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The combinatorial (19_4) configurations

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Abstract

A parallel backtrack search is carried out to classify, up to isomorphism, all combinatorial (19_4) configurations. A total of 269 224 652 such configurations were found. We prove that two of the combinatorial (19_4) configurations are not geometrically realizable over any field. Also we confirmed the computation of the 971 171 combinatorial (18_4) configurations which lacked an independent verification.

Keywords: (19_4)*-configurations, symmetric designs, finite geometries, AMDS-Code, parallel backtracking search with isomorph rejection.*

Math. Subj. Class.: 05B25, 14N20, 62K10, 68P10, 94B05.

1 Introduction

A combinatorial (v_k) configuration is an incidence structure consisting of a set of v points and b blocks such that each block is incident with exactly k points and each point is incident with exactly k blocks. We also require that every pair of points is incident with at most one block.

In this paper we are interested in the enumeration of all combinatorial configurations with $v = 19$ and $k = 4$, which is an open problem. Our approach is computational by means of a parallel search algorithm that generates, up to isomorphism, all combinatorial (19_4) configurations.

Very often combinatorial (v_k) configurations appear as theorems in geometry; for example, the fundamental Theorems of Pappus and Desargues give rise to (9_3) and (10_3) configurations. Conversely, for given v and k, it is interesting to investigate which (v_k) configurations are geometrically realizable as a set of points and lines in some projective plane over a field, and which of those correspond to geometric theorems; that is, geometric

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realizations for which the last incidence is given by a theorem. Also it is interesting to find combinatorial (v_k) configurations which cannot be realized over any projective space. We were able to find two configurations which cannot be geometrically realized over any field.

The enumeration of combinatorial (v_k) configurations is also important because they are the building blocks of more complex combinatorial structures such as block designs. The enumeration of combinatorial structures has recently received a lot of attention due to the diversity of its applications, from theorems in geometry and the design of experiments to the theory of error correcting codes and cryptology (see [4]). We were able to construct an error correcting code by embedding a combinatorial (19_4) configuration over the finite projective space $PG(3, 67)$.

2 Computational search

We label the points of our combinatorial configurations with the symbols $0, 1, \dots, 18$. The pencil at a point P in a configuration, consists of the set of blocks incident with P along with the points they contain.

Each configuration is initialized with the pencil at point 0. Then there are, up to isomorphism, five ways to add a fifth block (see Figure 1).

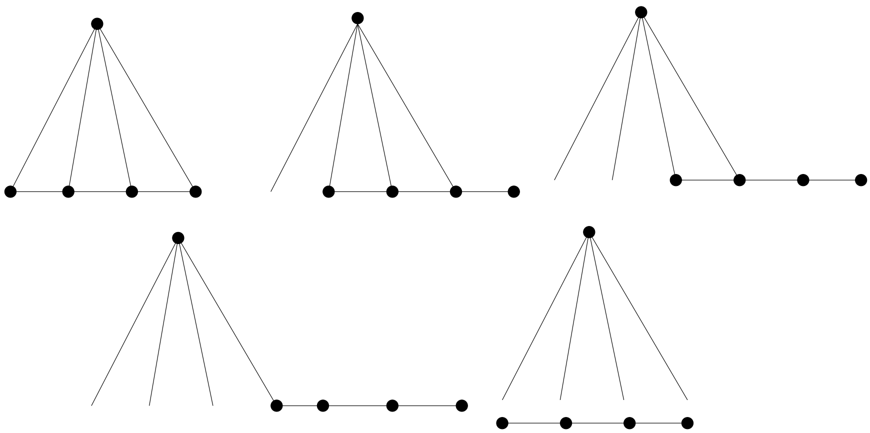

Figure 1: The five initial figures.

We call these five figures P_1, P_2, P_3, P_4, P_5 the initial figures. They define an ordering in the search tree. That is, when processing figure P_i, for $1 < i \leq 5$ we only allow adding blocks which yield configurations that do not contain subconfigurations isomorphic to P_j for $1 \leq j < i$.

We construct the configurations block by block. The decision to be made is which block should be added to the configuration being constructed. For each point in the configuration incident with less than 4 blocks, there is a fixed number of legal options for adding a block which is incident with it. We compute the number of options for each point, and we try to add a block that is incident with a point with the minimum number of options. Also a check is performed to find points in the configuration with $4 - s$ incidences with less than s options for an incident block. In the case there is at least one such point the configuration is not considered any further. This is a simple test that detects several configurations early

$\lvert Aut(C)\rvert$	$Frequency$
1	269195473
2	28474
3	492
4	172
6	32
8	5
12	3
18	2
$Total$	269224653

Table 1: Frequency of sizes of automorphism groups of combinatorial (19_4) configurations.

in the search tree that cannot be completed to a combinatorial (19_4) configuration and does not need to look exhaustively for all completions.

Due to the sizes of the search subtrees and computational infrastructure constraints, we found out that the enumeration was feasible if the partial configurations were distributed into several processors after 10 blocks have been added. We used a master/slave parallel model. Each slave processor attempts all possible completions of these partial configurations into a (19_4) configuration and creates a database of completed configurations. At the end, all these databases are merged by the master processor to eliminate possible isomorphic configurations.

To further reduce the search tree, we performed partial isomorph rejection at each level, using NAUTY (see [9]), to avoid the processing of isomorphic subproblems. Most of these ideas were adapted to our case from the *fast backtracking principles* found in ([10]).

After 39 182 hours of computation using the supercomputer KanBalam we found 269 224 652 nonisomorphic combinatorial (19_4) configurations. With our parallel approach the entire search took only 70 days. Our algorithm was tested for correctness against the known values of nonisomorphic combinatorial (v_4) and (v_3) configurations. In particular, we confirm the known counting of 971 171 combinatorial (18_4) configurations, (see [1], [2] and [3]), which lacked an independent verification.

Table 1 lists the possible sizes of the automorphism groups of the (19_4) configurations and their frequency.

3 The two combinatorial (19_4) configurations with largest automorphism group.

Let $\mathcal{C}$ be the combinatorial (19_4) configuration defined by block set given in Table 2.

The automorphism group of $\mathcal{C}$ has order 18. Such a group is generated by the following 3 generator set:

$$\begin{aligned}\{(1,15,16)(2,10,14)(3,13,7)(4,8,12)(5,11,9)(17,0,18),\\ (2,3)(7,10)(8,12)(9,11)(13,14)(15,16),\\ (2,10,14)(3,7,13)(4,5,0)(8,11,18)(9,17,12)\}\end{aligned}$$

However,

Theorem 3.1. $\mathcal{C}$ *is not geometrically realizable over any field.*

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
0	0	0	0	1	1	1	2	2	2	3	3	3	4	4	6	6	7	10
1	4	7	10	4	5	6	5	7	12	5	8	9	9	11	8	9	14	13
2	5	8	11	7	13	15	8	11	13	12	14	10	14	13	11	12	16	15
3	6	9	12	10	14	16	17	15	16	18	15	16	18	17	18	17	17	18

Table 2: The first (19_4) configuration with largest automorphism group.

Proof. Consider the following partial realization of $\mathcal{C}$ over $\mathbb{P}^2(\Bbbk)$, the projective plane over the field $\Bbbk$:

1. Choose the points 4, 6, 9 and 10 generic in $\mathbb{P}^2(\Bbbk)$. So, these points define the lines $e = \overline{4\,10}$, $m = \overline{9\,10}$, $n = \overline{4\,9}$, and $q = \overline{6\,9}$.

   Note that the symbols 6 and 10 do not share a block on $\mathcal{C}$ and so, we do not consider the corresponding line in $\mathbb{P}^2(\Bbbk)$. Also, although the symbols 4 and 6 belong to the block b in $\mathcal{C}$, we do not need block b for this argumentation.

2. Choose the points 1 in e, 18 in n, and 17 in q also generic on their respective lines. These choices define now the lines $g = \overline{1\,6}$, $o = \overline{4\,17}$, $p = \overline{6\,18}$, and $s = \overline{10\,18}$.

3. Consider the following intersection points: $11 := p \cap o$, $16 := g \cap m$, and $15 := g \cap s$, and the lines $i = \overline{11\,15}$ and $r = \overline{17\,16}$.

4. Finally, consider the point $7 := r \cap i$.

Since the symbols 4, 7, and 10 belong to the block e of $\mathcal{C}$, then the point 7 must also belong to the corresponding line e. If $\mathcal{C}$ were geometrically realizable over $\mathbb{P}^2(\Bbbk)$, then the points (except the point labeled 1) and lines considered in this construction would constitute a (10_3) geometric configuration $\mathcal{C}'$ in $\mathbb{P}^2(\Bbbk)$ (see Figure 2).

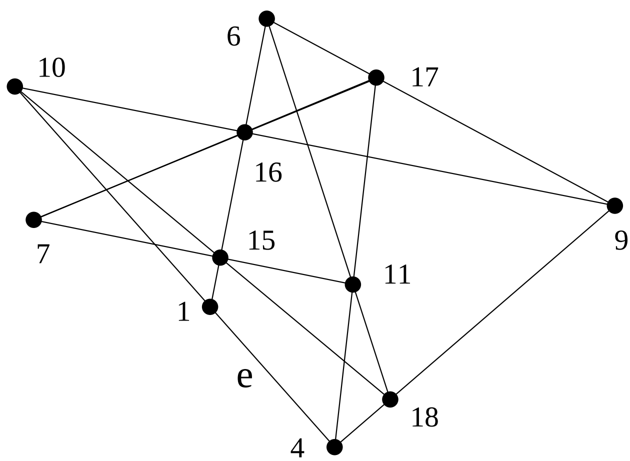

Figure 2: The point 7 does not belong in the line e.

Glynn's	A	B	C	D	E	F	G	H	I	J
$\mathcal{C}'$	7	10	4	16	15	18	11	9	17	6

Table 3: Correspondence between Glynn's construction and $\mathcal{C}'$.

Actually, $\mathcal{C}'$ would be the non-realizable (10_3) combinatorial configuration: In ([5], fig. 2 p. 73), D.G. Glynn gives all the incidences for this structure and the correspondence in this paper's Table 3 gives an isomorphism between them.

□

The following block set $\mathcal{D}$ corresponds to the other combinatorial (19_4) configuration with automorphism group of order 18:

0	0	0	0	1	1	1	2	2	2	3	3	3	4	4	5	5	6	6
1	4	7	10	4	5	6	7	8	9	10	11	12	7	8	7	9	8	9
2	5	8	11	13	15	17	13	14	16	13	14	15	10	11	12	11	12	10
3	6	9	12	14	16	18	15	17	18	18	16	17	16	18	14	17	13	15

One might proceed as in the former case or, alternatively, consider the following substructure of $\mathcal{D}$:

0	0	0	1	1	7	3	3	4	5
1	4	10	4	5	13	10	12	7	7
3	5	12	13	15	15	13	15	10	12

which also corresponds to the non-realizable (10_3) combinatorial configuration. This is so because both structures have the $6I + 0V + 4\triangle$ *restfigur* scheme (see ([7]), table 2.2.8 p. 74), which, in turn, is characteristic of such a configuration.

4 Embedding configurations in $PG(k-1, q)$ and codes

Let q be a prime power. An $\mathbb{F}_q-$linear error correcting code C of length n is an $\mathbb{F}_q-$linear subspace of $\mathbb{F}_q^n$. The elements of C are called words. The weight $wt(x)$ of a word x in C is the number of its non-zero coordinates. The minimum weight d of the code C is defined as the minimum of the weights of non-zero words occurring in C. For $x, y \in C$, we define the Hamming distance $d(x, y)$ between x and y as $wt(x - y)$. It follows that

$$d = \min\{d(x, y)|x, y \in C, x \neq y\}$$

and d is called the minimum distance of C. If k is the dimension of C as a vector space over $\mathbb{F}_q$, then we say that C is a

$$[n, k, d]_q$$

error correcting code. It is known that the parameters of a code C satisfy

$$n + 1 \geq k + d.$$

The above inequality is called the singleton bound. A code satisfying the previous inequality with equality is called a maximum distance separable code, or simply a MDS-Code. The singleton defect s of a code C is defined as

$$s = n + 1 - k - d.$$

A code with $s = 1$ is called an almost maximum distance separable code, or simply AMDS-code. See [8] for more on Coding Theory.

In our context, embedding a (v_k) configuration in $PG(k-1,q)$ means finding a set of v points and v hyperplanes in $PG(k-1,q)$ that preserve incidence in the configuration (see [6]). By embedding a combinatorial (v_k) configuration in $PG(k-1,q)$, we construct a linear error correcting code C as the linear $\mathbb{F}_q$-span of the rows of the matrix G whose v columns contains the coordinates of the v points of the configuration. If the embedding is such that no hyperplane of $PG(k-1,q)$ contains more than k points of the embedded configuration, then C has length v, dimension k and minimum distance $v-k$. Note that these parameters indicate that the code C is an AMDS code.

5 An example over $\mathbb{F}_{67}$

Let $\mathcal{C}$ be the (19_4) configuration defined by the following block set:

0	0	0	0	1	1	1	2	2	2	3	3	3	5	6	7	8	9	10
1	4	7	10	4	5	6	4	5	12	4	6	7	9	9	13	12	11	14
2	5	8	11	7	8	11	15	10	13	17	8	11	16	12	14	14	13	15
3	6	9	12	10	13	14	16	17	18	18	15	16	18	17	17	16	15	18

An embedding of the points of the configuration in $PG(3,67)$ is given by the columns of the following matrix:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
35	56	61	66	28	43	11	59	27	4	57	35	14	52	42	19	64	42	7
46	3	41	42	27	17	20	57	38	24	10	32	20	2	15	22	38	47	60
15	21	15	27	26	11	31	41	54	42	26	18	17	59	4	38	62	36	50
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

The code spanned by the rows of the previous matrix has parameters $[19,4,15]_{67}$ which is an AMDS linear code with 3564 words of weight 15.

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