ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA
https://doi.org/10.26493/1855-3974.2229.5f1
(Also available at http://amc-journal.eu)

Boundary-type sets of strong product of directed graphs*

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Received 26 January 2020, accepted 8 March 2021

Abstract

Let $D = (V, E)$ be a strongly connected digraph and let u and v be two vertices in D. The maximum distance $md(u, v)$ is defined as $md(u, v) = \max\{\overrightarrow{d}(u, v), \overrightarrow{d}(v, u)\}$, where $\overrightarrow{d}(u, v)$ denotes the length of a shortest directed $u - v$ path in D. This is a metric. The boundary, contour, eccentricity and periphery sets of a strongly connected digraph D with respect to this metric have been defined. The boundary-type sets of the strong product of two digraphs is investigated in this article.

Keywords: Maximum distance, boundary-type sets, strongly connected digraph, strong product.

Math. Subj. Class. (2020): 05C12, 05C20, 05C76

*Supported by Science and Engineering Research Board, a statutory body of Government of India under their Extra Mural Research Funding No. EMR/2015/002183. This research was partially supported by Kerala State Council for Science Technology and Environment of Government of Kerala under their SARD project grant Council(P) No. 436/2014/KSCSTE.

†Supported by Science and Engineering Research Board, a statutory body of Government of India under their MATRICS Scheme No. MTR/2018/000012.

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1 Introduction

Directed graphs or in short digraphs have immense applications in almost all areas of science and even in sociology. A directed network is a network in which each edge has a direction, pointing from one vertex to another. They can be represented as directed graphs.

Road traffic networks are the most frequently met examples of one-way networks. A two-way street is one in which vehicles are allowed to travel in both directions. The advantages of a one-way street network over a two-way street pattern are discussed in [14]. But when one-way traffic is introduced in a two-way network, the distance between places in one of the directions may increase. So the problem of designing a network is to minimize the distance between places and the cost of construction.

The one-way problem was first studied by Robbins [13]. It finds applications in various fields like computer science, biology, etc. In [2], directed graphs are used to analyze the local properties of internet connectivity. Neurons are connected in intricate communication networks established during development to convey sensory information from peripheral receptors of sensory neurons to the central nervous system and to convey commands from the central nervous system to effector organs [12].

The boundary-type sets of a graph, the *boundary, contour, eccentricity*, and *periphery sets* of a graph were studied in [5] and [7]. It is very difficult to identify the various boundary-type sets in large networks. So we try to decompose the network into smaller networks and identify the boundary-type sets. The four standard graph products, namely Cartesian, direct, strong, and lexicographic products can be extended to digraphs as well. Marc Hellmuth and Tilen Marc developed a polynomial-time algorithm for determining the prime factor decomposition of strong product of digraphs [11].

The directed distance defined in digraphs is generally not a metric. As we are concerned with the problem of designing the network to minimize the distance between places at a minimum cost, we consider the distance *maximum distance* or in short, *m-distance* which is a metric that was introduced by Chartrand and Tian in [8]. It gives the maximum of the directed distances in either direction and is denoted by $md(u, v)$. So minimizing $md(u, v)$ results in minimizing the distance between the nodes in both directions. An undirected graph G can be identified as a symmetric digraph, that is, one for which $(u, v) \in E(G)$ if and only if $(v, u) \in E(G)$, and the metric *md* is the usual distance metric in undirected graphs.

The boundary-type sets of the Cartesian product of two digraphs were studied in [6]. In this paper, a similar study is conducted for the strong product of digraphs.

2 Preliminaries

A *directed graph* or a *digraph* D consists of a non-empty finite set $V(D)$ of elements called vertices and a finite set $E(D)$ of ordered pairs of distinct vertices called arcs or edges [1]. We call $V(D)$ the vertex set and $E(D)$ the edge set of D. We write $D = (V, E)$ to denote the digraph D with vertex set V and edge set E. For an edge (u, v), the first vertex u of the ordered pair is the tail of the edge and the second vertex v is the head; together they are the endpoints. This definition of a digraph does not allow loops (edges whose head and tail coincide) or parallel edges (pairs of edges with the same tail and the same head). The underlying graph UD of a digraph D is the simple graph with the vertex set $V(D)$ and the unordered pair $(x, y) \in E(UD)$ if and only if either $(x, y) \in E(D)$ or $(y, x) \in E(D)$.

The following concepts are taken from [1].

For a vertex v in a digraph $D = (V, E)$, the neighborhoods are defined as follows: $N_D^+(v) = \{w \in V : (v, w) \in E\}$, $N_D^-(v) = \{u \in V : (u, v) \in E\}$. The sets $N_D^+(v)$, $N_D^-(v)$, and $N_D(v) = N_D^+(v) \bigcup N_D^-(v)$ are called the out-neighborhood, in-neighborhood, and neighborhood of v. These neighborhoods are called open neighborhoods of v. Similarly, we can define closed neighborhoods of v (neighbors including v). The closed neighborhood of v is denoted by $N_D[v]$. That is, $N_D[v] = N_D(v) \cup \{v\}$.

A *directed path* is a directed graph with $V(P) \neq \emptyset$ with distinct vertices $u_1, u_2, \ldots, u_k$ and edges $e_1, e_2, \ldots, e_{k-1}$ such that e_i is an edge directed from u_i to u_{i+1} for $1 \leq i \leq k-1$. In this article, a path will always mean a 'directed path'. A digraph is *strongly connected* or *strong* if, for each ordered pair (u, v) of vertices, there is a path from u to v. A digraph is *weakly connected* if its underlying graph is connected. A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. If $D_1, D_2, \ldots, D_t$ are the strong components of D, then $V(D_1) \cup V(D_2) \cup \cdots \cup V(D_t) = V(D)$ and $V(D_i) \cap V(D_j) = \emptyset$ for every $i \neq j$.

The *length* of a path is the number of edges in the path. The *directed distance* $\overrightarrow{d}(u, v)$ between two vertices $u, v \in V(D)$ is the length of the shortest directed path from u to v, or infinity if no such path exists. Note that this distance is not a metric, as generally $\overrightarrow{d}(u, v) \neq \overrightarrow{d}(v, u)$.

So in [8], Chartrand and Tian introduced two other distances between the vertices u and v in a strong digraph, namely the maximum distance $md(u, v) = \max\ \{\overrightarrow{d}(u, v), \overrightarrow{d}(v, u)\}$ and the sum distance $sd(u, v) = \overrightarrow{d}(u, v) + \overrightarrow{d}(v, u)$, both of which are metrics. In this article, we deal with the maximum distance, md.

Remark 2.1. $md(u, v)$ is denoted by $d(u, v)$ hereafter.

The *m-eccentricity* of a vertex v, the *m-radius* and the *m-diameter* of a digraph D are also defined in [8]. Consistent with our notation $d(u, v)$ for maximum distance between the vertices u and v, we denote them respectively as $\mathrm{ecc(v)}$, $\mathrm{rad}(D)$, and $\mathrm{diam(D)}$. Thus, $\mathrm{ecc}(v) = \max\limits_{u \in V(D)} \{d(v, u)\}$, $\mathrm{rad}(D) = \min\limits_{v \in V(D)} \{\mathrm{ecc}(v)\}$, and $\mathrm{diam}(D) = \max\limits_{v \in V(D)} \{\mathrm{ecc}(v)\}$, where $\mathrm{ecc}(v)$ denotes *m-eccentricity* of v.

If a digraph D is strongly connected, then the maximum distance between every pair of vertices is finite, and hence the m-eccentricity of every vertex in D is finite. Otherwise, D has more than one strong component, and the maximum distance between two vertices lying in different strong components of D is infinity. So if D is not strongly connected, then the m-eccentricity of every vertex in D is infinity.

2.1 Definitions of boundary-type sets

We define the boundary-type sets of a digraph D with respect to the metric maximum distance. Most of the following definitions are analogous to the definitions in [7]. Let D be a strong digraph and $u, v \in V(D)$. The vertex v is said to be a *boundary vertex* of u if no neighbor of v is further away from u than v. Hereafter, we denote $N_D(v)$ and $N_D[v]$ by $N(v)$ and $N[v]$, respectively.

A vertex v is called a *boundary vertex* of D if it is the boundary vertex of some vertex $u \in V(D)$.

Definition 2.2. The *boundary* $\partial(D)$ of D is the set of all of its boundary vertices

$$\partial(D) = \{v \in V(D) : \exists u \in V(D) \text{ such that } \forall w \in N(v),\ d(u, w) \leq d(u, v)\}.$$

Given $u, v \in V(D)$, the vertex v is called an *eccentric vertex* of u if no vertex in $V(D)$ is further away from u than v; that is, if $d(u,v) = \mathrm{ecc}(u)$. A vertex v is called an *eccentric vertex* of digraph D if it is the eccentric vertex of some vertex $u \in V(D)$.

Definition 2.3. The *eccentricity* $\mathrm{Ecc}(D)$ of a digraph D is the set of all of its eccentric vertices

$$\mathrm{Ecc}(D) = \{v \in V(D) :\ \exists u \in V(D) \text{ such that } \mathrm{ecc}(u) = d(u,v)\}.$$

In a similar way, we can define the *eccentricity* of any proper subset W of the vertex set $V(D)$:

$$\mathrm{Ecc}(W) = \{v \in V(D) :\ \exists u \in W \text{ such that } \mathrm{ecc}(u) = d(u,v)\}.$$

A vertex $v \in V(D)$ is called a *peripheral vertex* of D if no vertex in $V(D)$ has an eccentricity greater than $\mathrm{ecc}(v)$; that is, if the eccentricity of v is equal to the diameter $\mathrm{diam}(D)$ of D.

Definition 2.4. The *periphery* $\mathrm{Per}(D)$ of a digraph D is the set of all of its peripheral vertices

$$\mathrm{Per}(D) = \{v \in V(D) :\ \mathrm{ecc}(u) \le \mathrm{ecc}(v), \forall u \in V(D)\} = \{v \in V(D) :\ \mathrm{ecc}(v) = \mathrm{diam}(D)\}.$$

A vertex $v \in V(D)$ is called a *contour vertex* of digraph D if no neighbor vertex of v has an eccentricity greater than $\mathrm{ecc}(v)$. The following definition is from [5].

Definition 2.5. The *contour* $\mathrm{Ct}(D)$ of a digraph D is the set of all of its contour vertices

$$\mathrm{Ct}(D) = \{v \in V(D) :\ \mathrm{ecc}(u) \le \mathrm{ecc}(v),\ \forall u \in N(v)\}.$$

As in the case of undirected graphs [3] we have,

1. $\mathrm{Per}(D) \subseteq \mathrm{Ct}(D) \cap \mathrm{Ecc}(D)$,
2. $\mathrm{Ecc}(D) \cup \mathrm{Ct}(D) \subseteq \partial(D)$.

This is because a peripheral vertex is a vertex having the maximum eccentricity in the digraph D and so every peripheral vertex in D is a contour vertex in D as well as the eccentric vertex of a diametrical vertex in D.

If v is an eccentric vertex of a vertex u, then v is a boundary vertex of u. Also if v is a contour vertex, then $\mathrm{ecc}(u) \le \mathrm{ecc}(v)$ for all $u \in N(v)$. So there exists some vertex $w \in V(D)$ such that $d(w,u) \le d(w,v)$ for all $u \in N(v)$, and hence v is a boundary vertex of w.

The open neighborhood $N(v)$ can be replaced by the closed neighborhood $N[v]$ in the definitions of the boundary and the contour sets. This does not affect the definitions and is necessary for proving the relationship between the boundary and the contour sets of the strong product of two digraphs and its factors.

3 Strong Product of Directed Graphs

The strong product $D_1 \boxtimes D_2$ of two digraphs D_1 and D_2 with vertex sets $V(D_1) = \{u_1, u_2, \ldots, u_m\}$ and $V(D_2) = \{v_1, v_2, \ldots, v_n\}$ is the digraph having the vertex set $V(D_1) \times V(D_2)$ and with arc set $E(D_1 \boxtimes D_2)$ defined as follows. A vertex (u_i, v_r) is adjacent to (u_j, v_s) in $D_1 \boxtimes D_2$ if either

1. $(u_i, u_j) \in E(D_1)$, $v_r = v_s$, or

2. $u_i = u_j$, $(v_r, v_s) \in E(D_2)$, or
3. $(u_i, u_j) \in E(D_1)$, $(v_r, v_s) \in E(D_2)$.

The strong product of digraphs is commutative [10]. The distance between two vertices (g, h) and (g', h') in the strong product $G \boxtimes H$ of two graphs G and H is given in [9] as follows:
$d_{G\boxtimes H}((g,h),(g',h')) = \max\{d_G(g,g'), d_H(h,h')\}$.
So in the case of two digraphs D_1 and D_2, it follows that the directed distance $\overrightarrow{d}_{D_1\boxtimes D_2}((u_i,v_r),(u_j,v_s)) = \max\{\overrightarrow{d}_{D_1}(u_i,u_j), \overrightarrow{d}_{D_2}(v_r,v_s)\}$.

Lemma 3.1. *Let D_1 and D_2 be two strongly connected digraphs. Then*

$$d_{D_1\boxtimes D_2}((u_i,v_r),(u_j,v_s)) = \max\{d_{D_1}(u_i,u_j), d_{D_2}(v_r,v_s)\},$$
$$\mathrm{ecc}_{D_1\boxtimes D_2}(u_i,v_r) = \max\{\mathrm{ecc}_{D_1}(u_i), \mathrm{ecc}_{D_2}(v_r)\}.$$

Proof. $d_{D_1\boxtimes D_2}((u_i,v_r),(u_j,v_s))$
$= \max\{\overrightarrow{d}_{D_1\boxtimes D_2}((u_i,v_r),(u_j,v_s)), \overrightarrow{d}_{D_1\boxtimes D_2}((u_j,v_s),(u_i,v_r))\}$
$= \max\{\max\{\overrightarrow{d}_{D_1}(u_i,u_j), \overrightarrow{d}_{D_2}(v_r,v_s)\}, \max\{\overrightarrow{d}_{D_1}(u_j,u_i), \overrightarrow{d}_{D_2}(v_s,v_r)\}\}$
$= \max\{\max\{\overrightarrow{d}_{D_1}(u_i,u_j), \overrightarrow{d}_{D_2}(v_r,v_s), \overrightarrow{d}_{D_1}(u_j,u_i), \overrightarrow{d}_{D_2}(v_s,v_r)\}\}$
$= \max\{\max\{\overrightarrow{d}_{D_1}(u_i,u_j), \overrightarrow{d}_{D_1}(u_j,u_i)\}, \max\{\overrightarrow{d}_{D_2}(v_r,v_s), \overrightarrow{d}_{D_2}(v_s,v_r)\}\}$
$= \max\{d_{D_1}(u_i,u_j), d_{D_2}(v_r,v_s)\}$.

Hence it follows that
$\mathrm{ecc}_{D_1\boxtimes D_2}(u_i,v_r)$
$= \max\{d_{D_1\boxtimes D_2}((u_i,v_r),(u_j,v_s)) : (u_j,v_s) \in V(D_1 \boxtimes D_2)\}$
$= \max\{\max\{d_{D_1}(u_i,u_j), d_{D_2}(v_r,v_s)\} : u_j \in V(D_1), v_s \in V(D_2)\}$
$= \max\{\max\{d_{D_1}(u_i,u_j) : u_j \in V(D_1)\}, \max\{d_{D_2}(v_r,v_s) : v_s \in V(D_2)\}\}$
$= \max\{\mathrm{ecc}_{D_1}(u_i), \mathrm{ecc}_{D_2}(v_r)\}$.

□

Corollary 3.2. *Let D_1 and D_2 be two strongly connected digraphs. Then*

$$\mathrm{rad}(D_1 \boxtimes D_2) = \max\{\mathrm{rad}(D_1), \mathrm{rad}(D_2)\},$$
$$\mathrm{diam}(D_1 \boxtimes D_2) = \max\{\mathrm{diam}(D_1), \mathrm{diam}(D_2)\}.$$

Proof.

$$\begin{aligned}
\mathrm{rad}(D_1 \boxtimes D_2) &= \min_{(u_i,v_r)\in V(D_1\boxtimes D_2)}\{\mathrm{ecc}(u_i,v_r)\} \\
&= \min_{\substack{u_i\in V(D_1)\\ v_r\in V(D_2)}}\{\max\{\mathrm{ecc}_{D_1}(u_i), \mathrm{ecc}_{D_2}(v_r)\}\} \\
&= \max\{\min_{u_i\in V(D_1)}\{\mathrm{ecc}(u_i)\}, \min_{v_r\in V(D_2)}\{\mathrm{ecc}(v_r)\}\} \\
&= \max\{\mathrm{rad}(D_1), \mathrm{rad}(D_2)\}. \\
\mathrm{diam}(D_1 \boxtimes D_2) &= \max_{(u_i,v_r)\in V(D_1\boxtimes D_2)}\{\mathrm{ecc}(u_i,v_r)\} \\
&= \max_{\substack{u_i\in V(D_1)\\ v_r\in V(D_2)}}\{\max\{\mathrm{ecc}_{D_1}(u_i), \mathrm{ecc}_{D_2}(v_r)\}\} \\
&= \max\{\max_{u_i\in V(D_1)}\{\mathrm{ecc}(u_i)\}, \max_{v_r\in V(D_2)}\{\mathrm{ecc}(v_r)\}\} \\
&= \max\{\mathrm{diam}(D_1), \mathrm{diam}(D_2)\}.
\end{aligned}$$

□

The strong product of two directed graphs is strongly connected if and only if both the digraphs are strongly connected [9]. Also if G and H are two undirected graphs, $N_{G\boxtimes H}[(g,h)] = N_G[g] \times N_H[h]$ [9]. Since the neigbors of a vertex in a directed graph are exactly its neighbors in the underlying graph, it follows that $N_{D_1\boxtimes D_2}[(u_i,v_r)] = N_{G\boxtimes H}[(u_i,v_r)] = N_G[u_i] \times N_H[v_r] = N_{D_1}[u_i] \times N_{D_2}[v_r]$, where G and H are the underlying graphs of D_1 and D_2, respectively. In [4], Cáceres et al. presented a description of the boundary-type sets of two undirected graphs and the description of the boundary is as follows.

For two graphs G and H, $\partial(G \boxtimes H) = (\partial(G) \times V(H)) \bigcup (V(G) \times \partial(H))$. But this result does not hold in the case of directed graphs.

Consider the strong product, $D_1 \boxtimes D_2$ of the digraphs D_1 and D_2 in Figure 1. The eccentricity of each vertex is displayed near the vertex in red color. $\mathrm{Per}(D_1) = \mathrm{Ecc}(D_1) = \mathrm{Ct}(D_1) = \{u_1,u_4\}$, $\mathrm{Per}(D_2) = \mathrm{Ecc}(D_2) = \mathrm{Ct}(D_2) = \{v_1,v_2\}$, and $\mathrm{Per}(D_1 \boxtimes D_2) = \mathrm{Ecc}(D_1 \boxtimes D_2) = \mathrm{Ct}(D_1 \boxtimes D_2) = \{(u_1,v_1),(u_4,v_1),(u_1,v_2),(u_4,v_2)\}$. $\partial(D_1) = \{u_1,u_4\}$, $\partial(D_2) = \{v_1,v_2\}$, and $\partial(D_1 \boxtimes D_2) = \{(u_1,v_1),(u_4,v_1),(u_1,v_2),(u_4,v_2)\}$. The reason for $(u_2,v_1),(u_2,v_2),(u_3,v_1),(u_3,v_2) \notin \partial(D_1 \boxtimes D_2)$ is explained after the proof of Theorem 3.3.

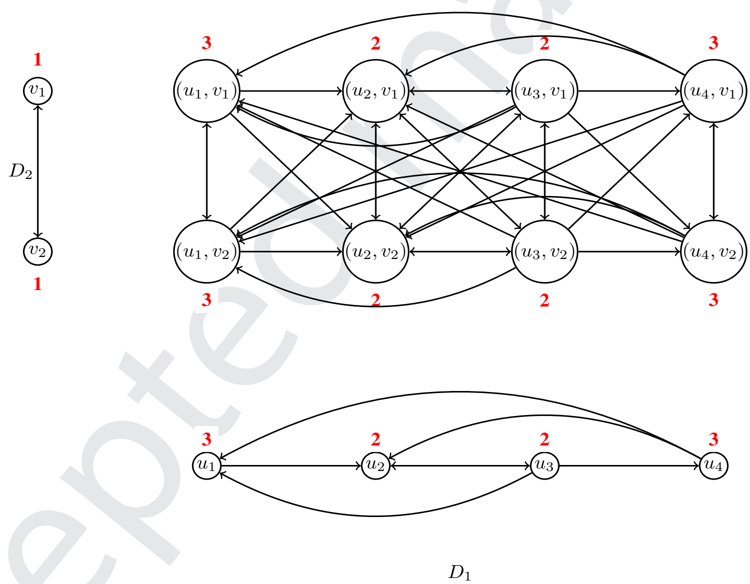

Figure 1: $D_1 \boxtimes D_2$

Now we present the results concerning the boundary-type sets of the strong product of two strongly connected digraphs. In all these results, D_1 and D_2 can be interchanged due to the commutativity of strong product of digraphs.

We have, $\partial(D_1 \boxtimes D_2) \subseteq [\partial(D_1) \times V(D_2)] \bigcup [V(D_1) \times \partial(D_2)]$.

To this end, let $(u_i,v_r) \in \partial(D_1 \boxtimes D_2)$. Then there exists a vertex $(u_j,v_s) \in V(D_1 \boxtimes D_2)$ such that $d((u_j,v_s),(u_i,v_r)) \geq d((u_j,v_s),(u_k,v_q))$ for every $(u_k,v_q) \in$

$N[(u_i, v_r)]$. This implies, $\max\{d(u_j, u_i), d(v_s, v_r)\} \geq \max\{d(u_j, u_k), d(v_s, v_q)\}$ for every $u_k \in N[u_i]$ and for every $v_q \in N[v_r]$. Hence $d(u_j, u_i) \geq d(u_j, u_k)$ for every $u_k \in N[u_i]$, or $d(v_s, v_r) \geq d(v_s, v_q)$ for every $v_q \in N[v_r]$. Thus, $u_i \in \partial(D_1)$ or $v_r \in \partial(D_2)$ or both. That is, if $(u_i, v_r) \in \partial(D_1 \boxtimes D_2)$, then at least one of the vertices u_i and v_r must be a boundary vertex in the corresponding factor graph.

Theorem 3.3. *Let D_1 and D_2 be two strongly connected digraphs. Then $\partial(D_1 \boxtimes D_2) = A_1 \bigcup A_2 \bigcup A_3$, where $A_1 = \partial(D_1) \times \partial(D_2)$,*
$A_2 = \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : u_i \in \partial(D_1), v_r \notin \partial(D_2)$, and $\exists v_t \in V(D_2)$ such that $d(v_t, v_q) \leq \mathrm{ecc}(u_i), \forall v_q \in N[v_r]\}$*, and*
$A_3 = \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : u_i \notin \partial(D_1), v_r \in \partial(D_2)$, and $\exists u_\ell \in V(D_1)$ such that $d(u_\ell, u_k) \leq \mathrm{ecc}(v_r), \forall u_k \in N[u_i]\}$.

Proof. Suppose that $(u_i, v_r) \in \partial(D_1 \boxtimes D_2)$.

Then there exists a vertex $(u_j, v_s) \in V(D_1 \boxtimes D_2)$ such that $d((u_j, v_s), (u_i, v_r)) \geq d((u_j, v_s), (u_k, v_q))$ for all vertices $(u_k, v_q) \in N[(u_i, v_r)]$. Since $d((u_j, v_s), (u_i, v_r)) = \max\{d(u_j, u_i), d(v_s, v_r)\}$ and $d((u_j, v_s), (u_k, v_q)) = \max\{d(u_j, u_k), d(v_s, v_q)\}$, we get $\max\{d(u_j, u_i), d(v_s, v_r)\} \geq \max\{d(u_j, u_k), d(v_s, v_q)\}$ for all $u_k \in N[u_i]$, $v_q \in N[v_r]$. We distinguish four cases.

1. $\max\{d(u_j, u_i), d(v_s, v_r)\} = d(u_j, u_i)$ and $d(v_s, v_r) \geq d(v_s, v_q)$ for all $v_q \in N[v_r]$
2. $\max\{d(u_j, u_i), d(v_s, v_r)\} = d(u_j, u_i)$ and $d(v_s, v_r) \geq d(v_s, v_q)$ does not hold for all $v_q \in N[v_r]$
3. $\max\{d(u_j, u_i), d(v_s, v_r)\} = d(v_s, v_r)$ and $d(u_j, u_i) \geq d(u_j, u_k)$ for all $u_k \in N[u_i]$
4. $\max\{d(u_j, u_i), d(v_s, v_r)\} = d(v_s, v_r)$ and $d(u_j, u_i) \geq d(u_j, u_k)$ does not hold for all $u_k \in N[u_i]$

In cases 1 and 3, $d(u_j, u_i) \geq d(u_j, u_k)$ for all $u_k \in N[u_i]$ and $d(v_s, v_r) \geq d(v_s, v_q)$ for all $v_q \in N[v_r]$. So $u_i \in \partial(D_1)$, $v_r \in \partial(D_2)$, and hence $(u_i, v_r) \in A_1$.

In case 2, $u_i \in \partial(D_1)$ and v_r is not a boundary vertex of v_s in D_2. If there exists any vertex v_t such that v_r is a boundary vertex of v_t, then we get $(u_i, v_r) \in A_1$. Otherwise, since $v_r \notin \partial(D_2)$, for every vertex $v_t \in V(D_2)$, there exists some vertex $v_q \in N[v_r]$ such that $d(v_t, v_r) < d(v_t, v_q)$. Hence if (u_i, v_r) is a boundary vertex of a vertex (u_ℓ, v_t) in $D_1 \boxtimes D_2$, then $d((u_\ell, v_t), (u_i, v_r)) = \max\{d(u_\ell, u_i), d(v_t, v_r)\} = d(u_\ell, u_i) > d(v_t, v_r)$, for otherwise $d(u_\ell, u_i) \leq d(v_t, v_r)$ and so we get $d((u_\ell, v_t), (u_i, v_r)) = d(v_t, v_r) < d(v_t, v_q) = d((u_\ell, v_t), (u_i, v_q))$, where $(u_i, v_q) \in N[(u_i, v_r)]$.

Let $(u_k, v_q) \in N[(u_i, v_r)]$. Then $d((u_\ell, v_t), (u_k, v_q)) = \max\{d(u_\ell, u_k), d(v_t, v_q)\}$. If (u_i, v_r) is a boundary vertex of (u_ℓ, v_t), then $\max\{d(u_\ell, u_i), d(v_t, v_r)\} \geq \max\{d(u_\ell, u_k), d(v_t, v_q)\}$. So the necessary condition for the vertex (u_i, v_r) such that $u_i \in \partial(D_1)$ and $v_r \notin \partial(D_2)$ to be a boundary vertex of the vertex (u_ℓ, v_t) in $D_1 \boxtimes D_2$ is $d(u_\ell, u_i) \geq d(v_t, v_q)$ for all $v_q \in N[v_r]$. Since $\mathrm{ecc}(u_i) \geq d(u_\ell, u_i)$ for all $u_\ell \in V(D_1)$, the necessary condition becomes $\mathrm{ecc}(u_i) \geq d(v_t, v_q)$ for all $v_q \in N[v_r]$. Thus, $(u_i, v_r) \in A_2$.

Thus in case 2, $(u_i, v_r) \in A_1 \cup A_2$.

In case 4, $v_r \in \partial(D_2)$ and u_i is not a boundary vertex of u_j in D_1. As in case 2, it follows that $(u_i, v_r) \in A_1 \cup A_3$.

Thus in all cases, we get $\partial(D_1 \boxtimes D_2) \subseteq A_1 \bigcup A_2 \bigcup A_3$.

Conversely, suppose that $(u_i, v_r) \in A_1 \bigcup A_2 \bigcup A_3$. First let $(u_i, v_r) \in A_1$. Then $u_i \in \partial(D_1)$ and $v_r \in \partial(D_2)$. So there exists vertices $u_j \in V(D_1)$, $v_s \in V(D_2)$ such that $d(u_j, u_i) \geq d(u_j, u_k)$ for every $u_k \in N[u_i]$, and $d(v_s, v_r) \geq d(v_s, v_q)$ for every $v_q \in N[v_r]$. Hence in $D_1 \boxtimes D_2$, $d((u_j, v_s),(u_i, v_r)) = \max\{d(u_j, u_i), d(v_s, v_r)\} \geq \max\{d(u_j, u_k), d(v_s, v_q)\} = d((u_j, v_s),(u_k, v_q))$ for all vertices $(u_k, v_q) \in N[(u_i, v_r)]$. Thus, $A_1 \subseteq \partial(D_1 \boxtimes D_2)$.

Now let $(u_i, v_r) \in A_2$. Then $u_i \in \partial(D_1), v_r \notin \partial(D_2)$ and there exists some vertex $v_t \in V(D_2)$ such that $d(v_t, v_q) \leq \operatorname{ecc}(u_i)$, for all $v_q \in N[v_r]$. Since $u_i \in \partial(D_1)$, there exists at least one vertex $u_j \in V(D_1)$ such that $d(u_j, u_i) \geq d(u_j, u_k)$ for every $u_k \in N[u_i]$. Of these vertices, let u_b be a vertex such that $d(u_b, u_i) = \operatorname{ecc}(u_i)$. Hence in $D_1 \boxtimes D_2$, $d((u_b, v_t),(u_i, v_r)) = \max\{d(u_b, u_i), d(v_t, v_r)\} \geq \max\{d(u_b, u_k), d(v_t, v_q)\} = d((u_b, v_t),(u_k, v_q))$ for all $(u_k, v_q) \in N[(u_i, v_r)]$, since $d(v_t, v_q) \leq \operatorname{ecc}(u_i) = d(u_b, u_i)$ for all $v_q \in N[v_r]$. Thus, (u_i, v_r) is a boundary vertex of (u_b, v_t) in $D_1 \boxtimes D_2$ and hence $A_2 \subseteq \partial(D_1 \boxtimes D_2)$.

By analogous arguments and since the strong product of digraphs is commutative, it follows that $A_3 \subseteq \partial(D_1 \boxtimes D_2)$.

Hence $A_1 \bigcup A_2 \bigcup A_3 \subseteq \partial(D_1 \boxtimes D_2)$. □

Now consider Figure 1. $ecc(v_1) = ecc(v_2) = 1$. $N[u_2] = N[u_3] = \{u_1, u_2, u_3, u_4\}$, $d(u_1, u_4) = 3$, $d(u_1, u_2) = d(u_1, u_3) = d(u_2, u_4) = d(u_3, u_4) = 2$, and $d(u_2, u_3) = 1$. $u_2 \notin \partial(D_1)$ and hence (u_2, v_1), $(u_2, v_2) \notin \partial(D_1 \boxtimes D_2)$, since there is no vertex $u_\ell \in V(D_1)$ such that $d(u_\ell, u_k)) \leq 1$ for all $u_k \in N[u_2]$. For similar reasons, (u_3, v_1), $(u_3, v_2) \notin \partial(D_1 \boxtimes D_2)$.

Consider the strong product of two connected undirected graphs. In the case of an undirected graph, the *maximum distance* between two vertices is the usual distance between the vertices. Also, since we deal with the distance between any two distinct vertices, it doesn't matter whether the undirected graphs are simple or not; that is, whether they contain loops or parallel edges. So we state the result for any two connected nontrivial (not equal to K_1) undirected graphs.

Remark 3.4. The description for the boundary set of the strong product of two graphs (undirected graphs) G and H presented in [4] holds only for the product of two **nontrivial graphs** G and H. To this end, let $H = K_1 = (\{v\}, \emptyset)$. We have, $\partial(K_1) = \{v\}$ (since all vertices of a complete graph are boundary vertices of the graph), and hence $\partial(G) \widehat{=} \partial(G \boxtimes K_1) = (\partial(G) \times \{v\}) \bigcup (V(G) \times \{v\}) \widehat{=} V(G)$, which is not true in general.

Corollary 3.5. *Let D_1 and D_2 be two nontrivial connected undirected graphs. Then*

$$\partial(D_1 \boxtimes D_2) = [\partial(D_1) \times V(D_2)] \bigcup [V(D_1) \times \partial(D_2)].$$

Proof. By Theorem 3.3, if D_1 and D_2 are two strongly connected digraphs, $\partial(D_1 \boxtimes D_2) = A_1 \bigcup A_2 \bigcup A_3$. Since D_1 and D_2 are given to be two nontrivial undirected graphs, $\operatorname{ecc}(u_i) \geq 1$ for all $u_i \in V(D_1)$, $\operatorname{ecc}(v_r) \geq 1$ for all $v_r \in V(D_2)$, $d(u_i, u_k) = 1$ for all $u_k \in N(u_i)$, and $d(v_r, v_q) = 1$ for all $v_q \in N(v_r)$. Thus, $A_1 = \partial(D_1) \times \partial(D_2)$, $A_2 = \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : u_i \in \partial(D_1),\ v_r \notin \partial(D_2)$, and $\exists v_t \in V(D_2)$ such that $d(v_t, v_q) \leq \operatorname{ecc}(u_i),\ \forall v_q \in N(v_r)\} = \partial(D_1) \times V(D_2)$, and $A_3 = \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : u_i \notin \partial(D_1),\ v_r \in \partial(D_2)$, and $\exists u_\ell \in V(D_1)$ such that $d(u_\ell, u_k) \leq \operatorname{ecc}(v_r),\ \forall u_k \in N(u_i)\} = V(D_1) \times \partial(D_2)$.

Therefore, $\partial(D_1 \boxtimes D_2) = A_1 \cup A_2 \cup A_3 = [\partial(D_1) \times V(D_2)] \bigcup [V(D_1) \times \partial(D_2)]$. □

Theorem 3.6. *Let D_1 and D_2 be two strongly connected digraphs.*

1. If $\operatorname{diam}(D_1) < \operatorname{diam}(D_2)$, *then* $\operatorname{Per}(D_1 \boxtimes D_2) = V(D_1) \times \operatorname{Per}(D_2)$.

2. If $\operatorname{diam}(D_1) = \operatorname{diam}(D_2)$*, then* $\operatorname{Per}(D_1 \boxtimes D_2) = [\operatorname{Per}(D_1) \times V(D_2)] \bigcup [V(D_1) \times \operatorname{Per}(D_2)]$.

Proof. 1. Let $\operatorname{diam}(D_2) = n$. Let $v_r \in \operatorname{Per}(D_2)$.

Then for all $u_i \in V(D_1)$, $\operatorname{ecc}(u_i, v_r) = \max\{\operatorname{ecc}(u_i), \operatorname{ecc}(v_r)\} = n$. Hence $(u_i, v_r) \in \operatorname{Per}(D_1 \boxtimes D_2)$. Also if $v_r \notin \operatorname{Per}(D_2)$, then since $\operatorname{ecc}(u_i, v_r) < n$, $(u_i, v_r) \notin \operatorname{Per}(D_1 \boxtimes D_2)$. Hence it follows that $\operatorname{Per}(D_1 \boxtimes D_2) = V(D_1) \times \operatorname{Per}(D_2)$.

2. Let $\operatorname{diam}(D_1) = \operatorname{diam}(D_2) = n$. If $u_i \in \operatorname{Per}(D_1)$, then for all $v_r \in V(D_2)$, $(u_i, v_r) \in \operatorname{Per}(D_1 \boxtimes D_2)$, since $\operatorname{ecc}(u_i, v_r) = \max\{\operatorname{ecc}(u_i), \operatorname{ecc}(v_r)\} = n$. Hence $(u_i, v_r) \in \operatorname{Per}(D_1 \boxtimes D_2)$. Similarly, if $v_r \in \operatorname{Per}(D_2)$, then for all $u_i \in V(D_1)$, $(u_i, v_r) \in \operatorname{Per}(D_1 \boxtimes D_2)$. Hence it follows that $[\operatorname{Per}(D_1) \times V(D_2)] \bigcup [V(D_1) \times \operatorname{Per}(D_2)] \subseteq \operatorname{Per}(D_1 \boxtimes D_2)$.

Conversely, if $(u_i, v_r) \in \operatorname{Per}(D_1 \boxtimes D_2)$, then $\operatorname{ecc}(u_i, v_r) = \max\{\operatorname{diam}(D_1), \operatorname{diam}(D_2)\} = n$. Thus, at least one of $\operatorname{ecc}(u_i)$ and $\operatorname{ecc}(v_r)$ must be necessarily equal to n. Hence $u_i \in \operatorname{Per}(D_1)$ or $v_r \in \operatorname{Per}(D_2)$, and therefore, $\operatorname{Per}(D_1 \boxtimes D_2) \subseteq [\operatorname{Per}(D_1) \times V(D_2)] \bigcup [V(D_1) \times \operatorname{Per}(D_2)]$. $\square$

Theorem 3.7. *Let* D_1 *and* D_2 *be two strongly connected digraphs.*

1. *If* $\operatorname{rad}(D_1) = \operatorname{rad}(D_2)$*, then*
$$\operatorname{Ecc}(D_1 \boxtimes D_2) = [\operatorname{Ecc}(D_1) \times V(D_2)] \bigcup [V(D_1) \times \operatorname{Ecc}(D_2)].$$

2. *If* $\operatorname{rad}(D_1) < \operatorname{rad}(D_2)$*, then*
$$\operatorname{Ecc}(D_1 \boxtimes D_2) = \left[\bigcup_{\operatorname{ecc}(u_i) \geq \operatorname{rad}(D_2)} \operatorname{Ecc}(u_i) \times V(D_2)\right] \bigcup \left[V(D_1) \times \operatorname{Ecc}(D_2)\right].$$

Proof. 1. First we will prove that $\operatorname{Ecc}(D_1 \boxtimes D_2) \subseteq [\operatorname{Ecc}(D_1) \times V(D_2)] \bigcup [V(D_1) \times \operatorname{Ecc}(D_2)]$. Let $(u_i, v_r) \in \operatorname{Ecc}(D_1 \boxtimes D_2)$. Then there exists a vertex (u_j, v_s) such that $\operatorname{ecc}(u_j, v_s) = d((u_j, v_s), (u_i, v_r)) = \max\{d(u_j, u_i), d(v_s, v_r)\}$. Since $\operatorname{ecc}(u_j, v_s) = \max\{\operatorname{ecc}(u_j), \operatorname{ecc}(v_s)\}$, and $\operatorname{ecc}(u_j) \geq d(u_j, u_i)$ and $\operatorname{ecc}(v_s) \geq d(v_s, v_r)$, at least one of $\operatorname{ecc}(u_j) = d(u_j, u_i)$ and $\operatorname{ecc}(v_s) = d(v_s, v_r)$ must hold. So necessarily u_i is an eccentric vertex of u_j, or v_r is an eccentric vertex of v_s.

Hence $(u_i, v_r) \in [\operatorname{Ecc}(D_1) \times V(D_2)] \bigcup [V(D_1) \times \operatorname{Ecc}(D_2)]$.

Let $\operatorname{rad}(D_1) = \operatorname{rad}(D_2) = n$. Let $u_i \in \operatorname{Ecc}(D_1)$. Then there exists a vertex $u_j \in V(D_1)$ such that $\operatorname{ecc}(u_j) = d(u_j, u_i)$. Consider the vertex $(u_i, v_r) \in V(D_1 \boxtimes D_2)$, where v_r is an arbitrary vertex in D_2. Since $\operatorname{rad}(D_2) = n$, there exists a vertex $v_s \in V(D_2)$ such that $\operatorname{ecc}(v_s) = n$. Hence $d(v_s, v_r) \leq n$ and so $\operatorname{ecc}(u_j, v_s) = \max\{\operatorname{ecc}(u_j), \operatorname{ecc}(v_s)\} = \max\{\operatorname{ecc}(u_j), n\} = \operatorname{ecc}(u_j)$. Thus, $d((u_j, v_s), (u_i, v_r)) = \max\{d(u_j, u_i), d(v_s, v_r)\} = \operatorname{ecc}(u_j) = \operatorname{ecc}(u_j, v_s)$. So (u_i, v_r) is an eccentric vertex of (u_j, v_s). Thus if $u_i \in \operatorname{Ecc}(D_1)$, then $(u_i, v_r) \in \operatorname{Ecc}(D_1 \boxtimes D_2)$ for all $v_r \in V(D_2)$. Similarly, we can prove that if $v_q \in \operatorname{Ecc}(D_2)$, then $(u_k, v_q) \in \operatorname{Ecc}(D_1 \boxtimes D_2)$ for all $u_k \in V(D_1)$.

Hence $[\operatorname{Ecc}(D_1) \times V(D_2)] \bigcup [V(D_1) \times \operatorname{Ecc}(D_2)] \subseteq \operatorname{Ecc}(D_1 \boxtimes D_2)$, and so the result holds.

2. Let $\operatorname{rad}(D_1) < \operatorname{rad}(D_2) = n$. Let $u_i \in V(D_1)$, $v_r \in V(D_2)$. Here two cases arise:

Case 1. $v_r \in \operatorname{Ecc}(D_2)$.

Then there exists a vertex $v_s \in V(D_2)$ such that $\operatorname{ecc}(v_s) = d(v_s, v_r)$. Let $u_p \in V(D_1)$ be such that $\operatorname{ecc}(u_p) = \operatorname{rad}(D_1)$. Then since $\operatorname{rad}(D_2) > \operatorname{ecc}(u_p)$, $\operatorname{ecc}(u_p, v_s) = \max\{\operatorname{ecc}(u_p), \operatorname{ecc}(v_s)\} = \operatorname{ecc}(v_s)$. Also, $d((u_p, v_s), (u_i, v_r)) =$

$\max\{d(u_p,u_i), d(v_s,v_r)\} = \operatorname{ecc}(v_s)$. Thus, (u_i,v_r) is an eccentric vertex of (u_p,v_s). So in this case, $V(D_1)\times \operatorname{Ecc}(D_2) \subseteq \operatorname{Ecc}(D_1 \boxtimes D_2)$.

Case 2. $v_r \notin \operatorname{Ecc}(D_2)$.

Let $v_q \in V(D_2)$ be such that $\operatorname{ecc}(v_q) = \operatorname{rad}(D_2)$. Let $\bigcup\limits_{\operatorname{ecc}(u_i)\geq \operatorname{rad}(D_2)} \operatorname{Ecc}(u_i) = A$. Let $u_k \in A$. Then there exists a vertex $u_p \in V(D_1)$ such that $\operatorname{ecc}(u_p) \geq \operatorname{rad}(D_2)$ and $\operatorname{ecc}(u_p) = d(u_p,u_k)$. Then $d((u_p,v_q),(u_k,v_r)) = \max\{d(u_p,u_k), d(v_q,v_r)\} = d(u_p,u_k) = \operatorname{ecc}(u_p) = \operatorname{ecc}(u_p,v_q)$ and hence (u_k,v_r) is an eccentric vertex of (u_p,v_q). Hence in this case, $\bigcup\limits_{\operatorname{ecc}(u_i)\geq \operatorname{rad}(D_2)} \operatorname{Ecc}(u_i)\times V(D_2) \subseteq \operatorname{Ecc}(D_1\boxtimes D_2)$.

$$\text{Thus, } \left[\bigcup_{\operatorname{ecc}(u_i)\geq \operatorname{rad}(D_2)} \operatorname{Ecc}(u_i)\times V(D_2)\right] \bigcup \left[V(D_1)\times \operatorname{Ecc}(D_2)\right] \subseteq \operatorname{Ecc}(D_1\boxtimes D_2).$$

Conversely, let $(u_k,v_r) \in \operatorname{Ecc}(D_1\boxtimes D_2)$. Then there exists a vertex $(u_j,v_s) \in V(D_1\boxtimes D_2)$ such that $\operatorname{ecc}(u_j,v_s) = d((u_j,v_s),(u_k,v_r)) = \max\{d(u_j,u_k), d(v_s,v_r)\} = \max\{\operatorname{ecc}(u_j), \operatorname{ecc}(v_s)\}$. If $v_r \in \operatorname{Ecc}(D_2)$, we get $(u_k,v_r) \in V(D_1)\times \operatorname{Ecc}(D_2)$.

Hence suppose that $(u_k,v_r) \in \operatorname{Ecc}(D_1\boxtimes D_2)$ and $v_r \notin \operatorname{Ecc}(D_2)$. Then for all $v_s \in V(D_2)$, $\operatorname{ecc}(v_s) > d(v_s,v_r)$. Thus, $\operatorname{ecc}(u_j,v_s) = \operatorname{ecc}(u_j) = d(u_j,u_k)$. If possible, suppose that $u_k \notin A = \bigcup\limits_{\operatorname{ecc}(u_i)\geq \operatorname{rad}(D_2)} \operatorname{Ecc}(u_i)$. Thus, there is no vertex u_j in D_1 such that $\operatorname{ecc}(u_j) = d(u_j,u_k)$ and $\operatorname{ecc}(u_j) \geq \operatorname{rad}(D_2)$. Hence if u_k is an eccentric vertex of u_j in D_1, then $d(u_j,u_k) < \operatorname{rad}(D_2)$. We have, $\operatorname{rad}(D_1\boxtimes D_2) = \max\{\operatorname{rad}(D_1), \operatorname{rad}(D_2)\} = \operatorname{rad}(D_2)$. Thus, (u_k,v_r) cannot be the eccentric vertex of any vertex $(u_j,v_s) \in D_1\boxtimes D_2$, since $d((u_j,v_s),(u_k,v_r)) = \max\{d(u_j,u_k),(v_s,v_r)\} \neq \operatorname{ecc}(u_j,v_s)$ in this case. This is a contradiction, and hence $u_k \in A$. Hence $(u_k,v_r) \in \bigcup\limits_{\operatorname{ecc}(u_i)\geq \operatorname{rad}(D_2)} \operatorname{Ecc}(u_i)\times V(D_2)$.

$$\text{Hence } \operatorname{Ecc}(D_1\boxtimes D_2) \subseteq \left[\bigcup_{\operatorname{ecc}(u_i)\geq \operatorname{rad}(D_2)} \operatorname{Ecc}(u_i)\times V(D_2)\right] \bigcup \left[V(D_1)\times \operatorname{Ecc}(D_2)\right].$$

□

Theorem 3.8. *Let D_1 and D_2 be two strongly connected digraphs. Then* $\operatorname{Ct}(D_1\boxtimes D_2) = A_1\cup A_2\cup A_3$, *where* $A_1 = [\operatorname{Ct}(D_1)\times \operatorname{Ct}(D_2)]$,
$A_2 = \{(u_i,v_r) \in V(D_1\boxtimes D_2):\ u_i \in \operatorname{Ct}(D_1),\ v_r \notin \operatorname{Ct}(D_2)$, *and* $\operatorname{ecc}(v_q) \leq \operatorname{ecc}(u_i)$ *for all* $v_q \in N[v_r]\}$,
$A_3 = \{(u_i,v_r) \in V(D_1\boxtimes D_2):\ u_i \notin \operatorname{Ct}(D_1),\ v_r \in \operatorname{Ct}(D_2)$, *and* $\operatorname{ecc}(u_k) \leq \operatorname{ecc}(v_r)$ *for all* $u_k \in N[u_i]\}$.

Proof. $(u_i,v_r) \in \operatorname{Ct}(D_1\boxtimes D_2)$ if and only if $\operatorname{ecc}(u_i,v_r) \geq \operatorname{ecc}(u_k,v_q)$ for all $(u_k,v_q) \in N[(u_i,v_r)]$;
if and only if $\max\{\operatorname{ecc}(u_i), \operatorname{ecc}(v_r)\} \geq \max\{\operatorname{ecc}(u_k), \operatorname{ecc}(v_q)\}$ for all $u_k \in N[u_i]$ and $v_q \in N[v_r]$;
if and only if one of the following three cases holds.

1. $\max\{\operatorname{ecc}(u_i), \operatorname{ecc}(v_r)\} = \operatorname{ecc}(u_i) = \operatorname{ecc}(v_r)$. Then, $\operatorname{ecc}(u_i) \geq \operatorname{ecc}(u_k)$ and $\operatorname{ecc}(v_r) \geq \operatorname{ecc}(v_q)$ for all $u_k \in N[u_i]$ and $v_q \in N[v_r]$.
2. $\max\{\operatorname{ecc}(u_i), \operatorname{ecc}(v_r)\} = \operatorname{ecc}(u_i) > \operatorname{ecc}(v_r)$. Then, $\operatorname{ecc}(u_i) \geq \operatorname{ecc}(u_k)$ for all $u_k \in N[u_i]$ and $\operatorname{ecc}(v_r) < \operatorname{ecc}(u_i)$, $\operatorname{ecc}(v_q) \leq \operatorname{ecc}(u_i)$ for all $v_q \in N(v_r)$.

3. $\max\{\mathrm{ecc}(u_i), \mathrm{ecc}(v_r)\} = \mathrm{ecc}(v_r) > \mathrm{ecc}(u_i)$. Then, $\mathrm{ecc}(v_r) \geq \mathrm{ecc}(v_q)$ for all $v_q \in N[v_r]$ and $\mathrm{ecc}(u_i) < \mathrm{ecc}(v_r)$, $\mathrm{ecc}(u_k) \leq \mathrm{ecc}(v_r)$ for all $u_k \in N(u_i)$.

In case 1, $(u_i, v_r) \in \mathrm{Ct}(D_1 \boxtimes D_2)$.

In case 2, $(u_i, v_r) \in \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : u_i \in \mathrm{Ct}(D_1),\ v_r \notin \mathrm{Ct}(D_2)$, and $\mathrm{ecc}(v_q) \leq \mathrm{ecc}(u_i)$ for all $v_q \in N[v_r]\}$.

In case 3, $(u_i, v_r) \in \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : u_i \notin \mathrm{Ct}(D_1),\ v_r \in \mathrm{Ct}(D_2)$, and $\mathrm{ecc}(u_k) \leq \mathrm{ecc}(v_r)$ for all $u_k \in N[u_i]\}$.

Thus we get, $\mathrm{Ct}(D_1 \boxtimes D_2) = A_1 \bigcup A_2 \bigcup A_3$. □

Consider the contour set of the strong product of two connected undirected graphs. As in the case of the boundary set, the result holds even when the undirected graphs are not simple.

Corollary 3.9. *Let D_1 and D_2 be two connected undirected graphs. Then*

$$\begin{aligned}\mathrm{Ct}(D_1 \boxtimes D_2) =&\{(u_i, v_r) \in V(D_1 \boxtimes D_2) :\ u_i \in \mathrm{Ct}(D_1),\ v_r \notin \mathrm{Ct}(D_2),\ \textit{and}\ \mathrm{ecc}(v_r) < \mathrm{ecc}(u_i)\}\\ &\bigcup\{(u_i, v_r) \in V(D_1 \boxtimes D_2) :\ u_i \notin \mathrm{Ct}(D_1),\ v_r \in \mathrm{Ct}(D_2),\ \textit{and}\ \mathrm{ecc}(u_i) < \mathrm{ecc}(v_r)\}\\ &\bigcup[\mathrm{Ct}(D_1) \times \mathrm{Ct}(D_2)].\end{aligned}$$

Proof. By Theorem 3.8, when D_1 and D_2 are two strongly connected digraphs, $\mathrm{Ct}(D_1 \boxtimes D_2) = A_1 \bigcup A_2 \bigcup A_3$. Since D_1 and D_2 are given to be undirected graphs, eccentricity of two adjacent vertices differ by atmost one. Hence $A_1 = \mathrm{Ct}(D_1) \times \mathrm{Ct}(D_2)$,

$$\begin{aligned}A_2 =&\{(u_i, v_r) \in V(D_1 \boxtimes D_2) :\ u_i \in \mathrm{Ct}(D_1),\ v_r \notin \mathrm{Ct}(D_2),\ \text{and}\ \mathrm{ecc}(v_q) \leq \mathrm{ecc}(u_i) \forall\ v_q \in N[v_r]\}\\ =&\{(u_i, v_r) \in V(D_1 \boxtimes D_2) :\ u_i \in \mathrm{Ct}(D_1),\ v_r \notin \mathrm{Ct}(D_2),\ \text{and}\ \mathrm{ecc}(v_r) + 1 \leq \mathrm{ecc}(u_i)\}\\ =&\{(u_i, v_r) \in V(D_1 \boxtimes D_2) :\ u_i \in \mathrm{Ct}(D_1),\ v_r \notin \mathrm{Ct}(D_2),\ \text{and}\ \mathrm{ecc}(v_r) < \mathrm{ecc}(u_i)\},\end{aligned}$$

and $A_3 = \{(u_i, v_r) \in V(D_1 \boxtimes D_2) :\ u_i \notin \mathrm{Ct}(D_1),\ v_r \in \mathrm{Ct}(D_2),\ \textit{and}\ \mathrm{ecc}(u_i) < \mathrm{ecc}(v_r)\}$, since $\max\limits_{u_k \in N[u_i]}\{\mathrm{ecc}(u_k)\} = ecc(u_i) + 1$. Hence it follows that

$$\begin{aligned}\mathrm{Ct}(D_1 \boxtimes D_2) =&\{(u_i, v_r) \in V(D_1 \boxtimes D_2) :\ u_i \in \mathrm{Ct}(D_1),\ v_r \notin \mathrm{Ct}(D_2),\ \text{and}\ \mathrm{ecc}(v_r) < \mathrm{ecc}(u_i)\}\\ &\bigcup\{(u_i, v_r) \in V(D_1 \boxtimes D_2) :\ u_i \notin \mathrm{Ct}(D_1),\ v_r \in \mathrm{Ct}(D_2),\ \textit{and}\ \mathrm{ecc}(u_i) < \mathrm{ecc}(v_r)\}\\ &\bigcup[\mathrm{Ct}(D_1) \times \mathrm{Ct}(D_2)].\end{aligned}$$

□

We have examined the boundary-type sets of the strong product of two strongly connected digraphs D_1 and D_2. Now suppose that at least one of D_1 and D_2, say D_1, is not strongly connected. Then the eccentricity of every vertex in D_1 is infinity, and hence the eccentricity of every vertex in $D_1 \boxtimes D_2$ is infinity. Thus, we have $\partial(D_1) = Per(D_1) = Ecc(D_1) = Ct(D_1) = V(D_1)$, and $\partial(D_1 \boxtimes D_2) = Per(D_1 \boxtimes D_2) = Ecc(D_1 \boxtimes D_2) = Ct(D_1 \boxtimes D_2) = V(D_1) \times V(D_2)$. Since $rad(D_1) = diam(D_1) = \infty$, the expression for $Per(D_1 \boxtimes D_2)$ in Theorem 3.6, and the expression for $Ecc(D_1 \boxtimes D_2)$ in Theorem 3.7 gives $V(D_1) \times V(D_2)$. Since $ecc(u_i) = \infty$ for all $u_i \in V(D_1)$, the expression for $\partial(D_1 \boxtimes D_2)$ in Theorem 3.3, and the expression for $Ct(D_1 \boxtimes D_2)$ in Theorem 3.8 also gives $V(D_1) \times V(D_2)$. Similar is the case when D_2 and both D_1 and D_2 are not strongly connected.

Thus, the results derived for the boundary-type sets of the strong product of two strongly connected digraphs D_1 and D_2 hold also when the digraphs D_1 and D_2 are not even weakly connected. So the results for the boundary-type sets of the strong product of two connected undirected graphs hold for any two arbitrary undirected graphs.

4 Conclusion

In this article, the relationship between the boundary-type sets of the strong product of two digraphs, and that of its factors is derived. As 'maximum distance' is the generalization of the usual distance in an undirected graph, these results hold for undirected graphs also. The results for the periphery and eccentricity sets of the strong product of two undirected graphs turn out to be the same as the results in [4]. The results for the boundary and contour sets in the undirected case, as described in [4], are also derived as special cases.

Acknowledgements

The authors thank the anonymous referees for their valuable suggestions, which helped improve this article.

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