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Noncommutative frames revisited

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Abstract

In this note we revisit noncommutative frames. Special attention is devoted to the study of join completeness and related properties in skew lattices.

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1 Introduction

In [5], the first author introduced noncommutative frames, motivated by a noncommutative topology constructed by Le Bruyn [7] on the points of the Connes–Consani Arithmetic Site [2], [3]. The definition of noncommutative frame fits in the general theory of skew lattices, a theory that goes back to Pascual Jordan [6] and is an active research topic starting with a series of papers of the third author [8] [9] [10]. For an overview of the primary results on skew lattices, we refer the reader to [12] or the earlier systematic survey [11].

Recall that a frame is a complete lattice which satisfies the infinite distributive laws. Noncommutative frames are noncommutative generalizations of frames, the precise definition is given in Section 1. Loosely speaking, a noncommutative frame is a frame of certain

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congruence classes, $\mathcal{D}$-classes. A noncommutative frame containing both the top and the bottom elements would necessarily be commutative. There are thus two natural ways of generalizing frames to the noncommutative setting: 1. We keep the bottom element, but replace the top element with a top $\mathcal{D}$-class. This approach is carried out in the present paper. 2. We keep the top element, but replace the bottom element with a bottom $\mathcal{D}$-class. This approach was carried out in [4]. Note that the two approaches are essentially different as they do not dualize one another.

The notion of completeness for noncommutative lattices is much more complex than for lattices. For example, join completeness and meet completeness turn out to be non-equivalent properties. The main purpose of this note is to study aspects of [join, meet] completeness for certain types of skew lattices as well as certain related properties, which we define and explore in Section 3. In Section 4 we study join completeness in terms of $\mathcal{D}$-classes. In Section 5 we state and prove a correction of Theorem 4.4 of [5], where the assumption of join completeness was erroneously omitted. Theorem 5.1 states that if S is a join complete, strongly distributive skew lattice with 0, then S is a noncommutative frame if and only if its commutative shadow $S/\mathcal{D}$ is a frame. Examples 3.2 and 3.4 show that the assumption of join completeness is indeed necessary.

2 Preliminaries

A *skew lattice* is a set A endowed with a pair of idempotent, associative operations $\wedge$ and $\vee$ which satisfy the absorption laws:

$$x \wedge (x \vee y) = x = x \vee (x \wedge y) \text{ and } (x \wedge y) \vee y = y = (x \vee y) \wedge y.$$

The terms meet and join are still used for $\wedge$ and $\vee$, but without assuming commutativity. Given skew lattices A and B, a *homomorphism* of skew lattices is a map $f : A \to B$ that preserves finite meets and joins, i.e. it satisfies the following pair of axioms:

- $f(a \wedge b) = f(a) \wedge f(b)$, for all $a, b \in A$;
- $f(a \vee b) = f(a) \vee f(b)$, for all $a, b \in A$.

A *natural partial order* is defined on any skew lattice A by: $a \leq b$ iff $a \wedge b = b \wedge a = a$, or equivalently, $a \vee b = b = b \vee a$. The *Green's equivalence relation* $\mathcal{D}$ is defined on A by: $a\mathcal{D}b$ iff $a \wedge b \wedge a = a$ and $b \wedge a \wedge b = b$, or equivalently, $a \vee b \vee a = a$ and $b \vee a \vee b = b$. By Leech's First Decomposition Theorem [8], relation $\mathcal{D}$ is a congruence on a skew lattice A and $A/\mathcal{D}$ is a maximal lattice image of A, also referred to as the *commutative shadow* of A.

Skew lattices are always *regular* in that they satisfy the identities:

$$a \wedge x \wedge a \wedge y \wedge a = a \wedge x \wedge y \wedge a \text{ and } a \vee x \vee a \vee y \vee a = a \vee x \vee y \vee a.$$

The following result is an easy consequence of regularity.

Lemma 2.1. *Let a, b, u, v be elements of a skew lattice A such that $\mathcal{D}_u \leq \mathcal{D}_a$, $\mathcal{D}_u \leq \mathcal{D}_b$, $\mathcal{D}_a \leq \mathcal{D}_v$ and $\mathcal{D}_b \leq \mathcal{D}_v$. Then:*

1. $a \wedge v \wedge b = a \wedge b$,
2. $a \vee u \vee b = a \vee b$.

A skew lattice is *strongly distributive* if it satisfies the identities:

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) \text{ and } x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

By a result of Leech [10], a skew lattice is strongly distributive if and only if it is symmetric, distributive and normal, where a skew lattice A is called:

- *symmetric* if for any $x, y \in A$, $x \vee y = y \vee x$ iff $x \wedge y = y \wedge x$;
- *distributive* if it satisfies the identities:

$$x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$$
$$x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x);$$

- *normal* if it satisfies the identity $x \wedge y \wedge z \wedge x = x \wedge z \wedge y \wedge x$.

Further, it is shown in [10] that a skew lattice A is normal if and only if given any $a \in A$ the set

$$a\!\downarrow\; = \{u \in A \,|\, u \leq a\}$$

is a lattice. For this reason, normal skew lattices are sometimes called *local lattices*. Given any comparable $\mathcal{D}$-classes $D < C$ in a normal skew lattice A and any $c \in C$ there exist a unique $d \in D$ such that $d < c$ with respect to the natural partial order.

Finally, a *skew lattice with* 0 is a skew lattice with a distinguished element 0 satisfying $x \vee 0 = x = 0 \vee x$, or equivalently, $x \wedge 0 = 0 = 0 \wedge x$.

Example 2.2. Let A, B be non-empty sets and denote by $\mathcal{P}(A, B)$ the set of all partial functions from A to B. We define the following operations on $\mathcal{P}(A, B)$:

$$\begin{aligned} f \wedge g &= f|_{\mathrm{dom}(f) \cap \mathrm{dom}(g)} \\ f \vee g &= g \cup f|_{\mathrm{dom}(f) \setminus \mathrm{dom}(g)}. \end{aligned}$$

Leech [10] proved that $(\mathcal{P}(A, B); \wedge, \vee)$ is a strongly distributive skew lattice with 0. Moreover, given $f, g \in (\mathcal{P}(A, B); \wedge, \vee)$ the following hold:

- $f \,\mathcal{D}\, g$ iff $\mathrm{dom}(f) = \mathrm{dom}(g)$;
- $f \leq g$ iff $f = g|_{\mathrm{dom}(f) \cap \mathrm{dom}(g)}$;
- $\mathcal{P}(A, B)/\mathcal{D} \cong \mathcal{P}(A)$, the Boolean algebra of subsets of A;
- $\mathcal{P}(A, B)$ is left-handed in that $x \wedge y \wedge x = x \wedge y$ and dually, $x \vee y \vee x = y \vee x$ hold.

A *commuting subset* of a skew lattice A is a nonempty subset $\{x_i \,|\, i \in I\} \subseteq A$ such that $x_i \wedge x_j = x_j \wedge x_i$ and $x_i \vee x_j = x_j \vee x_i$ hold for all $i, j \in I$. The following result is a direct consequence of the definitions.

Lemma 2.3. *Let A and B be skew lattices, $f : A \to B$ be a homomorphism of skew lattices, and $\{x_i \,|\, i \in I\} \subseteq A$ be a commuting subset of A. Then $\{f(x_i) \,|\, i \in I\}$ is a commuting subset of B.*

A skew lattice is said to be *join [meet] complete* if all commuting subsets have suprema [infima] with respect to the natural partial ordering. By a result of Leech [9], the choice axiom implies that any join complete symmetric skew lattice has a top $\mathcal{D}$-class. If it occurs, we denote the top $\mathcal{D}$-class of a skew lattice A by T (or T_A). Dually, if A is a meet complete symmetric skew lattice, then it always has a bottom $\mathcal{D}$-class, denoted by B (or B_A).

A *frame* is a lattice that has all joins (finite and infinite), and satisfies the infinite distributive law:

$$x \wedge \bigvee_i y_i = \bigvee_i (x \wedge y_i).$$

A *noncommutative frame* is a strongly distributive, join complete skew lattice A with 0 that satisfies the infinite distributive laws:

$$(\bigvee_i x_i) \wedge y = \bigvee_i (x_i \wedge y) \qquad \text{and} \qquad x \wedge (\bigvee_i y_i) = \bigvee_i (x \wedge y_i) \tag{2.1}$$

for all $x, y \in A$ and all commuting subsets $\{x_i \mid i \in I\}, \{y_i \mid i \in I\} \subseteq A$.

By a result of Bignall and Leech [1], any join complete, normal skew lattice A with 0 (for instance, any noncommutative frame) satisfies the following:

- A is meet complete, with the meet of a commuting subset C denoted by $\bigwedge C$;
- any nonempty subset $C \subseteq A$ has an infimum with respect to the natural partial order, to be denoted by $\bigcap C$ (or by $x \cap y$ in the case $C = \{x, y\}$);
- if C is a nonempty commuting subset of A, then $\bigwedge C = \bigcap C$.

We call the $\bigcap C$ the *intersection* of C.

A *lattice section* L of a skew lattice S is a subalgebra that is a lattice (i.e. both $\wedge$ and $\vee$ are commutative on L) and that intersects each $\mathcal{D}$-class in exactly one element. When it exists, a lattice section is a maximal commuting subset and it is isomorphic to the maximal lattice image, as shown by Leech in [8]. If a normal skew lattice S has a top $\mathcal{D}$-class T then given $t \in T$, $t{\downarrow} = \{x \in S \mid x \leq t\}$ is a lattice section of S; moreover, all lattice sections are of the form $t{\downarrow}$ for some $t \in T$. Further, it is shown in [8] that any symmetric skew lattice S such that $S/\mathcal{D}$ is countable has a lattice section.

We say that a commuting subset C in a symmetric skew lattice S *extends to a lattice section* if there exists a lattice section L of C such that $C \subseteq L$.

3 Comparison of completeness properties

Let S be a normal, symmetric skew lattice. We will consider the following four properties that S might have:

(JC) S is join complete;

(BA) S is *bounded from above*, i.e. for every commuting subset C there is an element $s \in S$ such that $c \leq s$ for all $c \in C$;

(EX) every commuting subset extends to a lattice section;

(LS) there exists a lattice section.

Note that the last two properties are trivially satisfied if S is commutative.

Proposition 3.1. *For normal, symmetric skew lattices, the following implications hold:*

$$\textbf{(JC)} \Rightarrow \textbf{(BA)} \Rightarrow \textbf{(EX)} \Rightarrow \textbf{(LS)}.$$

Proof. We only prove **(BA)** $\Rightarrow$ **(EX)**, the other two implications are trivial. Take a normal, symmetric skew lattice S, such that every commutative subset has a join. Let $C \subseteq S$ be a commuting subset. We have to prove that C extends to a lattice section. For every chain $C_0 \subseteq C_1 \subseteq \dots$ of commuting subsets, the union $\bigcup_{i=0}^{\infty} C_i$ is again a commuting subset. So by Zorn's Lemma, C is contained in a maximal commuting subset C'. Take an element $s \in S$ such that $s \geq c$ for all $c \in C'$. Then $s\!\downarrow$ contains C' and it is a commuting subset because S is normal. By maximality, $C' = s\!\downarrow$. Again by maximality, s is a maximal element for the natural partial order on S. This also means that s is in the top $\mathcal{D}$-class (if $y \in S$ has a $\mathcal{D}$-class with $[y] \not\leq [s]$, then $s \vee y \vee s > s$, a contradiction). So C' is a lattice section. □

We claim that the converse implications do not hold in general. We will give a counterexample to all three of them. In each case, the counterexamples are strongly distributive skew lattices with 0.

Example 3.2 (**(BA)** $\not\Rightarrow$ **(JC)**)**.** Consider the set $S = \mathbb{N} \cup \{\infty_a, \infty_b\}$ and turn S into a skew lattice by setting

$$x \wedge y = \min(x, y) \qquad x \vee y = \max(x, y)$$

whenever x or y is in $\mathbb{N}$ (∞_a and ∞_b are both greater than every natural number), and

$$\infty_a \wedge \infty_b = \infty_a = \infty_b \vee \infty_a$$
$$\infty_b \wedge \infty_a = \infty_b = \infty_a \vee \infty_b.$$

Then S is a left-handed strongly distributive skew lattice with 0. The commuting subsets of S are precisely the subsets that do not contain both ∞_a and ∞_b. Clearly, S is bounded from above (as well as meet complete). However, the commuting subset $\mathbb{N} \subseteq S$ does not have a join.

Note that there are commutative examples as well, for example the real interval $[0, 1]$ with join and meet given by respectively maximum and minimum. The element 1 is an upper bound for every subset, but the lattice is not join complete. However, we preferred an example where the commutative shadow $S/\mathcal{D}$ is join complete.

Example 3.3 (**(EX)** $\not\Rightarrow$ **(BA)**)**.** Here we give a commutative example. Take $S = \mathbb{N}$ with the meet and join given by respectively the minimum and maximum of two elements. Then **(EX)** is satisfied, but **(BA)** does not hold.

If S satisfies **(EX)** and $S/\mathcal{D}$ is bounded from above, then for any commuting subset $C \subseteq S$ we can find a lattice section $L \supseteq C$ and an element $y \in L$ such that $[y] \geq [c]$ for all $c \in C$. It follows that $y \geq c$ for all $c \in C$, so S is bounded from above. So any example as the one above essentially reduces to a commutative example.

Example 3.4 (**(LS)** $\not\Rightarrow$ **(EX)**)**.** Consider the subalgebra S of $\mathcal{P}(\mathbb{N}, \mathbb{N})$ consisting of all partial functions with finite image sets in $\mathbb{N}$. Note that $S/\mathcal{D} = \mathcal{P}(\mathbb{N})$. The skew lattice S has lattice sections, for example the subalgebra of all functions in $\mathcal{P}(\mathbb{N}, \mathbb{N})$ whose image set is $\{1\}$. The set of 1-point functions $\{n \mapsto n \mid n \in \mathbb{N}\}$ is clearly a commuting subset, but it cannot be extended to an entire lattice section.

Even the weakest property **(LS)**, the existence a lattice section, does not always hold for strongly distributive skew lattices.

Example 3.5 (**(LS)** does not hold)**.** Let S be the subalgebra of $\mathcal{P}(\mathbb{R}, \mathbb{N})$ consisting of all partial functions f such that $f^{-1}(n)$ is finite for all $n \in \mathbb{N}$. In particular, if $f \in S$, then the domain of f is at most countable. Conversely, for any at most countable subset $U \subseteq \mathbb{R}$ we can construct an element $f \in S$ with domain U. Suppose now that $Q \subseteq S$ is a lattice section. Then there is an entire function $q : \mathbb{R} \to \mathbb{N}$ such that every $f \in Q$ can be written as a restriction $f = q|_U$ with $U \subseteq \mathbb{R}$ at most countable. Take $n \in \mathbb{N}$ such that $q^{-1}(n)$ is infinite, and take a countably infinite subset $V \subseteq q^{-1}(n)$. Then $q|_V \notin S$, by definition. But this shows that there is no element $f \in Q$ with domain V, which contradicts that Q is a lattice section.

By [8], any symmetric skew lattice S with $S/\mathcal{D}$ at most countable has a lattice section. This shows that in the above example it is necessary that the commutative shadow $S/\mathcal{D}$ is uncountable.

4 Join completeness in terms of $\mathcal{D}$-classes

Let S be a normal, symmetric skew lattice. Recall that for an element $a \in S$, we write its $\mathcal{D}$-class as $[a]$. For a $\mathcal{D}$-class $u \leq [a]$, the unique element b with $b \leq a$ and $[b] = u$ will be called the *restriction of a to u*. We will denote the restriction of a to u by $a|_u$. For $u, v \leq [a]$ two $\mathcal{D}$-classes, we calculate that

$$(a|_u)|_v = a|_v \qquad \text{if } v \leq u,$$

and in particular

$$a|_u \leq a|_v \quad \Leftrightarrow \quad u \leq v.$$

Proposition 4.1. *Let S be a normal, symmetric skew lattice and take a commuting subset $\{a_i : i \in I\} \subseteq S$. Then the following are equivalent:*

1. *the join $\bigvee_{i\in I} a_i$ exists;*
2. *the join $\bigvee_{i\in I}[a_i]$ exists and there is a unique $a \in S$ with $[a] = \bigvee_{i\in I}[a_i]$ and $a_i \leq a$ for all $i \in I$.*

In this case, $a = \bigvee_{i\in I} a_i$. In particular, $\left[\bigvee_{i\in I} a_i\right] = \bigvee_{i\in I}[a_i]$.

Proof. (1) $\Rightarrow$ (2). We claim that $\left[\bigvee_{i\in I} a_i\right]$ is the join of the $\mathcal{D}$-classes $[a_i]$. Because taking $\mathcal{D}$-classes preserves the natural partial order, $[a_i] \leq \left[\bigvee_{i\in I} a_i\right]$ for all $i \in I$. If $\left[\bigvee_{i\in I} a_i\right]$ is not the join of the $[a_i]$'s, then we can find a $\mathcal{D}$-class $u < \left[\bigvee_{i\in I} a_i\right]$ such that $[a_i] \leq u$ for all $i \in I$. But then

$$a_i \leq \left.\left(\bigvee_{i\in I} a_i\right)\right|_u < \bigvee_{i\in I} a_i$$

for all $i \in I$, a contradiction. So $\bigvee_{i\in I}[a_i]$ exists and is equal to $\left[\bigvee_{i\in I} a_i\right]$. For the remaining part of the statement, it is a straightforward calculation to show that $a = \bigvee_{i\in I} a_i$ is the unique element with the given properties.

(2) $\Rightarrow$ (1). Write $u = \bigvee_{i\in I}[a_i]$. Let $b \in S$ be an element such that $a_i \leq b$ for all $i \in I$. Then $u \leq [b]$ and $a_i \leq b|_u$ for all $i \in I$. It follows that $a = b|_u$, in particular $a \leq b$. So a is the join of the a_i's. □

Corollary 4.2. *Let S be a normal, symmetric skew lattice. Suppose that S is bounded from above and that $S/\mathcal{D}$ is join complete. If every two elements $a, b \in S$ have an infimum $a \cap b$ for the natural partial order, then S is join complete.*

Proof. Let $\{a_i : i \in I\} \subseteq S$ be a commuting subset. Because S is bounded from above, we can take an element $s \in S$ such that $a_i \leq s$ for all $i \in I$. Set $u = \bigvee_{i\in I}[a_i]$. By Proposition 4.1 it is enough to show that there is a unique $a \in S$ with $[a] = u$ and $a_i \leq a$ for all $i \in I$. Existence follows by taking the restriction $s|_u$. To show uniqueness, take two elements a and a' with $[a] = [a'] = u$ and $a_i \leq a$, $a_i \leq a'$ for all $i \in I$. It follows that $[a \cap a'] = \bigvee_{i\in I}[a_i] = u$. But this shows that $a = a \cap a' = a'$. □

In Example 3.2, the two elements ∞_a and ∞_b do not have an infimum.

5 Noncommutative frames

The following is a correction of a result in [5], where the assumption of being join complete was erroneously omitted.

Theorem 5.1. *Let S be a join complete, strongly distributive skew lattice with 0. Then S is a noncommutative frame if and only if $S/\mathcal{D}$ is a frame.*

Proof. Suppose that $S/\mathcal{D}$ is a frame. We prove the infinite distributivity laws (2.1). Take $x \in S$ and let $\{y_i : i \in I\} \subseteq S$ be a commuting subset. It is enough to show that

$$x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I} x \wedge y_i$$

(the proof for the other infinite distributivity law is analogous). Using that S is strongly distributive, it is easy to compute that $y \leq z$ implies $x \wedge y \leq x \wedge z$. In particular, $x \wedge y_i \leq x \wedge \bigvee_i y_i$ for all $i \in I$. This shows:

$$\bigvee_{i\in I} x \wedge y_i \leq x \wedge \bigvee_{i\in I} y_i. \tag{5.1}$$

Further, we can use Proposition 4.1 to compute

$$\left[\bigvee_{i\in I} x \wedge y_i\right] = \bigvee_{i\in I}[x] \wedge [y_i] = [x] \wedge \bigvee_{i\in I}[y_i] = \left[x \wedge \bigvee_{i\in I} y_i\right],$$

where for the middle equality we use that $S/\mathcal{D}$ is a frame. Since left- and right-hand side in (5.1) are in the same $\mathcal{D}$-class, the inequality must be an equality, so that S is seen to be a noncommutative frame. Conversely, suppose that S is a noncommutative frame. Then S has a maximal $\mathcal{D}$-class, T_S. Let t be in T_S. Then $t{\downarrow}$ is a copy of $S/\mathcal{D}$. □

The extra assumption that S is join complete is necessary: the strongly distributive skew lattices from Examples 3.2 and 3.4 have a frame as commutative shadow, but they are not noncommutative frames, since they are not join complete.

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