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On a certain class of 1-thin distance-regular graphs

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Abstract

Let Γ denote a non-bipartite distance-regular graph with vertex set X, diameter $D \geq 3$, and valency $k \geq 3$. Fix $x \in X$ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. For any $z \in X$ and for $0 \leq i \leq D$, let $\Gamma_i(z) = \{w \in X : \partial(z, w) = i\}$. For $y \in \Gamma_1(x)$, abbreviate $D^i_j = D^i_j(x, y) = \Gamma_i(x) \cap \Gamma_j(y)$ $(0 \leq i, j \leq D)$. For $1 \leq i \leq D$ and for a given y, we define maps $H_i \colon D^i_i \to \mathbb{Z}$ and $V_i \colon D^i_{i-1} \cup D^{i-1}_i \to \mathbb{Z}$ as follows:

$$H_i(z) = |\Gamma_1(z) \cap D^{i-1}_{i-1}|, \qquad V_i(z) = |\Gamma_1(z) \cap D^{i-1}_{i-1}|.$$

We assume that for every $y \in \Gamma_1(x)$ and for $2 \leq i \leq D$, the corresponding maps H_i and V_i are constant, and that these constants do not depend on the choice of y. We further assume that the constant value of H_i is nonzero for $2 \leq i \leq D$. We show that every irreducible T-module of endpoint 1 is thin. Furthermore, we show Γ has exactly three irreducible T-modules of endpoint 1, up to isomorphism, if and only if three certain combinatorial conditions hold. As examples, we show that the Johnson graphs $J(n, m)$ where $n \geq 7$, $3 \leq m < n/2$ satisfy all of these conditions.

Keywords: Distance-regular graph, Terwilliger algebra, subconstituent algebra.

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1 Introduction

This paper is motivated by a desire to find a combinatorial characterization of the distance-regular graphs with exactly three irreducible modules (up to isomorphism) of the Terwilliger algebra with endpoint 1, all of which are thin (see Sections 2, 3 for formal definitions). This is a difficult problem which we will not complete in this paper. To begin, we find combinatorial conditions under which a distance-regular graph is 1-thin. When these combinatorial conditions hold, we identify additional combinatorial conditions that hold if and only if the distance-regular graph has exactly three irreducible T-modules of endpoint 1, up to isomorphism.

Let Γ denote a distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Let X denote the vertex set of Γ. For $x \in X$, let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. It is known that there exists a unique irreducible T-module with endpoint 0, and this module is thin [5, Proposition 8.4]. It is also known that Γ is bipartite or almost-bipartite if and only if Γ has exactly one irreducible T-module of endpoint 1, up to isomorphism, and this module is thin [4, Theorem 1.3]. Furthermore, Curtin and Nomura have shown that Γ is pseudo-1-homogeneous with respect to x with $a_1 \neq 0$ if and only if Γ has exactly two irreducible T-modules of endpoint 1, up to isomorphism, both of which are thin [4, Theorem 1.6].

For any $z \in X$ and any integer $i \geq 0$, let $\Gamma_i(z) = \{w \in X : \partial(z, w) = i\}$. For $y \in \Gamma_1(x)$ and integers $i, j \geq 0$, abbreviate $D^i_j = D^i_j(x, y) = \Gamma_i(x) \cap \Gamma_j(y)$. For $1 \leq i \leq D$ and for a given y, we define maps $H_i \colon D^i_i \to \mathbb{Z}$, $K_i \colon D^i_i \to \mathbb{Z}$ and $V_i \colon D^i_{i-1} \cup D^{i-1}_i \to \mathbb{Z}$ as follows:

$$H_i(z) = |\Gamma_1(z) \cap D^{i-1}_{i-1}|, \qquad K_i(z) = |\Gamma_1(z) \cap D^{i+1}_{i+1}|, \quad V_i(z) = |\Gamma_1(z) \cap D^{i-1}_{i-1}|.$$

Our main result is the following.

Theorem 1.1. *Let $\Gamma = (X, \mathcal{R})$ denote a non-bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$, and fix vertex $x \in X$. Assume that for every $y \in \Gamma_1(x)$ and for $2 \leq i \leq D$, the corresponding maps H_i and V_i are constant, and that these constants do not depend on the choice of y. Also assume that the constant value of H_i is nonzero for $2 \leq i \leq D$. Then Γ is 1-thin with respect to x.*

We need the following definition.

Definition 1.2. With the assumptions of Theorem 1.1, for $y \in \Gamma_1(x)$ let $D^i_j = D^i_j(x, y)$ $(0 \leq i, j \leq D)$ and let K_1 denote the corresponding map. Let $B = B(y)$ denote the adjacency matrix of the subgraph of Γ induced on D^1_1. Observe that $B \in \mathrm{Mat}_{D^1_1}(\mathbb{C})$, and so the rows and the columns of B are indexed by the elements of D^1_1. Let $\boldsymbol{j} \in \mathbb{C}^{D^1_1}$ denote the all-ones column vector with rows indexed by the elements of D^1_1.

With reference to Definition 1.2, we denote by P1, P2 and P3 the following properties of Γ:

- P1: There exists $y \in \Gamma_1(x)$ such that K_1 is not a constant.
- P2: For every $y, z \in \Gamma_1(x)$ with $\partial(y, z) \in \{0, 2\}$, the number of walks of length 3 inside $\Gamma_1(x)$ from y to z is a constant number, which depends only on $\partial(y, z)$ (and not on the choice of y, z).

P3: There exist scalars α, β such that for every $y \in \Gamma_1(x)$ we have

$$B^2\boldsymbol{j} = \alpha B\boldsymbol{j} + \beta\boldsymbol{j}.$$

We prove the following.

Theorem 1.3. *With reference to Definition 1.2, Γ has exactly three irreducible T-modules of endpoint 1, up to isomorphism, if and only if properties P1, P2, and P3 hold. We note these three T-modules are all thin by Theorem 1.1.*

Finally, we show that the Johnson graphs $J(n, m)$ where $n \geq 7,\ 3 \leq m < n/2$ satisfy the assumptions in Theorem 1.1 and the equivalent conditions in Theorem 1.3.

2 Preliminaries

In this section we review some definitions and basic results concerning distance-regular graphs. See the book of Brouwer, Cohen and Neumaier [2] for more background information.

Let $\mathbb{C}$ denote the complex number field and let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module*. We endow V with the Hermitian inner product $\langle\, ,\, \rangle$ that satisfies $\langle u, v\rangle = u^t\overline{v}$ for $u, v \in V$, where t denotes transpose and $^-$ denotes complex conjugation. For $y \in X$ let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. We observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V. The following will be useful: for each $B \in \mathrm{Mat}_X(\mathbb{C})$ we have

$$\langle u, Bv\rangle = \langle \overline{B}^t u, v\rangle \qquad (u, v \in V). \tag{2.1}$$

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set $\mathcal{R}$. Let ∂ denote the path-length distance function for Γ, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call D the *diameter* of Γ. For a vertex $x \in X$ and an integer $i \geq 0$ let $\Gamma_i(x)$ denote the set of vertices at distance i from x. We abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$ we say Γ is *regular with valency* k whenever $|\Gamma(x)| = k$ for all $x \in X$. We say Γ is *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y. The p^h_{ij} are called the *intersection numbers* of Γ.

For the rest of this paper we assume Γ is distance-regular with diameter $D \geq 3$. Note that $p^h_{ij} = p^h_{ji}$ for $0 \leq h, i, j \leq D$. For convenience set $c_i := p^i_{1,i-1}$ $(1 \leq i \leq D)$, $a_i := p^i_{1i}$ $(0 \leq i \leq D)$, $b_i := p^i_{1,i+1}$ $(0 \leq i \leq D-1)$, $k_i := p^0_{ii}$ $(0 \leq i \leq D)$, and $c_0 = b_D = 0$. By the triangle inequality the following hold for $0 \leq h, i, j \leq D$: (i) $p^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two; (ii) $p^h_{ij} \neq 0$ if one of h, i, j equals the sum of the other two. In particular $c_i \neq 0$ for $1 \leq i \leq D$ and $b_i \neq 0$ for $0 \leq i \leq D-1$.

We observe that Γ is regular with valency $k = k_1 = b_0$ and that $c_i + a_i + b_i = k$ for $0 \le i \le D$. Note that $k_i = |\Gamma_i(x)|$ for $x \in X$ and $0 \le i \le D$.

We recall the Bose-Mesner algebra of Γ. For $0 \le i \le D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in X). \tag{2.2}$$

We call A_i the ith *distance matrix* of Γ. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. We observe (ai) $A_0 = I$; (aii) $\sum_{i=0}^{D} A_i = J$; (aiii) $\overline{A_i} = A_i$ $(0 \le i \le D)$; (aiv) $A_i^t = A_i$ $(0 \le i \le D)$; (av) $A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h$ $(0 \le i, j \le D)$, where I (resp. J) denotes the identity matrix (resp. all 1's matrix) in $\mathrm{Mat}_X(\mathbb{C})$. Using these facts we find $A_0, A_1, \ldots, A_D$ is a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$. We call M the *Bose-Mesner algebra* of Γ. It turns out that A generates M [1, p. 190]. By [2, p. 45], M has a second basis $E_0, E_1, \ldots, E_D$ such that (ei) $E_0 = |X|^{-1}J$; (eii) $\sum_{i=0}^{D} E_i = I$; (eiii) $\overline{E_i} = E_i$ $(0 \le i \le D)$; (eiv) $E_i^t = E_i$ $(0 \le i \le D)$; (ev) $E_i E_j = \delta_{ij} E_i$ $(0 \le i, j \le D)$. We call $E_0, E_1, \ldots, E_D$ the *primitive idempotents* of Γ.

3 The Terwilliger algebra

Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \ge 3$ and valency $k \ge 3$. In this section we recall the dual Bose-Mesner algebra and the Terwilliger algebra of Γ. Fix a vertex $x \in X$. We view x as a "base vertex." For $0 \le i \le D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (y \in X).$$

We call E_i^* the ith *dual idempotent* of Γ with respect to x [11, p. 378]. We observe (i) $\sum_{i=0}^{D} E_i^* = I$; (ii) $\overline{E_i^*} = E_i^*$ $(0 \le i \le D)$; (iii) $E_i^{*t} = E_i^*$ $(0 \le i \le D)$; (iv) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \le i, j \le D)$. By these facts $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose-Mesner algebra* of Γ with respect to x [11, p. 378]. For $0 \le i \le D$ we have

$$E_i^* V = \mathrm{span}\{\hat{y} \mid y \in \Gamma_i(x)\}$$

so $\dim E_i^* V = k_i$. We call $E_i^* V$ the ith *subconstituent of* Γ with respect to x. Note that

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \qquad \text{(orthogonal direct sum)}.$$

Moreover E_i^* is the projection from V onto $E_i^* V$ for $0 \le i \le D$.

We recall the Terwilliger algebra of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. We call T the *Terwilliger algebra of* Γ with respect to x [11, Definition 3.3]. Recall M (resp. M^*) is generated by A (resp. $E_0^*, E_1^*, \ldots, E_D^*$) so T is generated by $A, E_0^*, E_1^*, \ldots, E_D^*$. We observe T has finite dimension. By construction T is closed under the conjugate-transpose map so T is semi-simple [11, Lemma 3.4(i)].

By a *T-module* we mean a subspace W of V such that $SW \subseteq W$ for all $S \in T$. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no T-modules other than 0 and W.

By [6, Corollary 6.2] any T-module is an orthogonal direct sum of irreducible T-modules. In particular the standard module V is an orthogonal direct sum of irreducible T-modules. Let W, W' denote T-modules. By an *isomorphism of T-modules* from W to W' we mean an isomorphism of vector spaces $\sigma\colon W \to W'$ such that $(\sigma S - S\sigma)W = 0$ for all $S \in T$. The T-modules W, W' are said to be *isomorphic* whenever there exists an isomorphism of T-modules from W to W'. By [3, Lemma 3.3] any two non-isomorphic irreducible T-modules are orthogonal. Let W denote an irreducible T-module. By [11, Lemma 3.4(iii)] W is an orthogonal direct sum of the nonvanishing spaces among $E_0^*W, E_1^*W, \ldots, E_D^*W$. By the *endpoint* of W we mean $\min\{i \mid 0 \le i \le D,\ E_i^*W \ne 0\}$. By the *diameter* of W we mean $|\{i \mid 0 \le i \le D,\ E_i^*W \ne 0\}| - 1$. We say W is *thin* if $\dim(E_i^*W) \le 1$ for $0 \le i \le D$. We say Γ is 1-*thin* with respect to x if every T-module with endpoint 1 is thin.

By [5, Proposition 8.3, Proposition 8.4] $M\hat{x}$ is the unique irreducible T-module with endpoint 0 and the unique irreducible T-module with diameter D. Moreover $M\hat{x}$ is the unique irreducible T-module on which E_0 does not vanish. We call $M\hat{x}$ the *primary module*. We observe that vectors s_i $(0 \le i \le D)$ form a basis for $M\hat{x}$, where

$$s_i = \sum_{y \in \Gamma_i(x)} \hat{y}. \tag{3.1}$$

Lemma 3.1. *Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \ge 3$ and distance matrices A_i $(0 \le i \le D)$. Fix a vertex $x \in X$ and let $E_i^* = E_i^*(x)$ $(0 \le i \le D)$ denote the dual idempotents with respect to x. For $0 \le h, i, j \le D$, the matrix $E_h^*A_iE_j^* = 0$ whenever any one of h, i, j is bigger than the sum of the other two.*

Proof. Routine using elementary matrix multiplication. □

The following result will be crucial later in the paper.

Lemma 3.2. *Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \ge 3$. Fix a vertex $x \in X$ and let $E_i^* = E_i^*(x)$ $(0 \le i \le D)$ denote the dual idempotents with respect to x. Let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Assume that (up to isomorphism) Γ has exactly three irreducible T-modules with endpoint 1, and that these modules are all thin. Let F_1, F_2, F_3, F_4, $F_5 \in T$ and pick an integer i, $1 \le i \le D$. Then the matrices*

$$E_i^*F_1E_1^*,\ E_i^*F_2E_1^*,\ E_i^*F_3E_1^*,\ E_i^*F_4E_1^*, E_i^*F_5E_1^*$$

are linearly dependent.

Proof. Let V_0 denote the primary module of Γ, and let V_ℓ $(1 \le \ell \le 3)$ denote pairwise non-isomorphic irreducible T-modules with endpoint 1. Define vectors v_ℓ $(0 \le \ell \le 3)$ as follows. If $E_i^*V_\ell = 0$, then let $v_\ell = 0$. Otherwise, let v_ℓ be an arbitrary nonzero vector of $E_i^*V_\ell$. Furthermore, for $0 \le \ell \le 3$ fix a nonzero $u_\ell \in E_1^*V_\ell$. As modules V_ℓ $(0 \le \ell \le 3)$ are thin, there exist scalars λ_j^ℓ $(1 \le j \le 5,\ 0 \le \ell \le 3)$ such that

$$E_i^*F_jE_1^*u_\ell = \lambda_j^\ell v_\ell.$$

Consider now the following homogeneous system of linear equations:

$$\begin{pmatrix} \lambda_1^0 & \lambda_2^0 & \lambda_3^0 & \lambda_4^0 & \lambda_5^0 \\ \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 & \lambda_5^3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.2}$$

Observe that the above system has a nontrivial solution, and let $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)^t$ denote one of its nontrivial solutions. We will now show that $\sum_{j=1}^5 \mu_j E_i^* F_j E_1^* = 0$. First, pick an arbitrary $u \in E_1^* V_\ell$, for some ℓ ($0 \le \ell \le 3$). As module V_ℓ is thin, there exists a scalar λ, such that $u = \lambda u_\ell$. Now we have

$$\sum_{j=1}^5 \mu_j E_i^* F_j E_1^* u = \lambda \sum_{j=1}^5 \mu_j E_i^* F_j E_1^* u_\ell = \lambda \sum_{j=1}^5 \mu_j \lambda_j^\ell v_\ell = \lambda v_\ell \sum_{j=1}^5 \mu_j \lambda_j^\ell = 0. \tag{3.3}$$

Assume now that W is an irreducible T-module with endpoint 1 and note that W is isomorphic to V_ℓ for some $1 \le \ell \le 3$. Pick arbitrary $w \in E_1^* W$. Let $\sigma \colon V_\ell \mapsto W$ be a T-module isomorphism and let $u \in E_1^* V_\ell$ be such that $w = \sigma(u)$. Now by (3.3) we have that

$$\sum_{j=1}^5 \mu_j E_i^* F_j E_1^* w = \sum_{j=1}^5 \mu_j E_i^* F_j E_1^* \sigma(u) = \sigma\left(\sum_{j=1}^5 \mu_j E_i^* F_j E_1^* u \right) = 0. \tag{3.4}$$

For $1 \le \ell \le 3$ let $\mathcal{V}_\ell$ denote the sum of all irreducible T-modules with endpoint 1, which are isomorphic to V_ℓ. Observe that

$$E_1^* V = E_1^* V_0 + E_1^* \mathcal{V}_1 + E_1^* \mathcal{V}_2 + E_1^* \mathcal{V}_3 \qquad \text{(orthogonal sum)}. \tag{3.5}$$

Pick now an arbitrary $v \in E_1^* V$. Note that by (3.5) v is a sum of vectors v_ξ, where ξ belongs to some index set Ξ, and each v_ξ is contained in $E_1^* W_\xi$, where W_ξ is either V_0, or isomorphic to V_ℓ for some $1 \le \ell \le 3$. By (3.4) we have that $\sum_{j=1}^5 \mu_j E_i^* F_j E_1^* v_\xi = 0$ for each $\xi \in \Xi$, and consequently $\sum_{j=1}^5 \mu_j E_i^* F_j E_1^* v = 0$. This shows that $\sum_{j=1}^5 \mu_j E_i^* F_j E_1^* = 0$. As at least one of μ_j ($1 \le j \le 5$) is nonzero (recall that $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)^t$ is a nontrivial solution of (3.2)), the result follows. □

4 The local eigenvalues

In order to discuss the thin irreducible T-modules with endpoint 1, we first recall some parameters called the local eigenvalues. We will use the notation from [7].

Definition 4.1. Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \ge 3$, valency $k \ge 3$ and adjacency matrix A. Fix a vertex $x \in X$. We let $\Delta = \Delta(x)$ denote the graph $(\breve{X}, \breve{\mathcal{R}})$, where

$$\breve{X} = \{y \in X \mid \partial(x, y) = 1\},$$
$$\breve{\mathcal{R}} = \{yz \mid y, z \in \breve{X},\ \partial(y, z) = 1\}.$$

The graph Δ has exactly k vertices and is regular with valency a_1. We let $\breve{A}$ denote the adjacency matrix of Δ. The matrix $\breve{A}$ is symmetric with real entries, and thus $\breve{A}$ is diagonalizable with real eigenvalues. We let $\eta_1, \eta_2, \ldots, \eta_k$ denote the eigenvalues of $\breve{A}$. We call $\eta_1, \eta_2, \ldots, \eta_k$ the *local eigenvalues of* Γ *with respect to* x.

We now consider the first subconstituent E_1^*V. We recall the dimension of E_1^*V is k. Observe E_1^*V is invariant under the action of $E_1^*AE_1^*$. We note that for an appropriate ordering of the vertices of Γ, we have

$$E_1^*AE_1^* = \begin{pmatrix} \breve{A} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\breve{A}$ is from Definition 4.1. Hence the action of $E_1^*AE_1^*$ on E_1^*V is essentially the adjacency map for Δ. In particular the action of $E_1^*AE_1^*$ on E_1^*V is diagonalizable with eigenvalues $\eta_1, \eta_2, \ldots, \eta_k$. We observe the vector s_1 from (3.1) is contained in E_1^*V. One may easily show that s_1 is an eigenvector for $E_1^*AE_1^*$ with eigenvalue a_1. Reordering the eigenvalues if necessary, we have $\eta_1 = a_1$. For the rest of this paper, we assume the local eigenvalues are ordered in this way. Now consider the the orthogonal complement of s_1 in E_1^*V. By (2.1), this space is invariant under multiplication by $E_1^*AE_1^*$. Thus the restriction of the matrix $E_1^*AE_1^*$ to this space is diagonalizable with eigenvalues $\eta_2, \eta_3, \ldots, \eta_k$.

Definition 4.2. Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$, valency $k \geq 3$ and adjacency matrix A. Fix a vertex $x \in X$, and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Let W denote a thin irreducible T-module with endpoint 1. Observe E_1^*W is a 1-dimensional eigenspace for $E_1^*AE_1^*$; let η denote the corresponding eigenvalue. We observe E_1^*W is contained in E_1^*V so η is one of $\eta_2, \eta_3, \ldots, \eta_k$. We refer to η as the *local eigenvalue* of W.

Theorem 4.3 ([14, Theorem 12.1])**.** *Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Fix a vertex $x \in X$, and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Let W denote a thin irreducible T-module with endpoint 1 and local eigenvalue η. Let W' denote an irreducible T-module. Then the following (i), (ii) are equivalent.*

(i) W and W' are isomorphic as T-modules.

(ii) W' is thin with endpoint 1 and local eigenvalue η.

Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Fix a vertex $x \in X$, and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Recall that in Section 3, we said that the standard module V is an orthogonal direct sum of irreducible T-modules. Let W denote an irreducible T-module. By the *multiplicity* of W, we mean the number of irreducible T-modules in the above decomposition which are isomorphic to W. It is well-known that this number is independent of the decomposition of V.

Theorem 4.4 ([14, Theorem 12.9])**.** *Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Fix a vertex $x \in X$, and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. With reference to Definition 4.1, the following are equivalent.*

(i) *For every i $(2 \leq i \leq k)$, there exists a thin irreducible T-module W of endpoint 1 with local eigenvalue η_i. Moreover, the multiplicity with which η_i appears in the list $\eta_2, \eta_3, \ldots, \eta_k$ is equal to the multiplicity with which W appears in the standard decomposition of V.*

(ii) *Γ is 1-thin with respect to x.*

With reference to Theorem 4.4, we note that if Γ is 1-thin with respect to x, then the number of non-isomorphic irreducible T-modules of endpoint 1 is equal to the number of distinct local eigenvalues in the list $\eta_2, \eta_3, \ldots, \eta_k$. We will need this fact later in the paper.

5 The matrices L, F, R

Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$. Fix a vertex $x \in X$. In this section we recall certain matrices L, F, R of the Terwilliger algebra $T = T(x)$.

Definition 5.1. Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$ and adjacency matrix A. Fix a vertex $x \in X$ and let $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$ denote the dual idempotents with respect to x. We define matrices $L = L(x)$, $F = F(x)$, $R = R(x)$ by

$$L = \sum_{h=1}^{D} E_{h-1}^* A E_h^*, \qquad F = \sum_{h=0}^{D} E_h^* A E_h^*, \qquad R = \sum_{h=0}^{D-1} E_{h+1}^* A E_h^*.$$

Note that $A = L + F + R$ [3, Lemma 4.4]. We call L, F, and R the *lowering matrix*, the *flat matrix*, and the *raising matrix of* Γ with respect to x, respectively.

Lemma 5.2. *Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. We fix $x \in X$ and let $L = L(x)$, $F = F(x)$ and $R = R(x)$ be as in Definition 5.1. For $y, z \in X$ the following (i)–(iii) hold.*

(i) *$L_{zy} = 1$ if $\partial(z, y) = 1$ and $\partial(x, z) = \partial(x, y) - 1$, and 0 otherwise.*

(ii) *$F_{zy} = 1$ if $\partial(z, y) = 1$ and $\partial(x, z) = \partial(x, y)$, and 0 otherwise.*

(iii) *$R_{zy} = 1$ if $\partial(z, y) = 1$ and $\partial(x, z) = \partial(x, y) + 1$, and 0 otherwise.*

Proof. Immediate from Definition 5.1 and elementary matrix multiplication. □

With the notation of Lemma 5.2, we display the (z, y)-entry of certain products of the matrices L, F and R. To do this we need another definition.

A sequence of vertices $[y_0, y_1, \ldots, y_t]$ of Γ is a *walk* in Γ if $y_{i-1}y_i$ is an edge for $1 \leq i \leq t$.

Lemma 5.3. *Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. We fix $x \in X$ and let $L = L(x)$, $F = F(x)$ and $R = R(x)$ be as in Definition 5.1. Choose $y, z \in X$ and let m denote a positive integer. Assume that $y \in \Gamma_i(x)$. Then the following (i)–(vi) hold.*

(i) *The (z, y)-entry of R^m is equal to the number of walks $[y = y_0, y_1, \ldots, y_m = z]$, such that $y_j \in \Gamma_{i+j}(x)$ for $0 \leq j \leq m$.*

(ii) *The (z, y)-entry of $R^m L$ is equal to the number of walks $[y = y_0, y_1, \ldots, y_{m+1} = z]$, such that $y_j \in \Gamma_{i-2+j}(x)$ for $1 \leq j \leq m + 1$.*

(iii) *The (z, y)-entry of LR^m is equal to the number of walks $[y = y_0, y_1, \ldots, y_{m+1} = z]$, such that $y_j \in \Gamma_{i+j}(x)$ for $0 \le j \le m$ and $y_{m+1} \in \Gamma_{i+m-1}(x)$.*

(iv) *The (z, y)-entry of $R^m F$ is equal to the number of walks $[y = y_0, y_1, \ldots, y_{m+1} = z]$, such that $y_j \in \Gamma_{i-1+j}(x)$ for $1 \le j \le m+1$.*

(v) *The (z, y)-entry of FR^m is equal to the number of walks $[y = y_0, y_1, \ldots, y_{m+1} = z]$, such that $y_j \in \Gamma_{i+j}(x)$ for $0 \le j \le m$ and $y_{m+1} \in \Gamma_{i+m}(x)$.*

(vi) *The (z, y)-entry of F^m is equal to the number of walks $[y = y_0, y_1, \ldots, y_m = z]$, such that $y_j \in \Gamma_i(x)$ for $0 \le j \le m$.*

Proof. Immediate from Lemma 5.2 and elementary matrix multiplication. □

6 The sets D^i_j

Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \ge 3$. In this section we display a certain partition of X that we find useful.

Definition 6.1. Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \ge 3$ and valency $k \ge 3$. Pick $x \in X$ and $y \in \Gamma(x)$. For $0 \le i, j \le D$ we define $D^i_j = D^i_j(x, y)$ by

$$D^i_j = \Gamma_i(x) \cap \Gamma_j(y).$$

For notational convenience we set $D^i_j = \emptyset$ if i or j is contained in $\{-1, D+1\}$. Please refer to Figure 1 for a diagram of this partition.

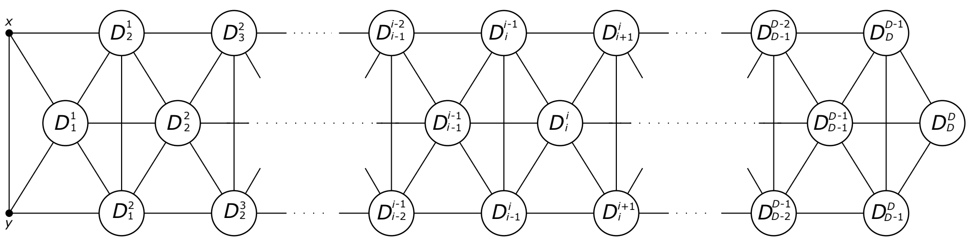

Figure 1: The partition with reference to Definition 6.1.

We now recall some properties of sets D^i_j.

Lemma 6.2 ([10, Lemma 4.2]). *With reference to Definition 6.1 the following (i), (ii) hold for $0 \le i, j \le D$.*

(i) $|D^i_j| = p^1_{ij}$.

(ii) $D^i_j = \emptyset$ *if and only if* $p^1_{ij} = 0$.

Observe that for $1 \le i \le D$ we have $p^1_{i,i-1} = c_i k_i / k \ne 0$ by [2, p. 134]. Therefore, D^i_{i-1} and D^{i-1}_i are nonempty for $1 \le i \le D$.

Lemma 6.3 ([9, Lemma 2.11]). *With reference to Definition 6.1 pick an integer i ($1 \le i \le D$). Then the following (i), (ii) hold.*

(i) Each $z \in D^i_{i-1}$ (resp. D^{i-1}_i) is adjacent to

(a) *precisely*	c_{i-1}	*vertices in* D^{i-1}_{i-2} *(resp.* D^{i-2}_{i-1}*),*
(b) *precisely*	$c_i - c_{i-1} - \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in* D^{i-1}_i *(resp.* D^i_{i-1}*),*
(c) *precisely*	$a_{i-1} - \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in* D^i_{i-1} *(resp.* D^{i-1}_i*),*
(d) *precisely*	b_i	*vertices in* D^{i+1}_i *(resp.* D^i_{i+1}*),*
(e) *precisely*	$a_i - a_{i-1} + \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in* D^i_i*.*

(ii) Each $z \in D^i_i$ is adjacent to

(a) *precisely*	$c_i - \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in* D^i_{i-1}*,*
(b) *precisely*	$c_i - \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in* D^{i-1}_i*,*
(c) *precisely*	$b_i - \lvert\Gamma(z) \cap D^{i+1}_{i+1}\rvert$	*vertices in* D^{i+1}_i*,*
(d) *precisely*	$b_i - \lvert\Gamma(z) \cap D^{i+1}_{i+1}\rvert$	*vertices in* D^i_{i+1}*,*
(e) *precisely*	$a_i - b_i - c_i + \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert + \lvert\Gamma(z) \cap D^{i+1}_{i+1}\rvert$	*vertices in* D^i_i*.*

In view of the above lemma we have the following definition.

Definition 6.4. With reference to Definition 6.1, for $1 \leq i \leq D$ we define maps $H_i\colon D^i_i \to \mathbb{Z}$, $K_i\colon D^i_i \to \mathbb{Z}$ and $V_i\colon D^i_{i-1} \cup D^{i-1}_i \to \mathbb{Z}$ as follows:

$$H_i(z) = |\Gamma(z) \cap D^{i-1}_{i-1}|, \qquad K_i(z) = |\Gamma(z) \cap D^{i+1}_{i+1}|, \quad V_i(z) = |\Gamma(z) \cap D^{i-1}_{i-1}|.$$

We have the following observation.

Lemma 6.5. *With reference to Definition 6.4, fix an integer i $(2 \leq i \leq D)$ and assume that there exist integers m_1, m_2, such that $V_i(z) = m_1$ for every $z \in D^{i-1}_i$ and $V_i(z) = m_2$ for every $z \in D^i_{i-1}$. Then $m_1 = m_2$.*

Proof. By Lemma 6.3(i) and using a simple double-counting argument we find that

$$|D^{i-1}_i|(c_i - c_{i-1} - m_1) = |D^i_{i-1}|(c_i - c_{i-1} - m_2).$$

As $|D^{i-1}_i| = |D^i_{i-1}| \neq 0$ by the comment below Lemma 6.2, the result follows. □

For the rest of the paper we assume the following situation.

Definition 6.6. Let $\Gamma = (X, \mathcal{R})$ denote a non-bipartite distance-regular graph with diameter $D \geq 3$, valency $k \geq 3$, and distance matrices A_i $(0 \leq i \leq D)$. We abbreviate $A := A_1$. Fix $x \in X$ and let $E^*_i = E^*_i(x)$ $(0 \leq i \leq D)$ denote the dual idempotents with respect to x. Let $T = T(x)$ denote the Terwilliger algebra with respect to x. Let $\Delta = \Delta(x)$ be as in Definition 4.1. Let matrices $L = L(x), F = F(x), R = R(x)$ be as defined in Definition 5.1. For $y \in \Gamma(x)$, let sets $D^i_j(x, y)$ $(0 \leq i, j \leq D)$ and the corresponding maps H_i, K_i, V_i $(1 \leq i \leq D)$ be as defined in Definition 6.1 and Definition 6.4. We assume that for every $y \in \Gamma(x)$ and for every $2 \leq i \leq D$, the corresponding maps H_i and V_i are constant, and that these constants do not depend on the choice of y. We denote the constant value of H_i (V_i, respectively) by h_i (v_i, respectively). We further assume that $h_i \neq 0$ for $2 \leq i \leq D$.

Remark 6.7. With reference to Definition 6.6, pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x, y)$ $(0 \leq i, j \leq D)$. Since Γ is assumed to be non-bipartite, $a_j \neq 0$ for some integer j $(1 \leq j \leq D)$. It follows that $D^j_j \neq \emptyset$ by Lemma 6.2(ii) and [2, p. 127]. But since each

$h_i \neq 0$ $(2 \le i \le D)$, we conclude each of sets $D^{j-1}_{j-1}, D^{j-2}_{j-2}, \ldots, D^1_1$ is nonempty. Since $D^1_1 \neq \emptyset$, we have $a_1 \neq 0$. Now by [2, Proposition 5.5.1], we find $a_i \neq 0$ for $1 \le i \le D-1$. Thus $D^i_i \neq \emptyset$ for $1 \le i \le D-1$. However, with our assumptions of Definition 6.6, it is possible that $a_D = 0$ and $D^D_D = \emptyset$. In this case, we make the convention that $h_D := 1$. Finally, we wish to make clear that while we are assuming the maps H_i and V_i are constant for $2 \le i \le D$, we are not making any such global assumptions about the maps K_i.

7 Some products in T

With reference to Definition 6.6, in this section we display the values of the entries of certain products in T.

Lemma 7.1. *With reference to Definition 6.6, pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$ $(0 \le i,j \le D)$. Pick an integer i $(1 \le i \le D)$, and let $z \in \Gamma_i(x)$. Then the following (i)–(iii) hold.*

(i) $$(R^{i-1})_{zy} = \begin{cases} c_{i-1}c_{i-2}\cdots c_1 & \text{if } z \in D^i_{i-1}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) $(R^iL)_{zy} = c_i c_{i-1} \cdots c_1$.

(iii) $$(LR^i)_{zy} = \begin{cases} b_i c_i c_{i-1} \cdots c_1 & \text{if } z \in D^i_{i-1}, \\ (b_i - K_i(z)) c_i c_{i-1} \cdots c_1 & \text{if } z \in D^i_i, \\ (c_{i+1} - c_i - v_{i+1}) c_i c_{i-1} \cdots c_1 & \text{if } z \in D^i_{i+1}. \end{cases}$$

Proof. First we observe that, by the triangle inequality, we have $\partial(y,z) \in \{i-1, i, i+1\}$.

(i): By Lemma 5.3(i), the (z,y)-entry of R^{i-1} is equal to the number of walks $[y = y_0, y_1, \ldots, y_{i-1} = z]$, such that $y_j \in \Gamma_{1+j}(x)$ for $0 \le j \le i-1$. Observe that there are no such walks if $\partial(y,z) \ge i$. If $\partial(y,z) = i-1$, then it is easy to see that $y_j \in \Gamma_{j+1}(x) \cap \Gamma_j(y) = D^{j+1}_j$ for $0 \le j \le i-1$. Lemma 6.3(i) now implies that the number of such walks is equal to $c_{i-1}c_{i-2}\cdots c_1$.

(ii): By Lemma 5.3(ii), the (z,y)-entry of R^iL is equal to the number of walks $[y = y_0, y_1, \ldots, y_{i+1} = z]$, such that $y_j \in \Gamma_{j-1}(x)$ for $1 \le j \le i+1$. Observe that this implies that $y_1 = x$. On the other hand, since $z \in \Gamma_i(x)$, there are $c_i c_{i-1} \cdots c_1$ walks $[x = y_1, y_2, \ldots, y_{i+1} = z]$, such that $y_j \in \Gamma_{j-1}(x)$ for $1 \le j \le i+1$. The result follows.

(iii): By Lemma 5.3(iii), the (z,y)-entry of LR^i is equal to the number of walks $[y = y_0, y_1, \ldots, y_{i+1} = z]$, such that $y_j \in \Gamma_{j+1}(x)$ for $0 \le j \le i$. It follows that $y_j \in D^{j+1}_j$ for $0 \le j \le i$. Furthermore, observe that by Lemma 6.3, z has exactly $c_{i+1} - c_i - v_{i+1}$ neighbours in D^{i+1}_i if $\partial(y,z) = i+1$ (that is, if $z \in D^i_{i+1}$), exactly $b_i - K_i(z)$ neighbours in D^{i+1}_i if $\partial(y,z) = i$ (that is, if $z \in D^i_i$), and exactly b_i neighbours in D^{i+1}_i if $\partial(y,z) = i-1$ (that is, if $z \in D^i_{i-1}$). Moreover, by Lemma 6.3(i), for any vertex $y_i \in D^{i+1}_i$, the number of walks $[y = y_0, y_1, \ldots, y_i]$, such that $y_j \in D^{j+1}_j$ for $0 \le j \le i$, is equal to $c_i c_{i-1} \cdots c_1$. The result follows. □

Lemma 7.2. *With reference to Definition 6.6, pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$ $(0 \le i,j \le D)$. Pick an integer i $(1 \le i \le D)$, and let $z \in \Gamma_i(x)$. Then the following (i), (ii) hold.*

$$\text{(i)}\ \ (R^{i-1}F)_{zy} = \begin{cases} \sum_{j=1}^{i-1} c_{i-1}c_{i-2}\cdots c_{j+1}v_{j+1}h_jh_{j-1}\cdots h_2 & \textit{if } z \in D^i_{i-1}, \\ h_ih_{i-1}\cdots h_2 & \textit{if } z \in D^i_i, \\ 0 & \textit{if } z \in D^i_{i+1}. \end{cases}$$

$$\text{(ii)}\ \ (FR^{i-1})_{zy} = \begin{cases} (a_{i-1}-v_i)c_{i-1}c_{i-2}\cdots c_1 & \textit{if } z \in D^i_{i-1}, \\ (c_i-h_i)c_{i-1}c_{i-2}\cdots c_1 & \textit{if } z \in D^i_i, \\ 0 & \textit{if } z \in D^i_{i+1}. \end{cases}$$

Proof. The proof is very similar to the proof of Lemma 7.1, so we omit the details. We only provide a sketch of the proof.

(i): We would like to count the number of walks of length $i-1$ from z to D^1_1. First, this number is 0 if $z \in D^i_{i+1}$. If $z \in D^i_i$, then this walk must pass through sets $D^{i-1}_{i-1}, D^{i-2}_{i-2}, \ldots, D^2_2$. Observe the number of such walks is equal to $h_ih_{i-1}\cdots h_2$. Finally, suppose $z \in D^i_{i-1}$. For any walk of length $i-1$ from z to D^1_1, there must exist some integer $1 \le j \le i-1$ such that this walk passes through sets $D^{i-1}_{i-2}, D^{i-2}_{i-3}, \ldots, D^{j+1}_j, D^j_j, D^{j-1}_{j-1}, \ldots, D^2_2, D^1_1$. By Lemma 6.3, the number of such walks (for a fixed j) is $c_{i-1}c_{i-2}\cdots c_{j+1}v_{j+1}h_jh_{j-1}\cdots h_2$. The result follows.

(ii): Here we note that z has 0 neighbours in D^i_{i-1} if $z \in D^i_{i+1}$, $c_i - h_i$ neighbours in D^i_{i-1} if $z \in D^i_i$, and $a_{i-1} - v_i$ neighbours in D^i_{i-1} if $z \in D^i_{i-1}$. Moreover, there are $c_{i-1}c_{i-2}\cdots c_1$ walks of length $i-1$ from each vertex of D^i_{i-1} to y. □

8 Proof of the main result

In this section we will prove our main result. With reference to Definition 6.6, we will show that Γ is 1-thin with respect to x.

Lemma 8.1. *With reference to Definition 6.6, fix an integer i ($1 \le i \le D$). Then there exist scalars λ_1, λ_2 such that*

$$E^*_i F R^{i-1} E^*_1 = \lambda_1 E^*_i R^{i-1} E^*_1 + \lambda_2 E^*_i R^{i-1} F E^*_1. \tag{8.1}$$

Proof. Let $z, y \in X$. We shall show the (z, y)-entry of both sides of (8.1) agree. Note that we may assume $z \in \Gamma_i(x), y \in \Gamma(x)$; otherwise the (z, y)-entry of both sides of (8.1) is zero. Let $D^\ell_j = D^\ell_j(x, y)$ ($0 \le \ell, j \le D$) and define scalars λ_1, λ_2 as follows:

$$\lambda_1 = a_{i-1} - v_i - \frac{(c_i - h_i)\sum_{j=1}^{i-1} c_{i-1}c_{i-2}\cdots c_{j+1}v_{j+1}h_jh_{j-1}\cdots h_2}{h_ih_{i-1}\cdots h_2},$$

$$\lambda_2 = \frac{(c_i - h_i)c_{i-1}c_{i-2}\cdots c_1}{h_ih_{i-1}\cdots h_2}.$$

Treating separately the cases where $z \in D^i_{i-1}, D^i_i, D^i_{i+1}$, it's now routine using Lemma 7.1(i) and Lemma 7.2 to check that the (z, y)-entry of both sides of (8.1) agree. □

Lemma 8.2. *With reference to Definition 6.6,*

$$E^*_i A_{i-1} E^*_1 = \frac{1}{c_1c_2\cdots c_{i-1}} E^*_i R^{i-1} E^*_1 \qquad (1 \le i \le D). \tag{8.2}$$

Proof. Let $z, y \in X$. Observe the (z, y)-entries of both sides of (8.2) are zero unless $z \in \Gamma_i(x), y \in \Gamma(x)$. When $z \in \Gamma_i(x), y \in \Gamma(x)$, the (z, y)-entries of both sides of (8.2) are equal by (2.2) and Lemma 7.1(i). The result follows. □

Lemma 8.3. *With reference to Definition 6.6, assume $v \in E_1^*V$ is an eigenvector for F. Then*

$$E_i^* A_i E_1^* v \in \operatorname{span}\{R^{i-1}v\} \qquad (1 \leq i \leq D). \tag{8.3}$$

Proof. We proceed by induction on i. For $i = 1$, the result is immediate since v is an eigenvector for F. Now assume the result is true for a fixed i, $1 \leq i \leq D - 1$. By [2, p. 127],

$$c_{i+1}A_{i+1} = AA_i - a_iA_i - b_{i-1}A_{i-1}.$$

Using this equation, Lemma 3.1, Definition 5.1, and Lemma 8.2, we find

$$\begin{aligned} c_{i+1}E_{i+1}^* A_{i+1} E_1^* v &= E_{i+1}^* A A_i E_1^* v - a_i E_{i+1}^* A_i E_1^* v \\ &= E_{i+1}^*(R + F + L) A_i E_1^* v - \frac{a_i}{c_1 c_2 \cdots c_i} E_{i+1}^* R^i E_1^* v \\ &= R E_i^* A_i E_1^* v + F E_{i+1}^* A_i E_1^* v - \frac{a_i}{c_1 c_2 \cdots c_i} E_{i+1}^* R^i E_1^* v. \end{aligned} \tag{8.4}$$

Observe $FE_{i+1}^* A_i E_1^* v = (c_1 c_2 \cdots c_i)^{-1} E_{i+1}^* F R^i E_1^* v$ by (8.2), and $E_{i+1}^* F R^i E_1^* v \in \operatorname{span}\{R^i v\}$ by Lemma 8.1 and the fact that v is an eigenvector for F. Using this information along with (8.4) and the inductive hypothesis, we find $E_{i+1}^* A_{i+1} E_1^* v \in \operatorname{span}\{R^i v\}$, as desired. □

Lemma 8.4. *With reference to Definition 6.6, let U denote the sum of all T-modules of endpoint 1. Assume $v \in E_1^*U$ is an eigenvector for F. Then $Lv = 0$ and $LR^i v \in \operatorname{span}\{R^{i-1}v\}$ for $1 \leq i \leq D - 1$.*

Proof. Since v is contained in a sum of irreducible T-modules of endpoint 1, we find $Lv = 0$. By [5, Propositions 8.3(ii), 8.4], the primary module is the unique irreducible T-module upon which J does not vanish. Thus $JE_1^* v = 0$, and for $1 \leq j \leq D - 1$,

$$\begin{aligned} 0 = E_j^* J E_1^* v &= E_j^* \Big(\sum_{t=0}^{D} A_t\Big) E_1^* v \\ &= E_j^* A_{j-1} E_1^* v + E_j^* A_j E_1^* v + E_j^* A_{j+1} E_1^* v. \end{aligned}$$

Thus $E_j^* A_{j+1} E_1^* v = -E_j^* A_{j-1} E_1^* v - E_j^* A_j E_1^* v$, and so by Lemma 8.2 and Lemma 8.3,

$$E_j^* A_{j+1} E_1^* v \in \operatorname{span}\{R^{j-1}v\} \qquad (1 \leq j \leq D - 1). \tag{8.5}$$

Now fix an integer i $(1 \leq i \leq D - 1)$. By [2, p. 127],

$$AA_i = c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1}.$$

Thus

$$E_i^* A A_i E_1^* v = c_{i+1} E_i^* A_{i+1} E_1^* v + a_i E_i^* A_i E_1^* v + b_{i-1} E_i^* A_{i-1} E_1^* v. \tag{8.6}$$

In view of (8.6), and using (8.5), (8.3), (8.2), we find

$$E_i^* A A_i E_1^* v \in \operatorname{span}\{R^{i-1}v\}. \tag{8.7}$$

Now using Definition 5.1 and (8.2),

$$\begin{aligned} E_i^* A A_i E_1^* v &= E_i^*(R+F+L) A_i E_1^* v \\ &= R E_{i-1}^* A_i E_1^* v + F E_i^* A_i E_1^* v + L E_{i+1}^* A_i E_1^* v \\ &= R E_{i-1}^* A_i E_1^* v + F E_i^* A_i E_1^* v + \frac{1}{c_1 c_2 \cdots c_i} L R^i v. \end{aligned}$$

Thus

$$L R^i v = c_1 c_2 \cdots c_i (E_i^* A A_i E_1^* v - R E_{i-1}^* A_i E_1^* v - F E_i^* A_i E_1^* v).$$

Recalling that v is an eigenvector for F, the result now follows from (8.7), (8.5), (8.3), (8.1). □

We now present our main result. With reference to Definition 6.6, let W denote an irreducible T-module of endpoint 1, and observe by Definition 5.1 that $F E_1^* W \subseteq E_1^* W$. Thus, there is a nonzero vector $v \in E_1^* W$ such that v is an eigenvector for F. We shall show W is thin.

Theorem 8.5. *With reference to Definition 6.6, let W denote an irreducible T-module with endpoint 1. Choose nonzero $v \in E_1^* W$ which is an eigenvector for F. Then the following set spans W:*

$$\{v, Rv, R^2 v, \ldots, R^{D-1} v\}. \tag{8.8}$$

In particular, W is thin.

Proof. We first show that W is spanned by the vectors in (8.8). Let W' denote the subspace of V spanned by the vectors in (8.8) and note that $W' \subseteq W$. We claim that W' is T-invariant. Observe that since $R E_j^* V \subseteq E_{j+1}^* V$ for $0 \le j \le D-1$, W' is invariant under the action of E_j^* for $0 \le j \le D$, and so W' is M^*-invariant. By definition and since $R E_D^* V = 0$, W' is invariant under R. From Lemma 8.1, Lemma 8.4, and the fact that v is an eigenvector for F, it follows that W' is also invariant under F and L. Since $A = R + F + L$ and since A generates M, W' is M-invariant. The claim follows. Hence W' is a T-module, and it is nonzero since $v \in W'$. By the irreducibility of W we have that $W' = W$. Since for $0 \le j \le D-1$ we have $R^j v \in E_{j+1}^* W$, it follows that W is thin. □

9 Special case – two modules with endpoint 1

With reference to Definition 6.6, in this section we consider the case where Γ has (up to isomorphism) exactly two irreducible T-modules with endpoint 1. Note that these modules are thin by Theorem 8.5. Observe that in this case it follows from the comments of Section 4 that the local graph $\Delta = \Delta(x)$ has either two or three distinct eigenvalues. In the former case Δ is a disjoint union of complete graphs (with order $a_1 + 1$), while in the latter case Δ is a strongly regular graph (see [8, Chapter 10, Lemma 1.5]). We observe that Δ has one of these two forms if and only if the map K_1 is constant for every $y \in \Gamma(x)$, and this constant does not depend on y.

Proposition 9.1. *With reference to Definition 6.6, assume that Δ is a disjoint union of $k/(a_1+1)$ cliques of order $a_1 + 1$. Let W denote an irreducible T-module with endpoint 1. Then W is thin with local eigenvalue a_1 or -1.*

Proof. Recall that W is thin by Theorem 8.5. Let η denote the local eigenvalue of W, and note that η is an eigenvalue of Δ by the comments of Section 4. But the eigenvalues of Δ are a_1 (with multiplicity $k/(a_1+1) > 1$) and -1 (with multiplicity $k - k/(a_1+1) = ka_1/(a_1+1)$). The result follows. $\square$

Proposition 9.2. *With reference to Definition 6.6, assume that Δ is a connected strongly regular graph with parameters (k, a_1, λ, v_2). Let W denote an irreducible T-module with endpoint 1. Then W is thin with local eigenvalue η_2 or η_3, where*

$$\eta_2, \eta_3 = \frac{\lambda - v_2 \pm \sqrt{(\lambda - v_2)^2 + 4(a_1 - v_2)}}{2}. \tag{9.1}$$

Proof. Recall that W is thin by Theorem 8.5. Let η denote the local eigenvalue of W, and recall that η is an eigenvalue of Δ. Therefore, by the well-known formula for the eigenvalues of a connected strongly regular graph, the eigenvalues of $\Gamma(x)$ are $\eta_1 = a_1$ (with multiplicity 1), and scalars η_2, η_3 from (9.1). The result follows. $\square$

Theorem 9.3. *With reference to Definition 6.6, assume that for every $y \in \Gamma(x)$ the map K_1 is constant, and that this constant does not depend on y. Then Γ has (up to isomorphism) exactly two irreducible T-modules with endpoint 1, both of which are thin. In particular, for every $1 \leq i \leq D-1$, the map K_i is constant, and this constant does not depend on y (in other words, Γ is pseudo-1-homogeneous with respect to x in the sense of Curtin and Nomura [4]).*

Proof. Recall that every irreducible T-module of Γ is thin by Theorem 8.5. Therefore, by Theorem 4.3, two irreducible T-modules with endpoint 1 are isomorphic if and only if they have the same local eigenvalue. As K_1 is constant and this constant does not depend on y, the local graph Δ is either a disjoint union of cliques of order a_1+1, or connected strongly regular graph. The first part of the above theorem now follows from Propositions 9.1 and 9.2. The second part follows from [4, Theorem 1.6]. $\square$

10 Special case – three modules with endpoint 1

With reference to Definition 6.6, in this section we consider the case where Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1. Note that these modules are thin by Theorem 8.5. It follows from the comments in Section 4 that this situation occurs if and only if the local graph Δ is either disconnected with exactly three distinct eigenvalues, or connected with exactly four distinct eigenvalues. Moreover, Δ is not connected if and only if $v_2 = 0$. But if $v_2 = 0$, then it is easy to see that Δ is a disjoint union of complete graphs (with order $a_1 + 1$), and has therefore 2 distinct eigenvalues. This shows that $v_2 \neq 0$, and so Δ is connected with exactly four distinct eigenvalues. To describe this case we need the following definition.

Definition 10.1. With reference to Definition 6.6, for $y \in \Gamma(x)$ let $B = B(y)$ denote the adjacency matrix of the subgraph of Γ induced on D_1^1. Observe that $B \in \mathrm{Mat}_{D_1^1}(\mathbb{C})$, and so the rows and the columns of B are indexed by the elements of D_1^1. Let $\boldsymbol{j} \in \mathbb{C}^{D_1^1}$ denote the all-ones column vector with rows indexed by the elements of D_1^1.

Lemma 10.2. *With reference to Definition 10.1, pick $y \in \Gamma(x)$. Then for every $z \in D_1^1$ we have*

$$K_1(z) = b_1 - a_1 + (B\boldsymbol{j})_z + 1.$$

Proof. Observe that $(B\boldsymbol{j})_z$ is equal to the number of neighbours that z has in D_1^1. Therefore, z has $a_1 - 1 - (B\boldsymbol{j})_z$ neighbours in D_2^1. But as z also has $K_1(z)$ neighbours in D_2^2 and no neighbours in D_2^3, it must have $b_1 - K_1(z)$ neighbours in D_2^1. The result follows. $\square$

With reference to Definition 10.1, we now describe three properties that Γ could have.

Definition 10.3. With reference to Definition 10.1, we denote by P1, P2 and P3 the following properties of Γ:

P1: There exists $y \in \Gamma(x)$ such that K_1 is not a constant.

P2: For every $y, z \in \Gamma(x)$ with $\partial(y,z) \in \{0,2\}$, the number of walks of length 3 from y to z in graph Δ is a constant number, which depends only on $\partial(y,z)$ (and not on the choice of y, z).

P3: There exist scalars α, β such that for every $y \in \Gamma(x)$ we have

$$B^2\boldsymbol{j} = \alpha B\boldsymbol{j} + \beta\boldsymbol{j}.$$

With reference to Definition 10.3, in the rest of this section we prove that Γ has properties P1, P2, P3 if and only if Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1.

Proposition 10.4. *With reference to Definition 10.3, assume that Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1. Then Γ has property P1.*

Proof. Assume on the contrary that K_1 is a constant for every $y \in \Gamma(x)$. We claim that this constant is independent of the choice of $y \in \Gamma(x)$. Pick $y \in \Gamma(x)$ and let $D_j^i = D_j^i(x,y)$. Denote the constant value of $K_1 = K_1(y)$ by $\kappa = \kappa(y)$. Observe that every vertex in D_2^1 has v_2 neighbours in D_1^1, and that every vertex in D_1^1 has $b_1 - \kappa$ neighbours in D_2^1. As $|D_2^1| = b_1$ and $|D_1^1| = a_1$, this gives us $a_1(b_1 - \kappa) = b_1 v_2$. This shows that κ is independent of the choice of $y \in \Gamma(x)$. By Theorem 9.3, Γ has up to isomorphism at most two irreducible modules with endpoint 1, a contradiction. This shows that Γ has property P1. $\square$

Lemma 10.5. *With reference to Definition 10.3, assume that Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1. Then*

$$E_1^* F^3 E_1^* = E_1^*\big(\mu_1 LR + \mu_2 RL + \mu_3 F + \mu_4 F^2\big)E_1^* \tag{10.1}$$

for some scalars μ_i $(1 \le i \le 4)$.

Proof. By Lemma 3.2, there exist scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, not all zero, such that

$$E_1^*\big(\lambda_1 LR + \lambda_2 RL + \lambda_3 F + \lambda_4 F^2 + \lambda_5 F^3\big)E_1^* = 0. \tag{10.2}$$

We claim that $\lambda_5 \neq 0$. Assume on the contrary that $\lambda_5 = 0$. By Proposition 10.4, there exists $y \in \Gamma(x)$ such that $K_1 = K_1(y)$ is not a constant. Pick such y and let $D_j^i = D_j^i(x,y)$. Let $z \in D_1^1$. We now compute the (z,y)-entry of (10.2). By Lemma 7.1(ii),(iii), the (z,y) entry of $E_1^* LR E_1^*$ ($E_1^* RL E_1^*$, respectively) is $b_1 - K_1(z)$ (1, respectively). By Lemma 5.3(vi), the (z,y)-entry of $E_1^* F E_1^*$ is 1, and the (z,y)-entry of $E_1^* F^2 E_1^*$ is equal to the number of neighbours of z in D^1_1. But by Lemma 10.2, the number of neighbours of z in D^1_1 is equal to $a_1 - 1 - b_1 + K_1(z)$. It follows from the above comments that

$$\lambda_1(b_1 - K_1(z)) + \lambda_2 + \lambda_3 + \lambda_4(a_1 - 1 - b_1 + K_1(z)) = 0.$$

Note that by the assumption the map K_1 is not constant, and so the above equality implies $\lambda_4 = \lambda_1$. Therefore $\lambda_1(a_1 - 1) + \lambda_2 + \lambda_3 = 0$.

We now compute the (y, y)-entry of (10.2). Similarly as above we get

$$\lambda_1(k - 1) + \lambda_2 = 0.$$

Finally, pick $z \in D^1_2$. By computing the (y, z)-entry of (10.2) we get

$$\lambda_1(c_2 - 1) + \lambda_2 = 0.$$

It follows easily from the above equations that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, a contradiction. This shows that $\lambda_5 \neq 0$ and so

$$E^*_1 F^3 E^*_1 = E^*_1\big(\mu_1 LR + \mu_2 RL + \mu_3 F + \mu_4 F^2\big)E^*_1,$$

where $\mu_i = -\lambda_i/\lambda_5$ for $1 \le i \le 4$. □

Theorem 10.6. *With reference to Definition 10.3, assume that Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1. Then Γ has properties P2 and P3.*

Proof. Note that for every $y, z \in \Gamma(x)$, the (z, y)-entry of $E^*_1 F^3 E^*_1$ is equal to the number of walks of length 3 from y to z in graph Δ. Pick $y, z \in \Gamma(x)$ such that $\partial(y, z) \in \{0, 2\}$. We compute the (z, y)-entry of (10.1). Using Lemma 5.3(vi) and Lemma 7.1(ii),(iii) we find that

$$(E^*_i F^3 E^*_1)_{zy} = \begin{cases} \mu_1 b_1 + \mu_2 + \mu_4 a_1 & \text{if } z = y, \\ \mu_1(c_2 - v_2 - 1) + \mu_2 + \mu_4 v_2 & \text{if } z \neq y. \end{cases}$$

This shows that Γ has property P2.

Pick now $y, z \in \Gamma(x)$ such that $\partial(y, z) = 1$ and let $D^i_j = D^i_j(x, y)$. Let K_1 denote the corresponding map, and let $B = B(y)$. Let $[y = y_0, y_1, y_2, y_3 = z]$ be a walk of length 3 from y to z in Δ. We will say that this walk is of type 0 if $y_2 = y$, of type 1 if $y_2 \in D^1_1$, and of type 2 if $y_2 \in D^1_2$. It is clear that we have a_1 walks of type 0 and $(a_1 - 1 - (B\boldsymbol{j})_z)v_2$ walks of type 2. Similarly, there are $(B^2\boldsymbol{j})_z$ walks of type 1. So there are in total

$$a_1 + (a_1 - 1 - (B\boldsymbol{j})_z)v_2 + (B^2\boldsymbol{j})_z$$

walks of length 3 from y to z in Δ.

We now compute the (z, y)-entry of the right side of (10.1). Using Lemma 7.1(iii) and Lemma 10.2, we find that the (z, y)-entry of $E^*_1 LR E^*_1$ is equal to

$$b_1 - K_1(z) = a_1 - (B\boldsymbol{j})_z - 1.$$

It is easy to see that the (z, y)-entries of $E^*_1 RL E^*_1$ and $E^*_1 F E^*_1$ are both equal to 1. Finally, the (z, y)-entry of $E^*_1 F^2 E^*_1$ is equal to the number of neighbours of z in D^1_1, that is to $(B\boldsymbol{j})_z$. It now follows from the above comments that

$$a_1 + (a_1 - 1 - (B\boldsymbol{j})_z)v_2 + (B^2\boldsymbol{j})_z = \mu_1(a_1 - (B\boldsymbol{j})_z - 1) + \mu_2 + \mu_3 + \mu_4(B\boldsymbol{j})_z.$$

This shows that

$$(B^2\boldsymbol{j})_z = \alpha(B\boldsymbol{j})_z + \beta$$

for some scalars α, β, which are independent of the choice of vertices y, z. This proves that Γ has property P3. $\square$

We now assume that Γ has properties P1, P2 and P3. We will show that this implies that Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1.

Definition 10.7. With reference to Definition 10.3, assume that Γ has properties P1, P2 and P3, and recall that $\breve{X} = \Gamma(x)$. Recall also that for any $y, z \in \breve{X}$ with $\partial(y, z) \in \{0, 2\}$, the number of walks of length 3 from y to z in Δ is a constant number, which depends just on the distance between y and z. We denote this number by w_0 if $y = z$ and by w_2 if $\partial(y, z) = 2$. Recall that $\breve{A} = \breve{A}(x) \in \mathrm{Mat}_{\breve{X}}(\mathbb{C})$ denotes the adjacency matrix of Δ. Furthermore, let $\breve{I}$ denote the identity matrix of $\mathrm{Mat}_{\breve{X}}(\mathbb{C})$ and let $\breve{J}$ denote the all-ones matrix of $\mathrm{Mat}_{\breve{X}}(\mathbb{C})$.

We now display the entries of $\breve{A}$, $\breve{A}^2$ and $\breve{A}^3$.

Proposition 10.8. *With reference to Definition 10.7, the following (i)–(iii) hold for all $z, y \in \breve{X}$.*

(i)

$$(\breve{A})_{zy} = \begin{cases} 1 & \text{if } \partial(y, z) = 1, \\ 0 & \text{otherwise}. \end{cases}$$

(ii)

$$(\breve{A}^2)_{zy} = \begin{cases} a_1 & \text{if } y = z, \\ (B\boldsymbol{j})_z & \text{if } \partial(y, z) = 1, \\ v_2 & \text{if } \partial(y, z) = 2, \end{cases}$$

where $B = B(y)$.

(iii)

$$(\breve{A}^3)_{zy} = \begin{cases} w_0 & \text{if } y = z, \\ a_1 + v_2(a_1 - 1) + (B\boldsymbol{j})_z(\alpha - v_2) + \beta & \text{if } \partial(y, z) = 1, \\ w_2 & \text{if } \partial(y, z) = 2, \end{cases}$$

where $B = B(y)$ and α, β are from Definition 10.3.

Proof. Recall that for $i \geq 0$, the (z, y)-entry of $\breve{A}^i$ is equal to the number of walks of length i from y to z in Δ. Parts (i), (ii) follow. We now prove part (iii).

Note that the result is clear if $y = z$ or if $\partial(y, z) = 2$. Therefore, assume $\partial(y, z) = 1$. Similarly as in the proof of Theorem 10.6, we split the walks of length 3 between y and z into three types, depending on whether the third vertex of the walk is equal to y, or is a neighbour of y, or is at distance 2 from y. There are a_1 walks of the first type, $(B^2\boldsymbol{j})_z$ walks of the second type, and $(a_1 - 1 - (B\boldsymbol{j})_z)v_2$ walks of the third type. Recall that by property P3 we have $B^2\boldsymbol{j} = \alpha B\boldsymbol{j} + \beta\boldsymbol{j}$, and so the result follows. $\square$

Proposition 10.9. *With reference to Definition 10.7, we have*

$$\breve{A}^3 = (\alpha - v_2)\breve{A}^2 + (a_1 + \beta + v_2(a_1 - 1 + \alpha - v_2) - w_2)\breve{A} \\ + (w_0 - w_2 + (\alpha - v_2)(v_2 - a_1))\breve{I} + (w_2 - (\alpha - v_2)v_2)\breve{J}, \tag{10.3}$$

where α, β are from Definition 10.3.

Proof. Pick $y, z \in \breve{X}$. It follows from Proposition 10.8 that the (z, y)-entry of the left side and the right side of (10.3) agree. This proves the proposition. □

Theorem 10.10. *With reference to Definition 10.7, Δ has exactly four distinct eigenvalues.*

Proof. Observe that Δ is connected and regular with valency a_1, so a_1 is an eigenvalue of Δ with multiplicity 1. The corresponding eigenvector is the all-ones vector in $\mathbb{C}^{\breve{X}}$, which we denote by $\breve{\boldsymbol{j}}$. Let θ denote an eigenvalue of Δ which is different from a_1, and let $\boldsymbol{w}$ denote a corresponding eigenvector. Note that $\boldsymbol{w}$ and $\breve{\boldsymbol{j}}$ are orthogonal, and so applying (10.3) to $\boldsymbol{w}$ we get

$$\theta^3\boldsymbol{w} = (\alpha - v_2)\theta^2\boldsymbol{w} + (a_1 + \beta + v_2(a_1 - 1 + \alpha - v_2) - w_2)\theta\boldsymbol{w} \\ + (w_0 - w_2 + (\alpha - v_2)(v_2 - a_1))\boldsymbol{w}.$$

As $\boldsymbol{w}$ is nonzero, we have

$$\theta^3 = (\alpha - v_2)\theta^2 + (a_1 + \beta + v_2(a_1 - 1 + \alpha - v_2) - w_2)\theta + w_0 - w_2 + (\alpha - v_2)(v_2 - a_1).$$

This shows that Δ could have at most four different eigenvalues. Now if Δ has fewer than four different eigenvalues, then Δ is strongly regular [8, Chapter 10, Lemma 1.5], and so $(B\boldsymbol{j})_z$ is constant for every $y, z \in \breve{X}$ with $z \in \Gamma(y)$, where $B = B(y)$ and $\boldsymbol{j}$ is from Definition 10.1. By Lemma 10.2, K_1 is constant for every $y \in \breve{X}$, contradicting property P1. □

Theorem 10.11. *With reference to Definition 10.7, Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint* 1.

Proof. Recall that Γ is 1-thin with respect to x by Theorem 8.5. The result now follows from Theorems 4.3, 4.4, and 10.10. □

11 Example: Johnson graphs

Pick a positive integer $n \geq 2$ and let m denote an integer ($0 \leq m \leq n$). The vertices of the Johnson graph $J(n, m)$ are the m-element subsets of $\{1, 2, \ldots, n\}$. Vertices x, y are adjacent if and only if the cardinality of $x \cap y$ is equal to $m - 1$. It follows that if x, y are arbitrary vertices of $J(n, m)$, then $\partial(x, y) = m - |x \cap y|$. Therefore, the diameter D of $J(n, m)$ is equal to $\min\{m, n - m\}$. Recall that $J(n, m)$ is distance-transitive (see [2, Theorem 9.1.2]), and so it is also distance-regular. It is well known that $J(n, m)$ is isomorphic to $J(n, n - m)$, so we will assume that $m \leq n/2$, which implies $D = m$. In fact, if n is even and $m = n/2$, then $J(2m, m)$ is 1-homogeneous (see [9]), and so we assume from here on that $m < n/2$. As we are also assuming that $D \geq 3$, we therefore have $m \geq 3$, $n \geq 7$. For more details on Johnson graphs, see [2, Section 9.1].

Pick adjacent vertices x, y of $J(n,m)$, and let $D^i_j = D^i_j(x,y)$ be as defined in Definition 6.1. For $1 \le i \le D$ let maps H_i, K_i and V_i be as defined in Definition 6.4. The main purposes of this section are to describe maps H_i, K_i and V_i in detail and to show $J(n,m)$ satisfies the assumptions of Definitions 6.6 and 10.7. As $J(n,m)$ is distance-transitive, it is also arc-transitive, and so we can assume that $x = \{1,2,\dots,m\}$, $y = \{2,3,\dots,m+1\}$. We start with a description of the sets D^i_j.

Proposition 11.1. *Pick positive integers n and m with $n \ge 7$, $3 \le m < n/2$, and let $x = \{1,2,\dots,m\}$, $y = \{2,3,\dots,m+1\}$ be adjacent vertices of $J(n,m)$. Let $D^i_j = D^i_j(x,y)$ be as defined in Definition 6.1. Then for $1 \le i \le D$, the set D^{i-1}_i (D^i_{i-1}, respectively) consists of vertices of the form $\{1\} \cup A \cup B$ ($\{m+1\} \cup A \cup B$, respectively), where $A \subseteq \{2,3,\dots,m\}$ with $|A| = m-i$ and $B \subseteq \{m+2,m+3,\dots,n\}$ with $|B| = i-1$.*

Proof. Routine. □

To describe sets D^i_i, we need the following definition.

Definition 11.2. Pick positive integers n and m with $n \ge 7$, $3 \le m < n/2$, and let $x = \{1,2,\dots,m\}$, $y = \{2,3,\dots,m+1\}$ be adjacent vertices of $J(n,m)$.

(i) For $1 \le i \le D-1$, define set $D^i_i(0)$ to be the set of vertices of the form $\{1,m+1\} \cup A \cup B$, where $A \subseteq \{2,3,\dots,m\}$ with $|A| = m-i-1$ and $B \subseteq \{m+2,m+3,\dots,n\}$ with $|B| = i-1$. We define $D^0_0(0) = D^D_D(0) = \emptyset$.

(ii) For $1 \le i \le D$, define set $D^i_i(1)$ to be the set of vertices of the form $A \cup B$, where $A \subseteq \{2,3,\dots,m\}$ with $|A| = m-i$, and $B \subseteq \{m+2,m+3,\dots,n\}$ with $|B| = i$. We define $D^0_0(1) = \emptyset$.

Please refer to Figure 2 for a diagram of this partition.

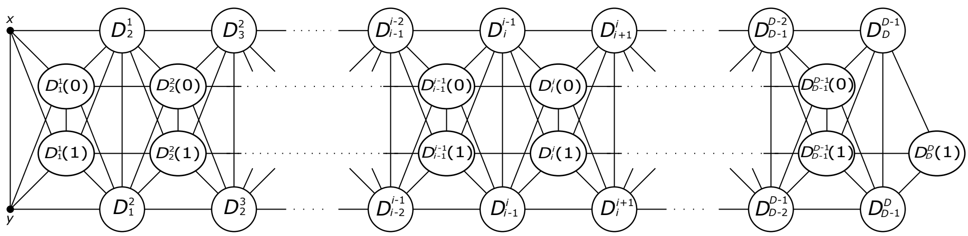

Figure 2: The partition with reference to Definition 11.2. For further information about which sets in the diagram are connected by edges, please refer to the propositions and corollaries later in this section.

Proposition 11.3. *Pick positive integers n and m with $n \ge 7$, $3 \le m < n/2$, and let $x = \{1,2,\dots,m\}$, $y = \{2,3,\dots,m+1\}$ be adjacent vertices of $J(n,m)$. Let $D^i_j = D^i_j(x,y)$ be as defined in Definition 6.1 and let $D^i_i(0)$, $D^i_i(1)$ be as in Definition 11.2. Then for $1 \le i \le D-1$ we have that D^i_i is a disjoint union of $D^i_i(0)$ and $D^i_i(1)$. Moreover, $D^D_D = D^D_D(1)$.*

Proof. Routine. □

We now first describe the maps V_i.

Proposition 11.4. *With the notation of Proposition 11.3, let the maps V_i be as defined in Definition 6.4. Then for $1 \le i \le D$ and any $z \in D_{i-1}^i \cup D_i^{i-1}$ we have*

$$V_i(z) = 2(i-1).$$

In particular, the maps V_i are constant.

Proof. Note that the result is clear for $i = 1$, so pick $2 \le i \le D$ and assume $z \in D_i^{i-1}$ (case $z \in D_{i-1}^i$ is treated similarly and we omit the details). First recall that by the definition of map V_i and by Proposition 11.3 we have

$$V_i(z) = |\Gamma(z) \cap D_{i-1}^{i-1}| = |\Gamma(z) \cap D_{i-1}^{i-1}(0)| + |\Gamma(z) \cap D_{i-1}^{i-1}(1)|.$$

Recall also that by Proposition 11.1 there exist subsets $A \subseteq \{2, 3, \dots, m\}$ with $|A| = m-i$ and $B \subseteq \{m+2, m+3, \dots, n\}$ with $|B| = i-1$, such that $z = \{1\} \cup A \cup B$. We first count the number of neighbours of z in $D_{i-1}^{i-1}(1)$. As vertices contained in $D_{i-1}^{i-1}(1)$ do not contain the number 1 as an element, vertex $w \in D_{i-1}^{i-1}(1)$ will be adjacent with z if and only if

$$w = A \cup B \cup \{\ell\}$$

for some $\ell \in \{2, 3, \dots, m\} \setminus A$. Therefore, there are exactly $m - 1 - (m - i) = i - 1$ neighbours of z in $D_{i-1}^{i-1}(1)$. We now count the number of neighbours of z in $D_{i-1}^{i-1}(0)$. As vertices contained in $D_{i-1}^{i-1}(0)$ contain numbers 1 and $m+1$ as elements, vertex $w \in D_{i-1}^{i-1}(0)$ will be adjacent with z if and only if

$$w = (\{1, m+1\} \cup A \cup B) \setminus \{\ell\}$$

for some $\ell \in B$. Therefore, there are exactly $i-1$ neighbours of z in $D_{i-1}^{i-1}(0)$. The result follows. □

Proposition 11.5. *With the notation of Proposition 11.3, for $1 \le i \le D-1$ and for any $z \in D_i^i(0)$ the following (i), (ii) hold.*

(i) $|\Gamma(z) \cap D_{i-1}^{i-1}(0)| = i(i-1)$.

(ii) $|\Gamma(z) \cap D_{i-1}^{i-1}(1)| = 0$.

Proof. Note that the result is clear for $i = 1$, so pick $2 \le i \le D-1$ and $z \in D_i^i(0)$. Recall that $z = \{1, m+1\} \cup A \cup B$ for some subsets $A \subseteq \{2, 3, \dots, m\}$ with $|A| = m - i - 1$ and $B \subseteq \{m+2, m+3, \dots, n\}$ with $|B| = i-1$.

(i): Note that $w \in D_{i-1}^{i-1}(0)$ is adjacent with z if and only if $w = \{1, m+1\} \cup A' \cup B'$, where $A' = A \cup \{\ell_1\}$ for some $\ell_1 \in \{2, 3, \dots, m\} \setminus A$ and $B' = B \setminus \{\ell_2\}$ for some $\ell_2 \in B$. We have $m - 1 - (m - i - 1) = i$ choices for ℓ_1 and $i - 1$ choices for ℓ_2. It follows that z has $i(i-1)$ neighbours in $D_{i-1}^{i-1}(0)$.

(ii): Recall that if w is an element of $D_{i-1}^{i-1}(1)$, then 1 and $m+1$ are not elements of w. On the other hand, 1 and $m+1$ are elements of z, and so z and w are not adjacent. □

Proposition 11.6. *With the notation of Proposition 11.3, for $1 \le i \le D$ and for any $z \in D_i^i(1)$ the following (i), (ii) hold.*

(i) $|\Gamma(z) \cap D_{i-1}^{i-1}(1)| = i(i-1)$.

(ii) $|\Gamma(z) \cap D_{i-1}^{i-1}(0)| = 0$.

Proof. Similar to the proof of Proposition 11.5. □

Corollary 11.7. *With the notation of Proposition 11.3, let the maps H_i be as defined in Definition 6.4. Then for $1 \leq i \leq D$ and any $z \in D_i^i$ we have*

$$H_i(z) = i(i-1).$$

In particular, the maps H_i are constant.

Proof. Immediate from Propositions 11.5 and 11.6 and since D_i^i is a disjoint union of $D_i^i(0)$ and $D_i^i(1)$. □

Proposition 11.8. *With the notation of Proposition 11.3, for $1 \leq i \leq D-1$ and for any $z \in D_i^i(0)$ the following (i), (ii) hold.*

(i) $|\Gamma(z) \cap D_{i+1}^{i+1}(0)| = (m-i-1)(n-m-i)$.

(ii) $|\Gamma(z) \cap D_{i+1}^{i+1}(1)| = 0$.

Proof. Pick $1 \leq i \leq D-1$ and $z \in D_i^i(0)$. Recall that $z = \{1, m+1\} \cup A \cup B$ for some subsets $A \subseteq \{2, 3, \ldots, m\}$ with $|A| = m-i-1$ and $B \subseteq \{m+2, m+3, \ldots, n\}$ with $|B| = i-1$.

(i): Note that $w \in D_{i+1}^{i+1}(0)$ is adjacent with z if and only if $w = \{1, m+1\} \cup A' \cup B'$, where $A' = A \setminus \{\ell_1\}$ for some $\ell_1 \in A$ and $B' = B \cup \{\ell_2\}$ for some $\ell_2 \in \{m+2, m+3, \ldots, n\} \setminus B$. We therefore have $m-i-1$ choices for ℓ_1 and $(n-m-1)-(i-1) = n-m-i$ choices for ℓ_2. It follows that z has $(m-i-1)(n-m-i)$ neighbours in $D_{i+1}^{i+1}(0)$.

(ii): Immediate from Proposition 11.6(ii). □

Proposition 11.9. *With the notation of Proposition 11.3, for $1 \leq i \leq D-1$ and for any $z \in D_i^i(1)$ the following (i), (ii) hold.*

(i) $|\Gamma(z) \cap D_{i+1}^{i+1}(1)| = (m-i)(n-m-i-1)$.

(ii) $|\Gamma(z) \cap D_{i+1}^{i+1}(0)| = 0$.

Proof. Similar to the proof of Proposition 11.8. □

Corollary 11.10. *With the notation of Proposition 11.3, let the maps K_i be as defined in Definition 6.4. Then for $1 \leq i \leq D-1$ and any $z \in D_i^i$ we have*

$$K_i(z) = \begin{cases} (m-i-1)(n-m-i) & \text{if } z \in D_i^i(0), \\ (m-i)(n-m-i-1) & \text{if } z \in D_i^i(1). \end{cases}$$

In particular, maps K_i are not constant.

Proof. The first part of the corollary follows immediately from Propositions 11.8 and 11.9 and since D_i^i is a disjoint union of $D_i^i(0)$ and $D_i^i(1)$. For the second part, observe that if K_i is a constant, then we have $n = 2m$, contradicting our assumption $m < n/2$. □

Proposition 11.11. *With the notation of Proposition 11.3, the following (i)–(iii) hold.*

(i) *Every* $z \in D^1_2$ *has* 1 *neighbour in* $D^1_1(0)$*,* 1 *neighbour in* $D^1_1(1)$*, and* $n-4$ *neighbours in* D^1_2*.*

(ii) *Every* $z \in D^1_1(0)$ *has* $n-m-1$ *neighbours in* D^1_2*,* $m-2$ *neighbours in* $D^1_1(0)$*, and no neighbours in* $D^1_1(1)$*.*

(iii) *Every* $z \in D^1_1(1)$ *has* $m-1$ *neighbours in* D^1_2*,* $n-m-2$ *neighbours in* $D^1_1(1)$*, and no neighbours in* $D^1_1(0)$*.*

Consequently, the partition $\{\{y\}, D^1_1(0), D^1_1(1), D^1_2\}$ *of* $\Gamma(x)$ *is equitable.*

Proof. First observe that it follows from the proof of Proposition 11.4 that each $z \in D^1_2$ has 1 neighbour in $D^1_1(0)$ and 1 neighbour in $D^1_1(1)$. Consequently, z has $a_1 - 2 = n - 4$ neighbours in D^1_2. Next observe that each vertex from $D^1_1(0)$ contains 1 and $m+1$ as elements, while 1 and $m+1$ are not elements of any vertex from $D^1_1(1)$. Consequently, there are no edges between vertices of $D^1_1(0)$ and $D^1_1(1)$. Furthermore, by Corollary 11.10, each vertex in $D^1_1(0)$ has $(m-2)(n-m-1)$ neighbours in D^2_2, while each vertex in $D^1_1(1)$ has $(m-1)(n-m-2)$ neighbours in D^2_2. The other claims of the above proposition now follow from the fact that intersection numbers a_1 and b_1 of $J(n,m)$ are equal to $n-2$ and $(m-1)(n-m-1)$, respectively. □

Theorem 11.12. *Pick positive integers* n *and* m *with* $n \geq 7$*,* $3 \leq m < n/2$*, and let* $\Gamma = J(n,m)$*. Pick* $x \in V(\Gamma)$ *and let* $T = T(x)$*. Then* Γ *has (up to isomorphism) exactly three irreducible* T*-modules with endpoint* 1*, and these modules are all thin.*

Proof. As Γ is arc transitive, it follows from Proposition 11.4 and Corollary 11.7 that maps V_i and H_i $(2 \leq i \leq D)$ are constant for every $y \in \Gamma(x)$, and that these constants are nonzero and independent of the choice of y. By Theorem 8.5, Γ is 1-thin. By Corollary 11.10, the map K_1 is not constant for any $y \in \Gamma(x)$. Pick $y, z \in \Gamma(x)$ and let $B = B(y)$ be as defined in Definition 10.1. It follows from Proposition 11.11 that the number of walks of length 3 from y to z in $\Delta = \Delta(x)$ depends only on the distance between y and z when $\partial(y,z) \in \{0,2\}$. Finally, by Proposition 11.11 we also have that $B^2\boldsymbol{j} = \alpha B\boldsymbol{j} + \beta\boldsymbol{j}$, where $\alpha = n-4$, $\beta = -(n-m-2)(m-2)$, and $\boldsymbol{j}$ is from Definition 10.1. Therefore Γ has properties P1, P2 and P3, and so, by Theorem 10.11, Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1. □

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