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Z-oriented triangulations of surfaces

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Abstract

The main objects of the paper are z-oriented triangulations of connected closed 2-dimensional surfaces. A z-orientation of a map is a minimal collection of zigzags which double covers the set of edges. We have two possibilities for an edge – zigzags from the z-orientation pass through this edge in different directions (type I) or in the same direction (type II). Then there are two types of faces in a triangulation: the first type is when two edges of the face are of type I and one edge is of type II and the second type is when all edges of the face are of type II. We investigate z-oriented triangulations with all faces of the first type (in the general case, any z-oriented triangulation can be shredded to a z-oriented triangulation of such type). A zigzag is homogeneous if it contains precisely two edges of type I after any edge of type II. We give a topological characterization of the homogeneity of zigzags; in particular, we describe a one-to-one correspondence between z-oriented triangulations with homogeneous zigzags and closed 2-cell embeddings of directed Eulerian graphs in surfaces. At the end, we give an application to one type of the z-monodromy.

Keywords: Directed Eulerian embedding, triangulation of a surface, zigzag, z-monodromy, z-orientation.

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1 Introduction

Petrie polygons are well-known objects described by Coxeter [6] (see also [13]). These are skew polygons in regular polyhedra such that any two consecutive edges, but not three, are on the same face. Analogs of Petrie polygons for graphs embedded in surfaces are called *zigzags* [7, 10] or *closed left-right paths* [9, 18]. These are sequences of oriented edges defined by the rule described above. Zigzags have many applications, for example, they are successfully exploited to enumerate all combinatorial possibilities for fullerenes [3].

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The case when a map, i.e an embedding of a graph in a surface, has a unique zigzag is very important [7, 9]. Following [7] we call such maps z*-knotted*. They have nice homological properties and are closely connected to the Gauss code problem [5, 9, 11].

The study of zigzags in 3-regular plane graphs, in particular fullerenes, is one of the main directions of [7]. A large class of z-knotted 3-regular plane graphs is obtained using a computer. The dual objects, i.e. spherical triangulations, have the same zigzag structure. Zigzags in triangulations of surfaces (not necessarily orientable) are investigated in [15, 16, 17]. By [16], every such triangulation admits a z-knotted shredding, i.e. it can be modified to a z-knotted triangulation of the same surface by triangulating of some of its faces.

A z*-orientation* of a map is a minimal collection of zigzags which double covers the set of edges [7]. In the z-knotted case, this collection contains only one zigzag and is unique up to reversing. For every z-orientation we have the following two types of edges: an edge is of type I if the distinguished zigzags pass through this edge in different directions and an edge is of type II if they pass through the edge in the same direction. It is not difficult to prove that for every face in a triangulation with fixed z-orientation one of the following possibilities is realized: the face contains precisely two edges of type I and the third edge is of type II (the first type of face) or all edges are of type II (the second type of face). We observe that every z-oriented triangulation can be shredded to a triangulation where all faces are of the first type (Section 2). In this paper, we restrict ourself to z-oriented triangulations with all faces of the first type.

Let Γ be such a triangulation of a surface M. Then the number of edges of type I is twice the number of edges of type II and we say that a zigzag is *homogeneous* if it contains precisely two edges of type I after each edge of type II. Denote by Γ_{II} the subgraph of Γ formed by all edges of type II. Our first result (Theorem 3.3) states that the following three conditions are equivalent:

(1) all zigzags of Γ are homogeneous,

(2) Γ_{II} is a closed 2-cell embedding of a simple Eulerian digraph such that every face is a directed cycle,

(3) each connected component of $M \setminus \Gamma_{II}$ is homeomorphic to an open 2-dimensional disk.

Note that directed Eulerian spherical embeddings are known also as *plane alternating dimaps*; they are investigated, for example, in [2, 8, 12]. Directed Eulerian embeddings in arbitrary surfaces are considered in [1, 4].

We will use the following structural property of Γ (without assumption that the zigzags are homogeneous): the connected components of $M \setminus \Gamma_{II}$ are open disks, cylinders or Möbius strips (the third type of components can be realized only for the non-orientable case) and all these possibilities are realized. We show that the existence of cylinders or Möbius strips contradicts the homogeneity of zigzags.

A z-monodromy of face is a permutation which acts on the oriented edges of this face, the z-monodromy of an edge e is the first oriented edge of the face which occurs in certain zigzag after e. By [16], there are precisely 7 types of z-monodromies (M1)–(M7). For each of the types (M3)–(M5) and (M7) there is a triangulation such that each face has the z-monodromy of this type. The types (M1) and (M2) are exceptional: all faces with z-monodromies of each of these types form a forest [17]. The case (M6) cannot be investigated by the methods of [17] and the authors left it as an open problem. It is easy to see that each face with the z-monodromy (M6) is of the first type for every z-orientation. Using this fact, we construct a series of toric triangulations where all faces have z-monodromies of type (M6).

2 Zigzags and z-orientations of triangulations of surfaces

Let M be a connected closed 2-dimensional surface (not necessarily orientable). A *triangulation* of M is a 2-cell embedding of a connected simple finite graph in M such that all faces are triangles [14, Section 3.1]. Then the following assertions are fulfilled: (1) every edge is contained in precisely two distinct faces, (2) the intersection of two distinct faces is an edge or a vertex or empty.

Let Γ be a triangulation of M. A *zigzag* in Γ is a *sequence* of edges $\{e_i\}_{i\in\mathbb{N}}$ satisfying the following conditions for every natural i:

- e_i and e_{i+1} are distinct edges of a certain face (then they have a common vertex, since every face is a triangle),
- the faces containing e_i, e_{i+1} and e_{i+1}, e_{i+2} are distinct and the edges e_i and e_{i+2} are non-intersecting.

Since Γ is finite, there is a natural number $n > 0$ such that $e_{i+n} = e_i$ for every natural i. In what follows, every zigzag will be presented as a cyclic sequence $e_1, \dots, e_n$, where n is the smallest number satisfying the above condition.

Every zigzag is completely determined by any pair of consecutive edges belonging to this zigzag and for any distinct edges e and e' on a face there is a unique zigzag containing the sequence e, e'. If $Z = \{e_1, \dots, e_n\}$ is a zigzag, then the reversed sequence $Z^{-1} = \{e_n, \dots, e_1\}$ also is a zigzag. A zigzag cannot contain a sequence $e, e', \dots, e', e$ which implies that $Z \neq Z^{-1}$ for any zigzag Z, i.e. a zigzag cannot be self-reversed (see, for example, [16]). We say that Γ is *z-knotted* if it contains precisely two zigzags Z and Z^{-1}, in other words, there is a single zigzag up to reversing.

Example 2.1. Zigzags in the Platonic solids (three of them are triangulations of the sphere) are skew polygons without self-intersections and are called *Petrie polygons*.

Example 2.2. Let BP_n be the n-gonal bipyramid, where $1, \dots, n$ denote the consecutive vertices of the base and the remaining two vertices are denoted by a, b (see Fig. 1 for $n = 3$).

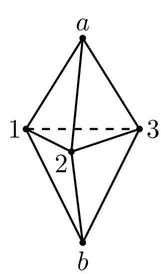

Figure 1:

(a). In the case when $n = 2k+1$, the bipyramid BP_n is z-knotted. If k is odd, then the unique (up to reversing) zigzag is

$$a1, 12, 2b, b3, \dots, a(n-2), (n-2)(n-1), (n-1)b, bn, n1,$$

$$1a, a2, 23, 3b, \dots, a(n-1), (n-1)n, nb,$$

$$b1, 12, 2a, a3, \dots, b(n-2), (n-2)(n-1), (n-1)a, an, n1,$$

$$1b, b2, 23, 3a, \dots, b(n-1), (n-1)n, na.$$

If k is even, then this zigzag is

$$a1, 12, 2b, b3, \dots, b(n-2), (n-2)(n-1), (n-1)a, an, n1,$$

$$1b, b2, 23, 3a, \dots, a(n-1), (n-1)n, nb,$$

$$b1, 12, 2a, a3, \dots, a(n-2), (n-2)(n-1), (n-1)b, bn, n1,$$

$$1a, a2, 23, 3b, \dots, b(n-1), (n-1)n, na.$$

(b). If $n = 2k$ and k is odd, then the bipyramid contains precisely two zigzags (up to reversing):

$$a1, 12, 2b, b3, 34, \dots, a(n-1), (n-1)n, nb,$$

$$b1, 12, 2a, a3, 34, \dots, b(n-1), (n-1)n, na$$

and

$$a2, 23, 3b, b4, 45, \dots, an, n1, 1b,$$

$$b2, 23, 3a, a4, 45, \dots, bn, n1, 1a.$$

(c). In the case when $n = 2k$ and k is even, there are precisely four zigzags (up to reversing):

$$a1, 12, 2b, \dots, b(n-1), (n-1)n, na;$$

$$b1, 12, 2a, \dots, a(n-1), (n-1)n, nb;$$

$$a2, 23, 3b, \dots, bn, n1, 1a;$$

$$b2, 23, 3a, \dots, an, n1, 1b.$$

See [15, 16] for more examples of z-knotted triangulations. Examples of z-knotted fullerenes can be found in [7].

Suppose that Γ contains precisely k distinct zigzags up to reversing. A *z-orientation* of Γ is a collection τ consisting of k distinct zigzags such that for each zigzag Z we have $Z \in \tau$ or $Z^{-1} \in \tau$. There are precisely 2^k distinct z-orientations of Γ. For every z-orientation $\tau = \{Z_1, \dots, Z_k\}$ the z-orientation $\tau^{-1} = \{Z_1^{-1}, \dots, Z_k^{-1}\}$ will be called *reversed* to τ. The triangulation Γ with a z-orientation τ will be denoted by (Γ, τ) and called a *z-oriented triangulation*.

Let τ be a z-orientation of Γ. For every edge e of Γ one of the following possibilities is realized:

- there is a zigzag $Z \in \tau$ such that e occurs in this zigzag twice and other zigzags from τ do not contain e,

- there are two distinct zigzags $Z, Z' \in \tau$ such that e occurs in each of these zigzags only once and other zigzags from τ do not contain e.

In the first case, we say that e is an *edge of type* I if Z passes through e twice in different directions; otherwise, e is said to be an *edge of type* II. Similarly, in the second case: e is an *edge of type* I if Z and Z' pass through e in different directions or e is an *edge of type* II if Z and Z' pass through e in the same direction. In what follows, edges of type II will be considered together with the direction defined by τ. A vertex of Γ is called *of type* I if it belongs only to edges of type I; otherwise, we say that this is a *vertex of type* II.

The following statements hold for any z-orientation τ of Γ.

Lemma 2.3. *For each vertex of type* II *the number of edges of type* II *which enter this vertex is equal to the number of edges of type* II *which leave it.*

Proof. The number of times that the zigzags from τ enter a vertex is equal to the number of times that these zigzags leave this vertex. □

Proposition 2.4. *For every face of* (Γ, τ) *one of the following possibilities is realized:*

(I) *the face contains two edges of type* I *and the third edge is of type* II*, see* Fig. 2(a)*;*

(II) *all edges of the face are of type* II *and form a directed cycle, see* Fig. 2(b).

A face in a triangulation is said to be *of type* I or *of type* II if the corresponding possibility is realized.

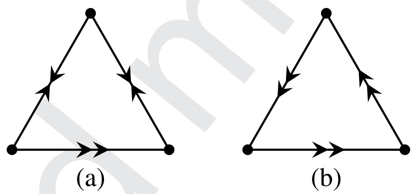

Figure 2:

Proof of Proposition 2.4. Consider a face whose edges are denoted by e_1, e_2, e_3. Without loss of generality we can assume that the zigzag containing the sequence e_1, e_2 belongs to τ. Let Z and Z' be the zigzags containing the sequences e_2, e_3 and e_3, e_1, respectively. Then $Z \in \tau$ or $Z^{-1} \in \tau$ and $Z' \in \tau$ or $Z'^{-1} \in \tau$. An easy verification shows that for each of these four cases we obtain (I) or (II). □

Example 2.5. If n is odd, then the bipyramid BP_n has a unique z-orientation (up to reversing), see Example 2.2(a). The edges ai and bi, $i \in \{1, \dots, n\}$ are of type I and the edges on the base of the bipyramid are of type II. The vertices a, b are of type I and the vertices on the base are of type II. All faces are of type I. The same happens for the case when $n = 2k$ and k is odd if the z-orientation is defined by the two zigzags presented in Example 2.2(b); however, all faces are of type II if we replace one of these zigzags by the reversed.

Example 2.6. Suppose that $n = 2k$ and k is even. Let Z_1, Z_2, Z_3, Z_4 be the zigzags from Example 2.2(c). For the z-orientation defined by these zigzags all faces are of type I. If the z-orientation is defined by Z_1, Z_2 and Z_3^{-1}, Z_4^{-1}, then all faces are of type II. In the case when the z-orientation is defined by Z_1, Z_2, Z_3 and Z_4^{-1}, there exist faces of both types.

Remark 2.7. If we replace a z-orientation by the reversed z-orientation, then the type of every edge does not change (but all edges of type II reverse the directions), consequently, the types of vertices and faces also do not change. For z-knotted triangulations there is a unique z-orientation (up to reversing) and we can determine the types of edges, vertices and faces without attaching to a z-orientation [15].

A triangulation Γ' of M is a *shredding* of the triangulation Γ if it is obtained from Γ by triangulating some faces of Γ such that all new vertices are contained in the interiors of these faces.

Proposition 2.8. *Any z-oriented triangulation admits a z-oriented shredding with all faces of type I.*

Proof. Let F be a face of type II in a z-oriented triangulation (Γ, τ) and let e_1, e_2, e_3 be edges of F. Suppose that the edges of F are oriented as in Fig. 3 and denote by σ the permutation $(1, 2, 3)$.

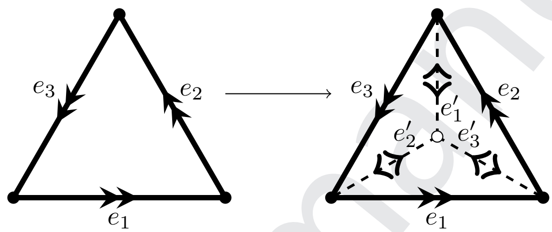

Figure 3:

Zigzags from τ passes through F precisely three times, so the face F separates them into 3 segments of type

$$e_{\sigma^{-1}(i)}, e_i, X_{ij}, e_j, e_{\sigma(j)},$$

where $i, j \in \{1, 2, 3\}$ and the sequence X_{ij} is a maximal part of a zigzag formed by edges occuring between e_i and e_j. Let $\mathcal{X}$ be the set of all such sequences X_{ij} for F and the z-orientation τ. Note that every $X_{ij} \in \mathcal{X}$ is completely determined by the beginning edge e_i and the final edge e_j. Now, we triangulate the face F by adding a vertex in the interior of F and three edges connecting this vertex with the vertices of F. We denote this new triangulation by Γ' and write e_i' for the new edge if it does not has a common vertex with e_i (see Fig. 3). Observe that for any $i \in \{1, 2, 3\}$ there exists a zigzag in Γ' containing a subsequence of the form

$$e_i, e'_{\sigma^{-1}(i)}, e'_i, e_{\sigma^{-1}(i)}, X_{\sigma^{-1}(i)j}$$

for certain $j \in \{1, 2, 3\}$ and $X_{\sigma^{-1}(i)j} \in \mathcal{X}$. The edge e_j which occurs in the zigzag directly after this subsequence is the same as the edge after $X_{\sigma^{-1}(i)j}$ in (Γ, τ), since $X_{\sigma^{-1}(i)j}$ does not contain edges of F. Therefore, zigzags of Γ' related to the three faces not contained in Γ pass through the edges coming from Γ in the same way as zigzags from τ. This implies the existence of a z-orientation of Γ' such that all edges from Γ do not change their types and the three new faces of Γ' contained in F are of type I. Recursively, we eliminate all faces of type II from (Γ, τ) and come to a z-oriented shredding of Γ with all faces of type I and such that the type of any edge from (Γ, τ) is preserved. $\square$

3 Homogeneous zigzags in triangulations with faces of type I

In this section, we will always suppose that Γ is a triangulation with fixed z-orientation τ such that all faces in Γ are of type I, i.e. each face contains precisely two edges of type I and the third edge is of type II. If m is the number of faces, then there are precisely m edges of type I and $m/2$ edges of type II. In other words, the number of edges of type I is twice the number of edges of type II. We say that a zigzag of Γ is *homogeneous* if it is a cyclic sequence $\{e_i, e'_i, e''_i\}_{i=1}^{n}$, where each e_i is an edge of type II and all e'_i, e''_i are edges of type I. If a zigzag is homogeneous, then the reversed zigzag also is homogeneous. Denote by Γ_{II} the subgraph of Γ formed by all vertices and by all edges of type II.

Example 3.1. The zigzags of $\Gamma = BP_n$ are homogeneous if n is odd (the z-knotted case) or n is even and the z-orientation is defined by the two zigzags from Example 2.2(b) or by the four zigzags from Example 2.2(c). Only a and b are vertices of type I and Γ_{II} is the directed cycle formed by the edges of the base of the bipyramid. Conversely, if all zigzags of Γ are homogeneous and there are precisely two vertices of type I, then Γ is a bipyramid (this statement is an easy consequence of Theorem 3.3 which will be presented later).

Example 3.2. Let Γ' be a triangulation of M with a z-orientation such that all faces are of type II (see [17, Example 4] for a z-knotted triangulation of $\mathbb{S}^2$ whose faces are of type II). As in the proof of Proposition 2.8, we consider the shredding Γ'' of Γ' which is obtained by adding a vertex in the interior of each face and three edges connecting this vertex with the vertices of the face. This triangulation Γ'' admits a z-orientation such that all faces are of type I. Every zigzag $e_1, e_2, e_3, \dots$ in Γ' is extended to a zigzag

$$e_1, e'_1, e''_1, e_2, e'_2, e''_2, e_3, \dots$$

in Γ'' which passes through edges of Γ' in the opposite directions. All e_i are of type II and all e'_i and e''_i are of type I. So, all zigzags in Γ'' are homogeneous.

An *Eulerian digraph* is a connected digraph such that indegree equals outdegree for every vertex.

Theorem 3.3. *The following three conditions are equivalent:*

(1) *All zigzags of Γ are homogeneous.*

(2) *Γ_{II} is a closed 2-cell embedding of a simple Eulerian digraph such that every face is a directed cycle.*

(3) *Each connected component of $M \setminus \Gamma_{II}$ is homeomorphic to an open 2-dimensional disk.*

The implication (2) $\Rightarrow$ (3) is obvious. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) will be proved in Section 4 and Section 5, respectively.

4 Proof of the implication (1) $\Rightarrow$ (2) in Theorem 3.3

Now, we generalize the construction described in Proposition 2.8 and Example 3.2. Let Γ' be a closed 2-cell embedding of a connected finite simple graph in the surface M. Then all faces of Γ' are homeomorphic to a closed 2-dimensional disk. For each face F we take a point v_F belonging to the interior of F. We add all v_F to the vertex set of Γ' and connect each v_F with every vertex of F by an edge. We obtain a triangulation of M which will be denoted by $\mathrm{T}(\Gamma')$.

The assumption that our 2-cell embedding is closed cannot be omitted. Indeed, if a certain face of Γ' is not homeomorphic to a closed 2-dimensional disk, then there is a pair of vertices connected by a double edge and $\mathrm{T}(\Gamma')$ is not a triangulation in our sense.

Proposition 4.1. *If all zigzags of Γ are homogeneous, then Γ_{II} is a closed 2-cell embedding of a simple Eulerian digraph such that every face is a directed cycle and $\Gamma = \mathrm{T}(\Gamma_{II})$. Conversely, if Γ' is a closed 2-cell embedding of a simple Eulerian digraph and every face is a directed cycle, then the triangulation $\mathrm{T}(\Gamma')$ admits a unique z-orientation such that all zigzags of $\mathrm{T}(\Gamma')$ are homogeneous and Γ' is the subgraph of $\mathrm{T}(\Gamma')$ formed by all vertices and edges of type II.*

Proof. (I). Let v be a vertex of Γ. Consider all faces containing v and take the edge on each of these faces which does not contain v. All such edges form a cycle which will be denoted by $C(v)$.

Suppose that all zigzags of Γ are homogeneous and consider any edge e_1 of type II. Let v_1 and v_2 be the vertices of this edge such that e_1 is directed from v_1 to v_2. We choose one of the two faces containing e_1 and take in this face the vertex v which does not belong to e_1. Let e_1' and e_1'' be the edges which contain v and occur in a certain zigzag $Z \in \tau$ immediately after e_1, see Fig. 4. Denote by e_2 the third edge of the face containing e_1' and e_1''. This edge contains v_2 and another one vertex, say v_3. Since Z is homogeneous, the edges e_1' and e_1'' are of type I, and consequently, e_2 is of type II. The zigzag which goes through e_1' from v to v_2 belongs to τ (this follows easily from the fact that Z goes through e_1' in the opposite direction and e_1' is an edge of type I). The latter guarantees that the edge e_2 is directed from v_2 to v_3. By our assumption, the edge e_3 which occurs in Z immediately after e_1' and e_1'' is of type II. This edge is directed from v_3 to a certain vertex v_4. So, e_1, e_2, e_3 are consecutive edges of the cycle $C(v)$ and each e_i is directed from v_i to v_{i+1}. Consider the zigzag from τ which contains the sequence e_2, e_1''. The next edge in this zigzag connects v and v_4 (the zigzag goes from v to v_4). Let e_4 be the edge which occurs in the zigzag after it. Then e_4 is an edge of type II (by our assumption), it belongs to $C(v)$ and leaves v_4. Recursively, we establish that $C(v)$ is a directed cycle formed by edges of type II and every edge containing v is of type I, i.e. v is a vertex of type I. Now, we consider the other face containing e_1 and take the vertex v' of this face which does not belong to e_1. Using the same arguments, we establish that v' is a vertex of type I and $C(v')$ is a directed cycle formed by edges of type II.

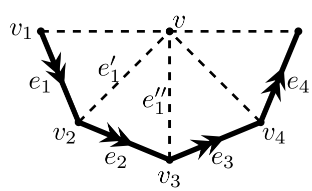

Figure 4:

For every vertex v of type I we can take a face containing v and the edge of this face which does not contain v. This edge is of type II (since the remaining two edges of the face are of type I). The above arguments show that the following assertions are fulfilled:

(1) vertices of type I exist and for every such vertex v the cycle $C(v)$ is a directed cycle formed by edges of type II;

(2) for every edge of type II there are precisely two vertices v and v' of type I such that this edge is contained in the cycles $C(v)$ and $C(v')$.

Similarly, for every edge e of type I we take a face containing e; this face contains an edge of type II which implies that e connects vertices of different types.

Consider Γ_{II}. Any two vertices of type II in Γ can be connected by a path formed by edges of type II which means that Γ_{II} is connected. Indeed, if a path between two vertices of type II goes through a vertex v of type I, then the edge going into v and the edge leaving v are incident to vertices in the same cycle $C(v)$ and so we can rewrite that part of the path to use edges from $C(v)$ instead of the edges through v. It is easy to see that Γ_{II} is a 2-cell embedding of a simple digraph such that every face is the directed cycle $C(v)$ for a certain vertex v of type I; in particular, this 2-cell embedding is closed. Lemma 2.3 implies that Γ_{II} is an Eulerian digraph. The equality $\Gamma = \mathrm{T}(\Gamma_{II})$ is obvious.

The following remark will be used to prove the second part of the theorem. The conditions (1) and (2) guarantee that every zigzag of Γ containing an edge of type II is homogeneous. Recall that the number of edges of type I is twice the number of edges of type II. This implies that there is no zigzag containing edges of type I only (since every edge occurs twice in a unique zigzag from τ or it occurs ones in precisely two distinct zigzags from τ). Therefore, every zigzag of Γ is homogeneous if (1) and (2) hold.

(II). Suppose that Γ' is a closed 2-cell embedding of a simple Eulerian digraph such that every face is a directed cycle.

Let $e_1, \dots, e_n$ be the directed cycle formed by all edges of a certain face of Γ'. For every $i \in \{1, \dots, n\}$ we define $j(i) = i + 2 (\mathrm{mod}\, n)$ and denote by e'_i and e''_i the edges containing the vertex v_F in $\mathrm{T}(\Gamma')$ and intersecting e_i and $e_{j(i)}$, respectively. Consider the zigzag of $\mathrm{T}(\Gamma')$ which contains the sequence $e_i, e'_i, e''_i, e_{j(i)}$. It passes through e_i and $e_{j(i)}$ according to the directions of these edges; and the same holds for every edge of Γ' which occurs in this zigzag. Such a zigzag exists for any pair formed by a face of Γ' and an edge on this face. The collection of all such zigzags is a z-orientation of $\mathrm{T}(\Gamma')$ with the following properties: all edges of Γ' are of type II and every v_F is a vertex of type I. This implies that $\mathrm{T}(\Gamma')$ satisfies the conditions (1) and (2) which gives the claim. □

Note that the second part of Proposition 4.1 will be used to prove the implication (3) ⇒ (1).

5 Structure of triangulations with faces of type I

In this section, we describe some structural properties of z-oriented triangulations with faces of type I. As an immediate consequence we obtain the implication (3) ⇒ (1).

As above, we suppose that (Γ, τ) is a z-oriented triangulation of M, where all faces are of type I. As above, we denote by Γ_{II} the subgraph of Γ consisting of all vertices and all edges of type II. From the previous section it follows that if the zigzags of (Γ, τ) are homogeneous, then connected components of $M \setminus \Gamma_{II}$ are homeomorphic to open 2-dimensional disks. Now, we describe the general case.

Theorem 5.1. *The following assertions are fullfiled:*

(1) *Connected components of $M \setminus \Gamma_{II}$ are homeomorphic to an open 2-dimensional disk, an open Möbius strip or an open cylinder.*

(2) *A connected component of $M \setminus \Gamma_{II}$ contains a vertex of type I if and only if it is an open 2-dimensional disk; such a vertex of type I is unique.*

Proof. Consider two distinct edges e_0 and e_1 of type I contained in a certain face F_1. There is precisely one face containing e_1 and distinct from F_1. Denote this face by F_2 and write e_2 for the other edge of type I on F_2. Recursively, we construct sequences of edges $\{e_i\}_{i\in\mathbb{N}\cup\{0\}}$ and faces $\{F_i\}_{i\in\mathbb{N}}$ such that e_{i-1} is the common edge of F_{i-1}, F_i for every $i \in \mathbb{N}$. For any pair of the faces F_{i-1}, F_i we distinguish the following two cases presented in Fig. 5. In the first case, the edges of type II of F_{i-1} and F_i have a common vertex (Fig. 5(a)). In the second case (Fig. 5(b)), the edges of type II are disjoint.

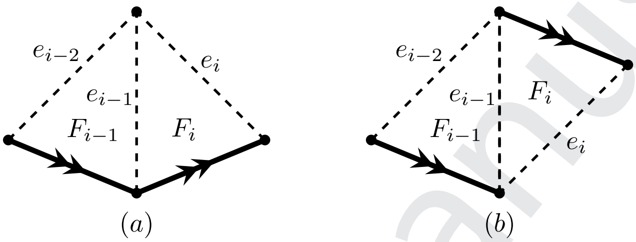

Figure 5:

Let n be the smallest natural number such that $e_n = e_0$ (such a number exists by finiteness). Therefore, the above sequences can be considered as cyclic sequences $\{e_i\}_{i=1}^n$ and $\{F_i\}_{i=1}^n$. The union $\mathcal{F} = \bigcup_{i=1}^{n} F_i$ will be called a *component* of (Γ, τ). The boundary of $\mathcal{F}$ consists of (not necessarily all) edges of type II belonging to faces F_i.

Denote by e_i^{II} the edge of type II belonging to F_i. We take n disjoint closed triangles $T_1, T_2, \ldots, T_n$. For any $i = 1, 2, \ldots, n$ there is a homeomorphism $h_i : F_i \to T_i$ transferring any vertex and any edge of F_i to a vertex and an edge of T_i, respectively. We identify $h_i(e_i)$ and $h_{i+1}(e_i)$ for any i in such a way that for every vertex v of e_i the vertices $h_i(v)$ and $h_{i+1}(v)$ are identified. We get a 2-dimensional surface $\mathcal{T}$ with boundary. The boundary of $\mathcal{T}$ is the union of the images of all edges of type II, i.e. $\partial\mathcal{T} = \bigcup_{i=1}^{n} h_i(e_i^{II})$. Note that $\mathcal{F}$ is not necessarily a surface (since it is possible that for distinct i, j the edges e_i^{II}, e_j^{II} have a common vertex). The interior of surface $\mathcal{T}$ is homeomorphic to one of the connected components of $M \setminus \Gamma_{II}$ and $\mathcal{F}$ can be obtained from $\mathcal{T}$ by gluing of some parts of the boundary.

Suppose that $h_i(e_i)$ and $h_{i+1}(e_i)$ are identified only for $i = 1, 2, \ldots, n-1$ (but not $h_1(e_0)$ and $h_n(e_n)$ from T_1 and T_n, respectively). Then we get a space homeomorphic to a closed 2-dimensional disk whose boundary contains $h_1(e_0), h_n(e_n)$. Now, to complete the construction of $\mathcal{T}$, we have to glue $h_1(e_0)$ and $h_n(e_n)$. Precisely one of the following possibilities is realized:

- A union of these sides is connected and by gluing of them we obtain that $\mathcal{T}$ is homeomorphic to a closed 2-dimensional disk (Fig. 6(1)).
- The sides are disjoint and by identification of them we get a surface homeomorphic to a closed Möbius strip (Fig. 6(2)) or a closed cylinder (Fig. 6(3)).

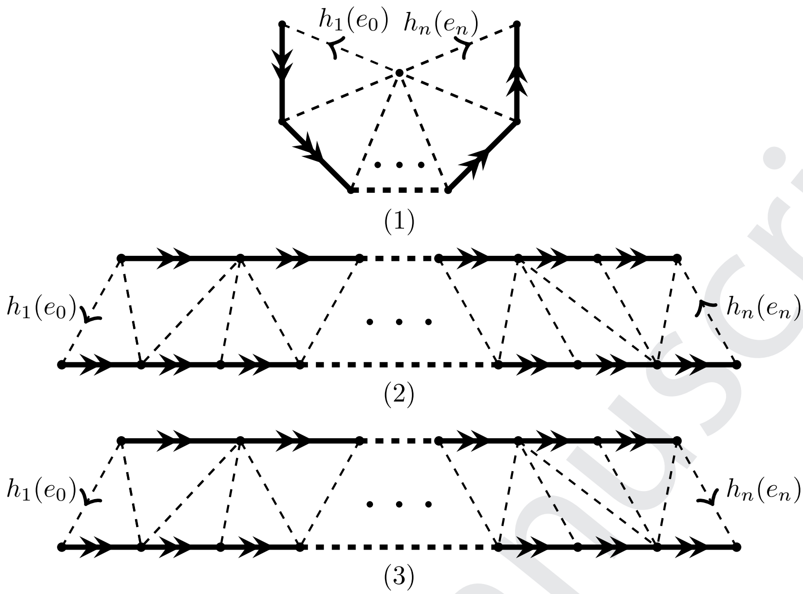

Figure 6:

Let v_i be the vertex of T_i corresponding to the vertex of F_i not belonging to the edge e_i^{II}. In the first case, the images of edges of type I have the common vertex which is the image of all $h_i(v_i)$; it is clear that this vertex corresponds to the vertex of type I from $\mathcal{F}$, see Fig. 6(1). In the remaining cases, any vertex $h_i(v_i)$ is contained in the boundary of $\mathcal{T}$ and correspond to a certain vertex of Γ_{II} (see Fig. 6(2) and 6(3)). So, we obtained the statements (1) and (2). □

If a connected component of $M \setminus \Gamma_{II}$ is homeomorphic to an open 2-dimensional disk, then the corresponding component of (Γ, τ) is homeomorphic to a closed 2-dimensional disk (if this component has some identifications at the boundary, then the vertex of type I in this component is joined by a double edge to a certain vertex at the boundary which is impossible, since we work with embeddings of simple graphs).

Proof of (3)⇒(1) in Theorem 3.3. Assume that each connected component of $M \setminus \Gamma_{II}$ is a disk. By the above remark, Γ_{II} is a closed 2-cell embedding. Lemma 2.3 shows that this is an embedding of simple Eulerian digraph. The second part of Theorem 5.1 states that each disk contains a unique vertex of type I; as in the proof of Theorem 5.1 we establish that its boundary is an oriented cycle. We have $\Gamma = \mathrm{T}(\Gamma_{II})$ and the second part of Proposition 4.1 gives the claim. □

The following three examples show that all possibilities for connected components of $M \setminus \Gamma_{II}$ are realized.

Example 5.2. Consider the following triangulation Γ of a torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ (see Fig. 7). The triangulation Γ admits the z-orientation such that all faces are of type I. The subgraph Γ_{II} has two connected components which are 6-cycles and $\mathbb{T} \setminus \Gamma_{II}$ consists of two connected components homeomorphic to open cylinders $(-1, 1) \times \mathbb{S}^1$.

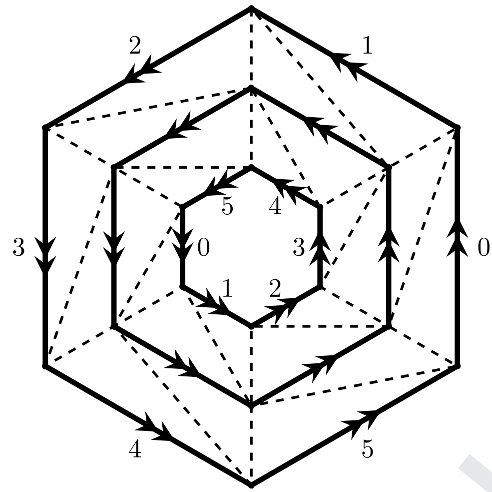

Figure 7:

In a similar way, we can construct a z-oriented toric triangulation with connected components of $\mathbb{T} \setminus \Gamma_{II}$ which are open cylinders of arbitrary length.

Example 5.3. Let $n \in \mathbb{N}$ and let Γ be the triangulation of a real projective plane obtained by gluing of boundaries of a Möbius strip and a closed 2-dimensional disk (see Fig. 8). According to the corresponding z-orientation all faces are of type I and the graph Γ_{II} consists of all edges marked by the double arrows and their vertices. Then $\mathbb{R}\mathrm{P}^2 \setminus \Gamma_{II}$ has two connected components. One of them is homeomorphic to an open 2-dimensional disk and the remaining to an open Möbius strip.

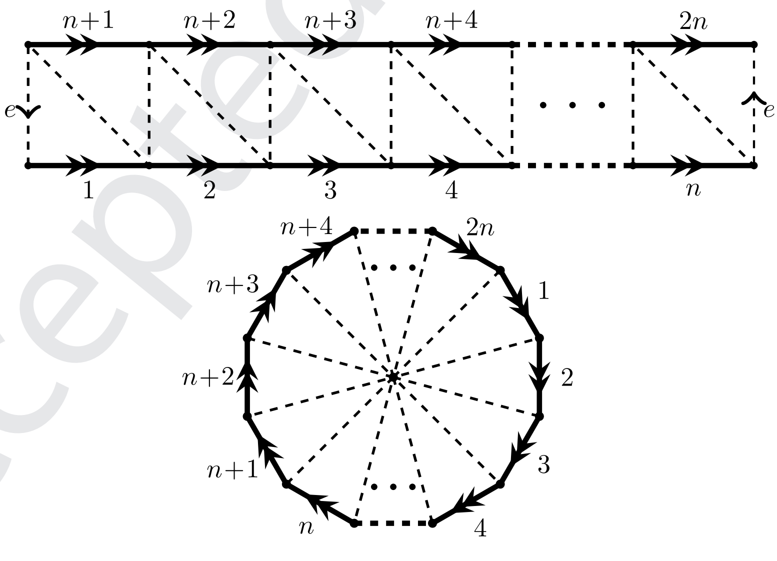

Figure 8:

Example 5.4. Suppose that Γ is the triangulation of a sphere obtained by the gluing of the two disks whose boundaries are cycles $e_1, e_2, \dots, e_6$ (see Fig. 9). There is a z-orientation τ such that all faces are of type I. Then $\mathbb{S}^2 \setminus \Gamma_{II}$ has precisely four connected components: three components are homeomorphic to an open 2-dimensional disk and the remaining to an open cylinder. The components of (Γ, τ) corresponding to the first three connected components are closed 2-dimensional disks. The fourth component of (Γ, τ) is homeomorphic to a closed cylinder $\mathbb{S}^1 \times D^1$, where two points at one of the connected components of the boundary are glued.

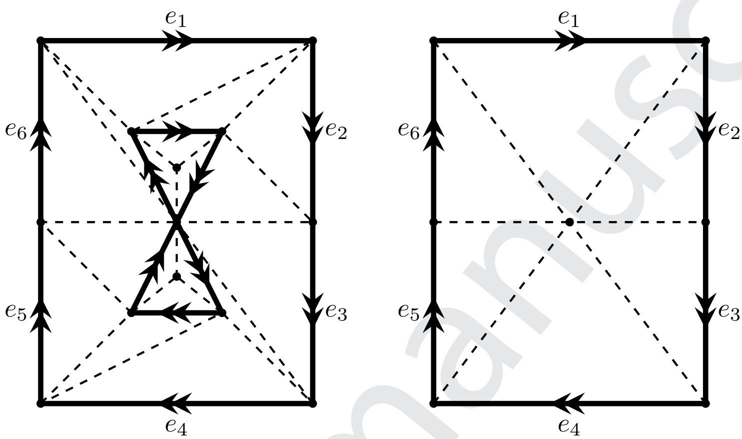

Figure 9:

6 Relations to z-monodromies

Let F be a face in Γ whose vertices are a, b, c. Consider the set $\Omega(F)$ of all oriented edges of F

$$\Omega(F) = \{ab, bc, ca, ac, cb, ba\},$$

where xy is the edge from $x \in \{a, b, c\}$ to $y \in \{a, b, c\}$. If $e = xy$ then we write yx by $-e$. Denote by D_F the following permutation of the set $\Omega(F)$

$$(ab, bc, ca)(ac, cb, ba).$$

The *z-monodromy* (see [16, 17]) of the face F is the permutation M_F defined as follows. For any $e \in \Omega(F)$ we take $e_0 \in \Omega(F)$ such that $D_F(e_0) = e$ and consider the zigzag containing the sequence e_0, e. We define $M_F(e)$ as the first element of $\Omega(F)$ contained in this zigzag after e.

By [16, Theorem 4.4], there are the following possibilities for the z-monodromy M_F and each of them is realized:

(M1) M_F is the identity,

(M2) $M_F = D_F$,

(M3) $M_F = (-e_1, e_2, e_3)(-e_3, -e_2, e_1)$,

(M4) $M_F = (e_1, -e_2)(e_2, -e_1)$, where e_3 and $-e_3$ are fixed points, (M5) $M_F = (D_F)^{-1}$,

(M6) $M_F = (-e_1, e_3, e_2)(-e_2, -e_3, e_1)$,

(M7) $M_F = (e_1, e_2)(-e_1, -e_2)$, where e_3 and $-e_3$ are fixed points

where (e_1, e_2, e_3) is one of the cycles in D_F.

Let G_i be the subgraph of the dual Γ^* formed by vertices corresponding to faces in Γ whose z-monodromies are of type (Mi), two vertices of G_i are adjacent if they are adjacent in Γ^*. By [17, Theorem 1], the subgraphs G_1 and G_2 are forests. For (M3), (M4), (M5) and (M7) the above statement fails: z-monodromies of all faces of the bipyramid BP_n are of type

- (M3) for $n = 2k + 1$ where k is odd,
- (M4) for $n = 2k + 1$ where k is even,
- (M7) for $n = 2k$ where k is odd,
- (M5) for $n = 2k$ where k is even.

Proposition 6.1. *If M_F is* (M6)*, then F is of type I for any z-orientation of* Γ.

Proof. Let e_1, e_2, e_3 be consecutive oriented edges of the face F. We suppose that the z-monodromy of F is (M6), i.e.

$$M_F = (-e_1, e_3, e_2)(-e_2, -e_3, e_1).$$

There are precisely two zigzags (up to reversing) which contain F

$$e_1, e_2, \dots, -e_1, -e_3, \dots \text{ and } e_2, e_3, \dots ;$$

since the edge corresponding to the pair $\{e_1, -e_1\}$ is passed in two different directions by the same zigzag, then it is of type I for any orientation of the zigzag. Therefore, F is of type I for any z-orientation. □

Lemma 6.2. *Let F be a face in (Γ, τ) such that there are precisely two zigzags from τ which contain edges from F. Then the following assertions are fulfilled:*

(1) There is a unique edge $e \in F$ which occurs in one of these zigzags twice,

(2) The type of e does not depend on the choice of z-orientation,

(3) If e is of type I, then M_F is (M6)*. If e is of type II, then M_F is* (M7).

Proof. (1). Any face occurs precisely thrice, as a pair of its adjacent edges, in zigzags from the z-orientation τ. By the assumption, there are precisely two zigzags from τ which pass through our face. This is possible only when one of these zigzags passes through it once and the second twice.

(2). The edge e can occur in the same zigzag twice in two ways: the zigzag passes through e the first time in one of directions and the second time in the opposite (type I) or the zigzag passes through e twice in the same direction (type II). It is easy to see that the type of e is the same for any z-orientation of Γ.

(3). By [16, Remark 4.9] the z-monodromy of the face F is (M6) or (M7). In the case (M6) the statement follows from Proposition 6.1. Let e_1, e_2, e_3 be consecutive edges of F and M_F be of type (M7), i.e.

$$M_F = (e_1, e_2)(-e_1, -e_2).$$

In this case, F occurs twice in the zigzag

$$e_2, e_3, \ldots, e_3, e_1, \ldots$$

and e_3 is of type II for any z-orientation of Γ. □

Now, we can construct a class of toric triangulations, where z-monodromies are of type (M6) for all faces. Our arguments are based on Lemma 6.2.

Example 6.3. Let n, m be odd numbers not less than 3 and let Γ_0 be a $n \times m$ grid where the opposite sides are identified. Then Γ_0 can be embedded into a torus in the natural way. Suppose that $\Gamma = \mathrm{T}(\Gamma_0)$ (see Fig. 10 for the case $n = m = 3$).

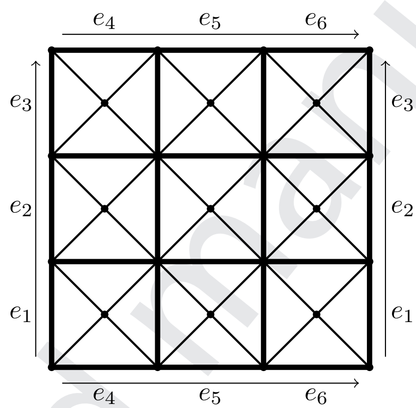

Figure 10:

Each zigzag of Γ determines a band formed by n or m squares from the grid (see Fig. 11 for a band consisting of 5 squares) and passes through each face of this band twice.

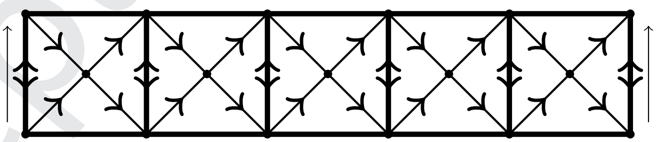

Figure 11:

Observe that the edges common for two consecutive squares from the grid are passed twice (they marked on Fig. 11 by the bold line) and are of type I for any z-orientation. Remaining edges are passed by the zigzag once. Therefore, all edges of subgraph Γ_0 are of type I and all faces of Γ are of type I for any z-orientation. It is clear that any edge incident to a vertex in the interior of a square occurs once in two different zigzags. Thus, for any face of Γ there are precisely two zigzags which pass it. Lemma 6.2 guarantees that z-monodromies of all faces of Γ are (M6).

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