \bigcup_{H\in\mathcal{A}} H$. We let $\mathcal{R}(\mathcal{A})$ denote the set of regions of $\mathcal{A}$.

Example 1.1. In the following picture, arrangement $\mathcal{A}_{\mathrm{p}}$ is an example of the so-called peelable arrangement, which is treated in Section 2. Here we see straight lines a, b, c form a hyperplane arrangement in the plane $\mathbb{R}^2$. There are 7 regions of $\mathcal{A}_{\mathrm{p}}$ which we denote by $1', 2', 3', 1, 2, 3, 4$. (We write it in this way for the example in Section 2.) The hyperplane arrangement $\mathcal{A}_{\mathrm{p}}^b$ contains two affine hyperplanes $A = b \cap a$, $B = b \cap c$ (two points in b).

Arrangement $\mathcal{A}_{\mathrm{p}}$ is also an example of the regular n-gon arrangement $\mathcal{G}_n$, which is treated in Section 4. It is a regular triangle arrangement. (Although in Figure 1 the central triangle is not so much like a equilateral triangle. This does not matter, because the Varchenko matrix that we are concerned with is a topological invariant.) As another example for the n-gon arrangement, a picture of the pentagon arrangement $\mathcal{G}_5$ is given in Section 4.

Arrangement $\mathcal{C}_4$ in Figure 1 is an example of arrangement $\mathcal{C}_n$, which is treated in Section 3.



(a) Arrangement $\mathcal{A}_\text{p}$ (b) Arrangement $\mathcal{C}_4$

Figure 1: Arrangements $\mathcal{A}_\text{p}$ and $\mathcal{C}_4$.

Definition 1.2. Let $\mathcal{A}$ be a finite hyperplane arrangement and $\mathcal{R}(\mathcal{A})$ its set of regions, and let a_H for $H \in \mathcal{A}$ be indeterminates. The *Varchenko matrix* $V = V(\mathcal{A})$ is indexed by $\mathcal{R}(\mathcal{A})$ with the entries given by

$$V_{RR'} = \prod_{H \in \text{Sep}_{\mathcal{A}}(R,R')} a_H, \tag{1.1}$$

where $\text{Sep}_{\mathcal{A}}(R, R')$ is the set of hyperplanes in $\mathcal{A}$ which separate R and R'. We write $V_q = V_q(\mathcal{A})$ for $V(\mathcal{A})$ when we set each $a_H = q$, an indeterminate, and call V_q the q*-Varchenko matrix* of $\mathcal{A}$.

Thus $(V_q)_{RR'} = q^{\#\text{Sep}(R,R')}$. Also note that $V(\mathcal{A})$ and $V_q(\mathcal{A})$ are symmetric matrices with 1's on the main diagonal.

We are interested mostly in the q-Varchenko matrix V_q. We are going to prove that $V_q(\mathcal{A})$ has an SNF over the ring $\mathbb{Z}[q]$ for the peelable arrangements (in Section 2), arrangement $\mathcal{C}_n$ (in Section 3) and regular n-gon arrangement $\mathcal{G}_n$ (in Section 4). (Since this ring is not a PID, an SNF does not a priori exist.) In Section 5, we compute the SNF of the Varchenko matrices for two arrangements which are not included in the previous sections.

2 Peelable hyperplane arrangements

Example 2.1. Let us look at the arrangement $\mathcal{A}_\text{p}$ in Example 1.1. Its Varchenko matrix $V_q = V_q(\mathcal{A}_\text{p})$ is

$$V_q = \left[\begin{array}{ccc|ccc|c} 1 & q & q^2 & q & q^2 & q^3 & q^2 \\ q & 1 & q & q^2 & q & q^2 & q^3 \\ q^2 & q & 1 & q^3 & q^2 & q & q^2 \\ \hline q & q^2 & q^3 & 1 & q & q^2 & q \\ q^2 & q & q^2 & q & 1 & q & q^2 \\ q^3 & q^2 & q & q^2 & q & 1 & q \\ \hline q^2 & q^3 & q^2 & q & q^2 & q & 1 \end{array}\right],$$

where the columns are indexed by the regions in the order $1', 2', 3', 1, 2, 3, 4$, and so are the rows. We will briefly show that this matrix has an SNF. We write V_q as a block matrix the way it is partitioned:

$$V_q = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}.$$

Notice that $(B_1, C_1) = q(B_2, C_2)$ and $A_2 = qA_1$. (This is not a coincidence. We see that $[B_1, C_1]$ is the submatrix indexed by $1', 2', 3'$ (rows) and $1, 2, 3, 4$ (columns), while $[B_2, C_2]$ is the submatrix indexed by $1, 2, 3$ (rows) and $1, 2, 3, 4$ (columns). There is one more line, line b, to separate regions i' and j' than regions i and j.) We can multiply by the following matrix on the left to cancel B_1:

$$P = \begin{bmatrix} I_3 & -qI_3 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have

$$PV_q = \begin{bmatrix} A_1 - qA_2 & 0 & 0 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}.$$

As V_q is a symmetric matrix, so is PVP^{t}. We thus have

$$PV_qP^{\mathrm{t}} = \begin{bmatrix} A_1 - qA_2 & 0 & 0 \\ 0 & B_2 & C_2 \\ 0 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} (1-q^2)A_1 & 0 \\ 0 & M_1 \end{bmatrix},$$

where we write

$$M_1 = \begin{pmatrix} B_2 & C_2 \\ B_3 & C_3 \end{pmatrix},$$

and we use that $A_2 = qA_1$. The matrix A_1 is the q-Varchenko matrix of $\mathcal{A}_{\mathrm{p}}^b$. (See Example 1.1 for the notation $\mathcal{A}_{\mathrm{p}}^b$). The matrix M_1 is the q-Varchenko matrix of $\mathcal{A}_{\mathrm{p}} - \{b\}$. We can use induction to transform PV_qP^{t} into an SNF.

This example motivates us to define a peelable hyperplane in an arrangement.

Definition 2.2. Let $\mathcal{A}$ be a finite hyperplane arrangement and H be a hyperplane in $\mathcal{A}$. We say that H is *peelable* (from $\mathcal{A}$) if there is one side H_f of H such that if R is a region of $\mathcal{A}$ and R is in H_f, then $\bar{R} \cap H$ is the closure of a region of $\mathcal{A}^H$.

For example, the hyperplane b is peelable from $\mathcal{A}_{\mathrm{p}}$ in Example 1.1. Let us see why this is. On the side above b there are three regions $1'$, $2'$ and $3'$. For each one of these regions, the intersection of its closure with b is actually a closure of a region of $\mathcal{A}_{\mathrm{p}}^b$. For instance, the closure of region $2'$ intersects b at a line section AB, and this line section is actually a closure of a region of $\mathcal{A}_{\mathrm{p}}^b$. (In fact $\mathcal{A}_{\mathrm{p}}^b$ has 3 regions: the part to the left of A, the part between A and B, and the part to the right of B.)

Theorem 2.3. *Assume that H is peelable from $\mathcal{A}$. Then there is a matrix P with entries in $\mathbb{Z}[q]$ such that $\det(P) = 1$ and*

$$PV_q(\mathcal{A})P^t = \begin{bmatrix} (1-q^2)V_q(\mathcal{A}^H) & 0 \\ 0 & V_q(\mathcal{A} - \{H\}) \end{bmatrix}.$$

Remark 2.4. Under the same assumption, a similar result can be given for the Varchenko matrix $V(\mathcal{A})$, and the proof is almost the same. Using this result, we can prove that the Varchenko matrix $V(\mathcal{A})$ associated to a peelable hyperplane arrangement (as defined below) has a "diagonal form" in $\mathbb{Z}[a_H : H \in \mathcal{A}]$, that is, we can find matrices P, Q whose determinants are units and $PV(\mathcal{A})Q$ is a diagonal matrix. Let us mention that, subsequent to our work, Gao and Zhang [5] gave a necessary and sufficient condition on an arrangment $\mathcal{A}$ for $V(\mathcal{A})$ to have a diagonal form.

The main idea of the proof of this theorem is in the previous example. We will give a rigorous proof in a while, in order to make sure there is no gap that might have occurred when we move from the more visualizable two-dimensional example.

Iteratively using this result, the Varchenko matrices of a special type of hyperplane arrangement can be shown to have an SNF.

Definition 2.5. Let $\mathcal{A} = \{H_1, H_2, \ldots, H_m\}$ be a finite hyperplane arrangement. We inductively define $\mathcal{A}$ to be *peelable* as follows.

1. If $m = 1$ then $\mathcal{A} = \{H_1\}$ is peelable.
2. If there is one peelable hyperplane H in $\mathcal{A}$ such that both $\mathcal{A} - \{H\}$ and $\mathcal{A}^H$ are peelable, then we say that $\mathcal{A}$ is peelable.

Now it is easy to see that we have the following result.

Corollary 2.6. *The q-Varchenko matrix $V_q(\mathcal{A})$ of a peelable hyperplane arrangement $\mathcal{A}$ has an SNF over $\mathbb{Z}[q]$. Moreover, its SNF is of the form*

$$\operatorname{diag}((1-q^2)^{n_1}, (1-q^2)^{n_2}, \ldots, (1-q^2)^{n_r}),$$

where $0 \le n_1 \le n_2 \le \cdots \le n_r$ is a sequence of nonnegative integers and r is the number of regions of $\mathcal{A}$.

We will need the following two results, which are not hard to prove.

Lemma 2.7. *Let H be a hyperplane in $\mathcal{A}$. Assume that R is a region such that $\bar{R} \cap H$ contains a point which is an interior point of some region R_1 in $\mathcal{A}_H$. Then $\bar{R}_1 = \bar{R} \cap H$.*

Lemma 2.8. *Let H be a hyperplane in $\mathcal{A}$. Assume that R is a region such that $\bar{R} \cap H$ is the closure of some region of $\mathcal{A}_H$. Then there is a unique region R' on the other side of H such that $\bar{R}' \cap H = \bar{R} \cap H$.*

To simplify the wording of the proof of Theorem 2.3, we introduce a new notation.

Definition 2.9. Let $\mathcal{A}$ be a hyperplane arrangement. Let $\mathcal{R}_1, \mathcal{R}_2$ be two subsets of $\mathcal{R}(\mathcal{A})$. We denote by $V_q(\mathcal{R}_1, \mathcal{R}_2)$ the submatrix of $V_q(\mathcal{A})$ with rows indexed by $\mathcal{R}_1$ and column indexed by $\mathcal{R}_2$.

Now let us prove Theorem 2.3. Assume that H is peelable from $\mathcal{A}$ and H_f is a side of H with the properties as in Definition 2.2. Let $R_1, R_2, \ldots, R_s$ be the set of the regions in H_f. Let H, $\mathcal{A}$, H_f be as in the Definition 2.2. Let $\mathcal{R}_1' = \{1', 2', \ldots, r'\}$ denote the set of regions in H_f, and let $\mathcal{R}_1 = \{1, 2, \ldots, r\}$ denote the corresponding regions on the other side of H_f as given by the previous lemma. Let $\mathcal{R}_2 = \{r+1, \ldots, r+s\}$ be the set of other regions. Let $\mathcal{R}' = \{1, 2, \ldots, r+t\}$, i.e., $\mathcal{R}'$ is the union of $\mathcal{R}_1$ and $\mathcal{R}_2$. It is not difficult to prove the following facts:

$$\begin{aligned} V_q(\mathcal{R}'_1, \mathcal{R}'_1) &= V_q(\mathcal{A}^H) \\ V_q(\mathcal{R}', \mathcal{R}') &= V_q(\mathcal{A} - \{H\}) \\ V_q(\mathcal{R}'_1, \mathcal{R}') &= qV_q(\mathcal{R}_1, \mathcal{R}') \\ V_q(\mathcal{R}'_1, \mathcal{R}'_1) &= qV_q(\mathcal{R}_1, \mathcal{R}'_1). \end{aligned}$$

The q-Varchenko matrix $V = V(\mathcal{A})$ has the following block matrix form:

$$V_q(\mathcal{A}) = \begin{bmatrix} V_q(\mathcal{R}'_1, \mathcal{R}'_1) & V_q(\mathcal{R}'_1, \mathcal{R}_1) & V_q(\mathcal{R}'_1, \mathcal{R}_2) \\ V_q(\mathcal{R}_1, \mathcal{R}'_1) & V_q(\mathcal{R}_1, \mathcal{R}_1) & V_q(\mathcal{R}_1, \mathcal{R}_2) \\ V_q(\mathcal{R}_2, \mathcal{R}'_1) & V_q(\mathcal{R}_2, \mathcal{R}_1) & V_q(\mathcal{R}_2, \mathcal{R}_2) \end{bmatrix}.$$

Now an argument similar to Example 2.1 can be applied to prove the theorem.

3 The case that all lines go through the same point

From now on, we consider hyperplane arrangements in $\mathbb{R}^2$. Define $\mathcal{C}_n$ to be the arrangement consisting of n lines intersecting in a common point in $\mathbb{R}^2$. We prove that the q-Varchenko matrix $V(n)$ associated to $\mathcal{C}_n$ has a Smith normal form (over $\mathbb{Z}[q]$, as usual). This matrix has the form

$$V(n) = \begin{bmatrix} 1 & q & q^2 & q^3 & \cdots & q^n & q^{n-1} & \cdots & q \\ q & 1 & q & q^2 & \cdots & q^{n-1} & q^n & \cdots & q^2 \\ & & & & \vdots & & & \vdots & \\ q & q^2 & q^3 & q^4 & \cdots & q^{n-1} & q^{n-2} & \cdots & 1 \end{bmatrix}.$$

Remark 3.1. This matrix is an example of circulant matrices $C(c_1, c_2, \ldots, c_n)$ which is defined by

$$C(c_1, c_2, \ldots, c_n) = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_n \\ c_n & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_n & c_1 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_n & c_1 \end{bmatrix}. \tag{3.1}$$

We see that $V(n)$ is circulant because the regions of $\mathcal{C}_n$ are in a circular mode. Similar but more complicated situations occur in the regular n-gon arrangement, which is considered in the next section.

Proposition 3.2. *Let n be a positive integer. Then the Varchenko matrix $V(n)$ has the following Smith normal form over $\mathbb{Z}[q]$:*

$$\operatorname{diag}(1, \underbrace{1-q^2, \ldots, 1-q^2}_{n}, (1-q^2)^2, \underbrace{(1-q^2)(1-q^{2n}), \ldots, (1-q^2)(1-q^{2n})}_{n-2}). \tag{3.2}$$

Proof. First successively apply the row operations $r_i - qr_{i-1}$ $(i = n, n-1, \ldots, 2)$, $r_{n+i} - qr_{n+i+1}$ $(i = 1, 2, \ldots, n-1)$, $r_{2n} - qr_1$. This transforms $V(n)$ into the block matrix

$$\begin{bmatrix} 1 & \alpha & q \\ O & M & O \\ 0 & \beta & 1-q^2 \end{bmatrix},$$

where M is a $2(n-1) \times 2(n-1)$ matrix, α, β are row vectors, O is a zero column vector and β's components are all multiples of $1-q^2$. It's easy to see that we only need to find the Smith normal form of M. Factoring $1-q^2$ out of M, one finds that

$$M = (1-q^2) \begin{bmatrix} A & B \\ B^{\mathrm{t}} & A^{\mathrm{t}} \end{bmatrix},$$

where

$$A = \sum_{k=0}^{n-2} q^k T^k, \quad B = \sum_{k=0}^{n-2} q^{n-1-k} (T^{\mathrm{t}})^k$$

and $T = (t_{ij})$ with $t_{i,j} = \delta_{i+1,j}$. Note that A is a unitriangular matrix; in particular, it is invertible in $\mathbb{Z}[q]$. Multiplying M on the left by

$$P = \begin{bmatrix} I & O \\ -B^{\mathrm{t}} A^{-1} & I \end{bmatrix},$$

we transform M into

$$(1-q^2) \begin{bmatrix} A & B \\ O & A^{\mathrm{t}} - B^{\mathrm{t}} A^{-1} B \end{bmatrix}.$$

We see that we only need to find the Smith normal form of $A^{\mathrm{t}} - B^{\mathrm{t}} A^{-1} B$, but it can be seen from the following lemma that its SNF is

$$\operatorname{diag}(1-q^2, \underbrace{1-q^{2n}, \ldots, 1-q^{2n}}_{n-2}). \tag{3.3}$$

Now the SNF of $V(n)$ follows. □

Lemma 3.3. *Let $m \times m$ matrix $T = (t_{ij})$ with $t_{i,j} = \delta_{i+1,j}$. Let*

$$A = \sum_{k=0}^{m-1} q^k T^k, \quad B = \sum_{k=0}^{m-1} q^{m-k} (T^t)^k.$$

Then the matrix

$$C = (I_m - qT^t)(A^t - B^t A^{-1} B)$$

is equal to a matrix with first row

$$(1-q^2, q^3 - q^{2m+1}, q^4 - q^{2m}, q^5 - q^{2m-1}, \ldots, q^{m+1} - q^{m+3}),$$

the other diagonal entries all equal to $1 - q^{2m+2}$, and all other entries zero.

Proof. First $A^{-1} = I_m - qT$, so

$$B^{\mathrm{t}} A^{-1} = q^m I_m + \sum_{k=1}^{m-1} (q^{m-k} - q^{m+2-k}) T^k.$$

Then one computes $B^{\mathrm{t}}A^{-1}B$ and finds it is equal to

$$M = \begin{bmatrix} q^2 & q^3 - q^{2m+1} & q^4 - q^{2m} & \dots & q^{m+1} - q^{m+3} \\ q^3 & q^4 & q^5 - q^{2m+1} & \dots & q^{m+2} - q^{m+4} \\ q^4 & q^5 & q^6 & \dots & q^{m+3} - q^{m+5} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q^m & q^{m+1} & q^{m+2} & \dots & q^{2m-1} - q^{2m+1} \\ q^{m+1} & q^{m+2} & q^{m+3} & \dots & q^{2m} \end{bmatrix}.$$

Now let $N = (I_m - qT^{\mathrm{t}})M$. We find that the first row of N is the same as that of M, the other diagonal entries of N are all equal to q^{2m+2} and all other entries are zero. Now we see that C is as claimed in the lemma since

$$C = (I_m - qT^{\mathrm{t}})A^{\mathrm{t}} - (I_m - qT^{\mathrm{t}})M = I_m - (I_m - qT^{\mathrm{t}})M = I_m - N. \qquad \square$$

4 The case of regular n-gon arrangement $\mathcal{G}_n$

Let $\mathcal{G}_n$ be the arrangement in $\mathbb{R}^2$ obtained by extending the sides of a regular n-gon. Let $V_q(\mathcal{G}_n)$ be the Varchenko matrix associated to $\mathcal{G}_n$. We are going to prove the following

Theorem 4.1. *Let $V_q(\mathcal{G}_n)$ be the Varchenko matrix associated to the regular n-gon arrangement $\mathcal{G}_n$. A Smith normal form of $V_q(\mathcal{G}_n)$ over $\mathbb{Z}[q]$ is*

$$\mathrm{diag}(1, \underbrace{1 - q^2, \dots, 1 - q^2}_{n}, \underbrace{(1 - q^2)^2, \dots, (1 - q^2)^2}_{(p-1)n}), \tag{4.1}$$

where p is the integer part of $(n + 1)/2$.

The above result can be proved by using some results and tools in [3, 12]. But we want to prove it directly. First, it is easy to calculate the number of regions of the arrangement $\mathcal{G}_n$. For instance, one uses the formula that the number of regions is one more than the sum of the number of the lines and the number of intersection points.

Lemma 4.2. *The number of the regions associated to the regular n-gon arrangement P_n is $np + 1$, where p is the integer part of $(n + 1)/2$.*

The main idea of the proof of Theorem 4.1 is to group the regions by their shapes. We then write the Varchenko matrix as a block matrix. The columns of each block are labeled by regions of a same shape and so are the rows of a block. For regions of the same shape, we order them clockwise. The key property of this treatment is that each block is a circulant matrix. Once we write the block matrix down, it will be relatively easy to do cancelations and turn it into an SNF, although it takes some space to write the process down. To show how to write the block matrix, we consider the example of $\mathcal{G}_5$. Then in the proof we write the block matrix for general n and then do the cancelation.

Example 4.3. We mark the regions of $\mathcal{G}_5$ (see Figure 2) as in the following.

They are regions $\Delta_i^{(j)}$ ($i = 1, 2, \dots, 5$; $j = 1, 2, 3$) together with a unmarked central region. Note that we mark the regions according to their shape. Precisely, for each j, the shape of the $\Delta_i^{(j)}$ for $i = 1, 2, \dots, 5$ are the same. Let us call them the regions of type

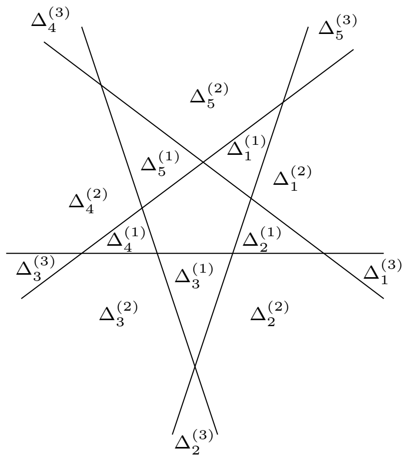


Figure 2: Arrangement $\mathcal{G}_5$.

j. For regions of the same type, we label them clockwise as $\Delta_1^{(j)}, \Delta_2^{(j)}, \ldots, \Delta_5^{(j)}$. We call $\Delta_1^{(j)}$ the leading region of the type j regions. The union of the three leading regions $\Delta_1^{(1)}, \Delta_1^{(2)}, \Delta_1^{(3)}$ is the region inside an exterior angle of the pentagon. (So is the union of three region $\Delta_i^{(1)}, \Delta_i^{(2)}, \Delta_i^{(3)}$.) We obtain the Varchenko matrix

$$V_q(\mathcal{G}_5) = \begin{bmatrix} 1 & Q_1 & Q_2 & Q_3 \\ Q_1^{\mathrm{t}} & E_{11} & E_{12} & E_{13} \\ Q_2^{\mathrm{t}} & E_{21} & E_{22} & E_{23} \\ Q_3^{\mathrm{t}} & E_{31} & E_{32} & E_{33} \end{bmatrix},$$

where the first (block) column is indexed by the central non-marked region. For $j = 2, 3, 4$, the jth block column is indexed by the type j regions. The block rows are indexed in the same way. For example, the rows of the matrix E_{12} are indexed by the type 1 regions and the columns of it are indexed by type 2 regions. Because regions of the same type are ordered in a circular mode, the blocks E_{ij} should all be circulant matrices (see Remark 3.1). In fact, it can be checked that the blocks are as follows

$$\begin{aligned} &Q_k = (q^k, q^k, q^k, q^k, q^k) \quad \text{for } k = 1, 2, 3, \\ &E_{11} = C(1, q^2, q^2, q^2, q^2) \quad E_{12} = C(q, q^3, q^3, q^3, q) \quad E_{13} = C(q^2, q^4, q^4, q^2, q^2), \\ &E_{22} = C(1, q^2, q^4, q^4, q^2) \quad E_{23} = C(q, q^3, q^5, q^3, q) \quad E_{33} = C(1, q^2, q^4, q^4, q^2). \end{aligned}$$

We then use Gaussian elimination (in blocks) to turn the matrix into an SNF. For instance, at the beginning, we subtract the q times of the third block row $(Q_2^{\mathrm{t}}\ E_{21}\ E_{22}\ E_{23})$ from the fourth block row $(Q_3^{\mathrm{t}}\ E_{31}\ E_{32}\ E_{33})$.

Proof. We write the Varchenko matrix of $\mathcal{G}_n$ in the following form of block matrix:

$$V_q(\mathcal{G}_n) = \begin{bmatrix} 1 & Q_1 & Q_2 & \dots & Q_p \\ Q_1^{\mathrm{t}} & E_{11} & E_{12} & \dots & E_{1p} \\ Q_2^{\mathrm{t}} & E_{21} & E_{22} & \dots & E_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_p^{\mathrm{t}} & E_{p1} & E_{p2} & \dots & E_{pp} \end{bmatrix}$$

where $E_{kl} = E_{lk}^{\mathrm{t}}$, Q_k is the row vector

$$Q_k = \underbrace{(q^k, q^k, \dots, q^k)}_{n} \tag{4.2}$$

and E_{ij} $(i \le j)$ is a circulant matrix:

$$E_{ij} = C\Big(q^{j-i}, q^{j-i+2}, q^{j-i+4}, \dots, q^{j-i+2(i-1)}, \underbrace{q^{i+j}, \dots, q^{i+j}}_{n+1-i-j},$$
$$q^{j+i-2}, q^{j+i-4}, \dots, q^{j+i-2(i-1)}, \underbrace{q^{j-i}, \dots, q^{j-i}}_{j-i}\Big).$$

Now we apply Gaussian elimination to $V_q(\mathcal{G}_n)$ and transform it into the desired diagonal form. We do this in blocks and we will use the multiplication of elementary block matrices to realize the elimination. We proceed in four steps.

Step 1: We first apply some row eliminations. Let

$$R_1 = \begin{bmatrix} 1 & & & & \\ -qI_{n\times 1} & I_n & & & \\ & & I_n & & \\ & & & \ddots & \\ & & & & I_n \end{bmatrix} \quad \text{and} \quad R_k = \begin{bmatrix} 1 & & & & & \\ & I_n & & & & \\ & & \ddots & & & \\ & & -qI_n & I_n & & \\ & & & & \ddots & \\ & & & & & I_n \end{bmatrix}$$

for $k \ge 2$, where $I_{n\times 1}$ is a column of n 1's and R_k comes from the (block) identity matrix by adding the $-q$ times of its $(k-1)$th block row to it's kth block row. Now compute the matrix $M_1 = R_1 R_2 \cdots R_p V_q(P_n)$.

Step 2: We apply some column eliminations. Let

$$S_1 = \begin{bmatrix} 1 & -qI_{1\times n} & & & \\ & I_n & & & \\ & & I_n & & \\ & & & \ddots & \\ & & & & I_n \end{bmatrix} \quad \text{and} \quad S_k = \begin{bmatrix} 1 & & & & & \\ & I_n & & & & \\ & & \ddots & & & \\ & & & I_n & -qI_n & \\ & & & & \ddots & \\ & & & & & I_n \end{bmatrix}$$

for $k \ge 2$, where $I_{1\times n}$ is a row of n 1's and S_k comes from the (block) identity matrix by adding the $-q$ times of its $(k-1)$th block column to its kth block column. (So $S_k = T_k^{\mathrm{t}}$.) Now compute the matrix $M_2 = M_1 S_p \cdots S_2 S_1$.

Step 3: We apply some more row eliminations. Let

$$T_k = \begin{bmatrix} {}^{1}\, I_n & & & \\ & \ddots & & \\ & -qJ & I_n & \\ & & & \ddots \\ & & & & I_n \end{bmatrix} \quad \text{with} \quad J = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where T_k come from the (block) identify matrix by adding the $-qJ$ times the kth block row to the $(k+1)$th block row. Now compute $M_3 = T_1 \cdots T_{p-1} M_2$. We find the Varchenko matrix is now transformed to

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & D & N_{12} & N_{13} & \dots & N_{1p-1} & N_{1p} \\ 0 & 0 & D' & N_{23} & \dots & N_{2p-1} & N_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & D' & N_{p-1p} \\ 0 & 0 & 0 & 0 & \dots & 0 & D' \end{bmatrix},$$

where $D = (1-q^2)I_n, D' = (1-q^2)^2 I_n$ and all non-diagonal entries are the multiple of the diagonal entry on the same row. This ensures that we can do the following:

Step 4: We apply more column eliminations to cancel the non-diagonal entries. This does not change the diagonal entries of M_3. We finish the proof as the diagonal of M_3 is the same as that of (4.1). □

5 Two more examples

We now simply say that a hyperplane arrangement $\mathcal{A}$ has SNF if its Varchenko matrix $V_q(\mathcal{A})$ has an SNF over $\mathbb{Z}[q]$. We can use Theorem 2.3 to give more examples of hyperplane arrangements who have SNF. For example, starting from an arrangement which has SNF, for instance C_n, we can keep adding straight lines to it. As long as every time the line added does not separate the set of intersection points of the previous arrangement, the new arrangement will have SNF. This helps us to construct lots of examples of hyperplane arrangements having SNF. We now give two examples which can not be constructed this way. We found that they both have SNF.

1. The *Shi arrangement* $\mathcal{S}_3$ with hyperplanes $x_i - x_j = 0, 1$ for $1 \le i < j \le 3$. We write the multiplicity of a diagonal element in brackets following that entry. For instance, $1 - q^2$ [3] indicates that $1 - q^2$ occurs three times as a diagonal element of the SNF. The diagonal elements of the SNF of $V_q(\mathcal{S}_3)$ are 1 [1], $1-q^2$ [6], $(1-q^2)^2$ [6], and $(1-q^2)(1-q^6)$ [3].
2. Define a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^3$ by the equations $x = 0$, $y = 0$, $z = 0$, $x - y - z = 0$. We verified that its q-Varchenko matrix has an SNF over $\mathbb{Z}[q]$, with diagonal entries 1 [1], $1-q^2$ [4], $(1-q^2)^2$ [6], $(1-q^2)^3$ [2], and $(1-q^2)^2(1-q^8)$ [1].

Based on the previous examples, it is natural to consider the following problem.

Problem 5.1. Do all hyperplane arrangements have SNF?

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