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# Two families of pseudo metacirculants*

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**Abstract**

A split weak metacirculant which is not metacirculant is simply called a *pseudo metacirculant*. In this paper, two infinite families of pseudo metacirculants are constructed.

*Keywords: Metacirculant, metacyclic, split weak metacirculant.*

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## 1 Introduction

Metacirculant graphs were introduced by Alspach and Parsons [1]. In 2008 Marušič and Šparl [11] gave an equivalent definition of metacirculant graphs as follows. Let $m \geq 1$ and $n \geq 2$ be integers. A graph $\Gamma$ of order $mn$ is called [11] an $(m, n)$*-metacirculant graph* (in short $(m, n)$*-metacirculant*) if it has an automorphism $\sigma$ of order $n$ such that $\langle\sigma\rangle$ is semiregular on the vertex set of $\Gamma$, and an automorphism $\tau$ normalizing $\langle\sigma\rangle$ and cyclically permuting the $m$ orbits of $\langle\sigma\rangle$ such that $\tau$ has a cycle of size $m$ in its cycle decomposition. A graph is called a *metacirculant* if it is an $(m, n)$-metacirculant for some $m$ and $n$.

It follows from the definition above that a metacirculant $\Gamma$ has a vertex-transitive automorphism group $\langle\sigma, \tau\rangle$ which is metacyclic. If we, instead, require that the graph has a vertex-transitive metacyclic group of automorphisms, then we obtained the so-called *weak metacirculants*, which were introduced by Marušič and Šparl [11] in 2008. In [10] Li *et al.* initiated the study of relationship between metacirculants and weak metacirculants, and they divided the weak metacirculants into the following two subclasses: A weak metacirculant which has a vertex-transitive split metacyclic automorphism group is called *split weak metacirculant*. Otherwise, a weak metacirculant $\Gamma$ is called a *non-split weak metacirculant* if its full automorphism group does not contain any split metacyclic subgroup which is vertex-transitive. In [10, Lemma 2.2] it was proved that every metacirculant is a split weak

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metacirculant, but it was unknown whether the converse of this statement is true. In [14] Sanming Zhou and the second author asked the following question:

**Question 1.1** ([14, Question A])**.** Is it true that any split weak metacirculant is a metacirculant?

If the graph under consideration is of prime-power order, it was shown by the authors [5, 14] that the answer to the above question is positive. However, in [6] we show that there do exist infinitely many split weak metacirculants which are not metacirculants. For convenience, we shall say that a split metacirculant is a *pseudo metacirculant* if it is not metacirculant. To the best of our knowledge, up to now the only known pseudo metacirculants are the graphs constructed in [6]. So it might be interesting to find some other families of pseudo metacirculants. Furthermore, it seems that the existence of pseudo metacirculants is closely related to the orders of graphs. Motivated by this, it is natural to consider the following problem.

**Problem 1.2.** Characterize those integers $n$ for which there is a pseudo metacirculant of order $n$.

There are some partial answers to Problem 1.2. By [5, 14], there do not exist a pseudo metacirculant with a prime-power order. The construction of pseudo metacirculants in [6] shows that for two any primes $p, q$ with $q \mid p-1$, there exists a pseudo metacirculant of order $p^m q$ for each $m \geq 3$. In this paper, two new infinite families of pseudo metacirculants are constructed. Our construction implies that for any primes $p, q$, if either $p^{\lfloor \frac{m}{2} \rfloor + 1} \mid q-1$ or $p = 2$ and $4 \mid q-1$, then there exists a pseudo metacirculant of order $p^m q$ with $m \geq 3$.

Our research also shows that the three families of pseudo metacirculants constructed in [6] and this paper are crucial for solving Problem 1.2, and we shall use them to give a complete solution of Problem 1.2 in our subsequent paper [4].

## 2 Preliminaries

### 2.1 Definitions and notation

For a positive integer $n$, we denote by $C_n$ the cyclic group of order $n$, by $\mathbb{Z}_n$ the ring of integers modulo $n$, by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to $n$, and by $D_{2n}$ the dihedral group of order $2n$. For two groups $M$ and $N$, $N : M$ denotes a semidirect product of $N$ by $M$. Given a group $G$, denote by $1$, $\mathrm{Aut}(G)$, and $Z(G)$ the identity element, full automorphism group and center of $G$, respectively. Denote by $o(x)$ the order of an element $x$ of $G$. For a subgroup $H$ of $G$, denote by $C_G(H)$ the centralizer of $H$ in $G$. A group $G$ is called *metacyclic* if it contains a normal cyclic subgroup $N$ such that $G/N$ is cyclic. In other words, a metacyclic group $G$ is an extension of a cyclic group $N \cong C_n$ by a cyclic group $G/N \cong C_m$, written as $G \cong C_n.C_m$. If this extension is split, namely $G \cong C_n : C_m$, then $G$ is called a *split metacyclic group*.

Let $G$ be a permutation group on a set $\Omega$ and $\alpha \in \Omega$. Denote by $G_\alpha$ the stabilizer of $\alpha$ in $G$, that is, the subgroup of $G$ fixing the point $\alpha$. We say that $G$ is *semiregular* on $\Omega$ if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if $G$ is transitive and semiregular. For any subset $\Delta$ of $\Omega$, use $G_\Delta$ and $G_{(\Delta)}$ to denote the subgroups of $G$ fixing $\Delta$ setwise and pointwise, respectively. A *block of imprimitivity* of $G$ on $\Omega$ is a subset $\Delta$ of $\Omega$ with $1 < |\Delta| < |\Omega|$ such that for any $g \in G$, either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$. In this case the *blocks* $\Delta^g,\ g \in G$ form a *$G$-invariant partition* of $\Omega$.

All graphs in this paper are finite, simple and undirected. For a graph $\Gamma$, we denote its vertex set and edge set by $V(\Gamma)$ and $E(\Gamma)$, respectively. Given two adjacent vertices $u, v$ of $\Gamma$, denote by $\{u, v\}$ the edge between $u$ and $v$. Denote by $\Gamma(v)$ the neighbourhood of $v$, and by $\Gamma[B]$ the subgraph of $\Gamma$ induced by a subset $B$ of $V(\Gamma)$. An *$s$-cycle* in $\Gamma$, denoted by $\mathrm{C}_s$, is an $(s+1)$-tuple of pairwise distinct vertices $(v_0, v_1, \ldots, v_s)$ such that $\{v_{i-1}, v_i\} \in E(\Gamma)$ for $1 \leq i \leq s$ and $\{v_s, v_0\} \in E(\Gamma)$. Denote by $\mathbf{K}_n$ the complete graph of order $n$, and by $\mathbf{K}_{n,n}$ the complete bipartite graph with biparts of cardinality $n$. The full automorphism group of $\Gamma$ is denoted by $\mathrm{Aut}(\Gamma)$.

### 2.2 Quotient graph

Let $\Gamma$ be a connected vertex-transitive graph, and let $G \leq \mathrm{Aut}(\Gamma)$ be vertex-transitive on $\Gamma$. A partition $\mathcal{B}$ of $V(\Gamma)$ is said to be *$G$-invariant* if for any $B \in \mathcal{B}$ and $g \in G$ we have $B^g \in \mathcal{B}$. For a $G$-invariant partition $\mathcal{B}$ of $V(\Gamma)$, the *quotient graph* $\Gamma_{\mathcal{B}}$ is defined as the graph with vertex set $\mathcal{B}$ such that, for any two different vertices $B, C \in \mathcal{B}$, $B$ is adjacent to $C$ if and only if there exist $u \in B$ and $v \in C$ which are adjacent in $\Gamma$. Let $N$ be a normal subgroup of $G$. Then the set $\mathcal{B}$ of orbits of $N$ on $V(\Gamma)$ is a $G$-invariant partition of $V(\Gamma)$. In this case, the symbol $\Gamma_{\mathcal{B}}$ will be replaced by $\Gamma_N$, and the original graph $\Gamma$ is said to be a *cover* of $\Gamma_N$ if $\Gamma$ and $\Gamma_N$ have the same valency.

### 2.3 Cayley graph

Given a finite group $G$ and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on $G$ with respect to $S$ is a graph with vertex set $G$ and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. For any $g \in G$, $R(g)$ is the permutation of $G$ defined by $R(g)\colon x \mapsto xg$ for $x \in G$. Set $R(G) := \{R(g) \mid g \in G\}$. It is well-known that $R(G)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. This concept was introduced by Xu in [13], and for more results about normal Cayley graphs, we refer the reader to [7].

The following proposition determines the normalizer of $R(G)$ in the full automorphism group of $\mathrm{Cay}(G, S)$.

**Proposition 2.1** ([8, Lemma 2.1]). *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph on $G$ with respect to $S$. Then $N_{\mathrm{Aut}(\Gamma)}(R(G)) = R(G) : \mathrm{Aut}(G, S)$, where $\mathrm{Aut}(G, S)$ is the group of automorphisms of $G$ fixing the set $S$ setwise.*

### 2.4 Coset graph

Let $G$ be a group and for a subgroup $H$ of $G$, let $\Omega = [G : H] = \{Hx \mid x \in G\}$, the set of right cosets of $H$ in $G$. For $g \in G$, define $R_H(g)\colon Hx \mapsto Hxg$, $x \in G$, and set $R_H(G) = \{R_H(g) \mid g \in G\}$. The map $g \mapsto R_H(g)$, $g \in G$, is a homomorphism from $G$ to $S_\Omega$ and it is called the *coset action* of $G$ relative to $H$. The kernel of the coset action is $H_G = \cap_{g \in G} H^g$, the largest normal subgroup of $G$ contained in $H$, and $G/H_G \cong R_H(G)$. It is well-known that any transitive action of $G$ on $\Omega$ is equivalent to the coset action of $G$ relative the subgroup $G_\alpha$ for any given $\alpha \in \Omega$. If $H_G = 1$, we say that $H$ is *core-free* in $G$.

Let $D$ be a union of several double-cosets of the form $HgH$ with $g \notin H$ such that $D = D^{-1}$. The *coset graph* $\Gamma = \mathrm{Cos}(G, H, D)$ of $G$ with respect to $H$ and $D$ is defined as the graph with vertex set $V(\Gamma) = [G : H]$, and edge set $E(\Gamma) = \{\{Hg, Hdg\} \mid g \in G, d \in D\}$. It is easy to see that $\Gamma$ is well defined and has valency $|D|/|H|$, and $\Gamma$ is connected if and only if $D$ generates $G$. Further, $R_H(G) \leq \mathrm{Aut}(\Gamma)$, and hence $\Gamma$ is vertex-transitive.

Let $\mathrm{Aut}(G, H, D) = \{\alpha \in \mathrm{Aut}(G) \mid H^\alpha = H, D^\alpha = D\}$. For any $\alpha \in \mathrm{Aut}(G, H, D)$, define $\alpha_H \colon Hx \mapsto Hx^\alpha$, $x \in G$, and consider the action of $\mathrm{Aut}(G, H, D)$ on $[G : H]$ induced by $\alpha \mapsto \alpha_H$. It follows that $\mathrm{Aut}(G, H, D)/L \cong \mathrm{Aut}(G, H, D)_H$, where $\mathrm{Aut}(G, H, D)_H = \{\alpha_H \mid \alpha \in \mathrm{Aut}(G, H, D)\}$ and $L$ is the kernel of the action. Furthermore, it is easy to see that $\mathrm{Aut}(G, H, D)_H \leq \mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(G, H, D)_H$ fixes the vertex $H$. For $h \in H$, let $\sigma(h)$ be the inner automorphism of $G$ induced by $h$, that is, $\sigma(h) \colon g \mapsto h^{-1}gh$, $g \in G$. One may show that $\sigma(H) = \{\sigma(h) \mid h \in H\}$ is a subgroup of $\mathrm{Aut}(G, H, D)$ and hence $R_H(H) = \{R_H(h) \mid h \in H\}$ is a subgroup of $\mathrm{Aut}(G, H, D)_H$.

The following proposition determines the normalizer of $R_H(G)$ in the full automorphism group of $\mathrm{Cos}(G, H, D)$.

**Proposition 2.2** ([12, Lemma 2.10]). *Let $G$ be a finite group, $H$ a core-free subgroup of $G$ and $D$ a union of several double-cosets $HgH$ such that $H \nsubseteq D$. Let $\Gamma = \mathrm{Cos}(G, H, D)$ and $A = \mathrm{Aut}(\Gamma)$. Then $R_H(G) \cong G$, $\mathrm{Aut}(G, H, D)_H \cong \mathrm{Aut}(G, H, D)$, $R_H(H) \cong \sigma(H)$, and $N_A(R_H(G)) = R_H(G)\,\mathrm{Aut}(G, H, D)_H$ with $R_H(G) \cap \mathrm{Aut}(G, H, D)_H = R_H(H)$.*

Below, we prove a technical lemma.

**Lemma 2.3.** *Let $G$ be a finite group and let $H$ be a core-free subgroup of $G$. Let $S$ be a set of non-identity elements of $G$ such that $S$ is self-inverse, and let $T$ be any self-inverse subset of $S$. Let $D = HSH$ and let $C = HTH$. Set*

$$\Gamma = \mathrm{Cos}(G, H, D) \text{ and } \Sigma = \mathrm{Cos}(G, H, C).$$

*If the vertex-stabilizer $\mathrm{Aut}(\Gamma)_H$ fixes $\Sigma(H) = \{Hd \mid d \in C\}$ setwise, then $\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\Sigma)$.*

*Proof.* Let $A = \mathrm{Aut}(\Gamma)$. Suppose that $A_H$ fixes $\Sigma(H) = \{Hd \mid d \in C\}$ setwise. Then for any $g \in G$, we have $A_{Hg} = (A_H)^{R_H(g)}$, and so $A_{Hg}$ fixes the following set setwise:

$$\{Hd \mid d \in C\}^{R_H(g)} = \{Hdg \mid d \in C\} = \Sigma(Hg).$$

Take $x \in A$ and take any edge $e = \{Hg, Hdg\}$ of $\Sigma$. To show $A \leq \mathrm{Aut}(\Sigma)$, it suffices to show that $e^x \in E(\Sigma)$. Since $G$ acts transitively on $V(\Gamma)$ by right multiplication, there exists $g' \in G$ such that $(Hg)^x = Hgg'$, and then $(Hg)^{xR_H((g')^{-1})} = Hg$. It follows that $(Hdg)^{xR_H((g')^{-1})} \in \Sigma(Hg)$ and so $(Hdg)^x \in (\Sigma(Hg))^{R_H((g'))} = \Sigma(Hgg') = \Sigma((Hg)^x)$. Hence, we have $e^x \in E(\Sigma)$, and consequently, $A \leq \mathrm{Aut}(\Sigma)$. □

### 2.5 Circulants

A *circulant* of order $n$ is a Cayley graph over a cyclic group of order $n$. The following proposition gives a classification of arc-transitive circulants. Before stating this result, we introduce several concepts.

If a graph $\Gamma$ has $n > 1$ connected components, each of which is isomorphic to a graph $\Sigma$, then we shall write $\Gamma = n\Sigma$. The *lexicographic* (or *wreath*) *product* of graphs $\Gamma_1$ and $\Gamma_2$ is a graph $\Gamma_1 \circ \Gamma_2$ with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ such that $\{(x_1, x_2), (y_1, y_2)\} \in E(\Gamma_1 \circ \Gamma_2)$ if and only if either $x_1 = y_1$ and $\{x_2, y_2\} \in E(\Gamma_2)$, or $\{x_1, y_1\} \in E(\Gamma_1)$. If $\Gamma_2$ is of order $m$ with $V(\Gamma_2) = \{y_1, y_2, \ldots, y_m\}$, then we have a natural embedding of $m\Gamma_1$ in

$\Gamma_1 \circ \Gamma_2$, where, for $1 \leq i \leq m$, the $i$th copy of $\Gamma_1$ is the subgraph induced by the subset of vertices of $\Gamma_1 \circ \Gamma_2$. The *deleted lexicographic product* of graphs $\Gamma_1$ and $\Gamma_2$, denoted by $\Gamma_1 \circ \Gamma_2 - m\Gamma_1$, is the graph obtained by deleting from $\Gamma_1 \circ \Gamma_2$ the edges of $m\Gamma_1$.

**Proposition 2.4** ([9, Theorem 1]). *Let $\Gamma$ be a connected arc-transitive circulant of order $n$. Then one of the following holds:*

(a) $\Gamma = \mathbf{K}_n$*;*

(b) $\Gamma$ *is a normal circulant;*

(c) $\Gamma = \Sigma \circ \bar{\mathbf{K}}_d$*, where $n = md$ and $\Sigma$ is a connected arc-transitive circulant of order $m$;*

(d) $\Gamma = \Sigma \circ \bar{\mathbf{K}}_d - d\Sigma$*, where $n = md$, $d > 3$, $\gcd(\mathrm{d}, \mathrm{m}) = 1$ and $\Sigma$ is a connected arc-transitive circulant of order $m$.*

## 3 Pseudo metacirculants–Family A

**Consruction A.** *Let $p$ be a prime such that $4 \mid p-1$ and let $n \geq 2$ be an integer. If $p > 5$, then let $r \in \mathbb{Z}_p^*$ be such that $r^2 \equiv -1 \pmod p$, and if $p = 5$, then let $r = 2$. Let*

$$G_{2,n,p,r} = \langle a, b, c \mid a^{2^n} = b^p = c^4 = 1, ab = ba, c^{-1}ac = a^{-1}, c^{-1}bc = b^r \rangle.$$

*Let*

$$\Gamma_{2,n,p,r} = \mathrm{Cos}(G_{2,n,p,r}, H, H\{(ab)^{\pm 1}, (abc)^{\pm 1}, (bc)^{\pm 1}, c^{\pm 1}\}H),$$

*where $H = \langle a^{2^{n-1}}c^2 \rangle$.*

We first prove a lemma.

**Lemma 3.1.** *Let $G = G_{2,n,p,r}$, $H = \langle a^{2^{n-1}}c^2 \rangle$, and let $D = H\{(ab)^{\pm 1}, (abc)^{\pm 1}, (bc)^{\pm 1}, c^{\pm 1}\}H$. If $n = 2$ and $p > 5$, then $\mathrm{Aut}(G, H, D)$ contains exactly one involution.*

*Proof.* Since $H \cong C_2$, by Proposition 2.2, $\mathrm{Aut}(G, H, D)$ has an involution. Take an involution $\alpha \in \mathrm{Aut}(G, H, D)$. Since $\langle b \rangle$ is a normal Sylow $p$-subgroup of $G$, $\langle b \rangle$ is characteristic in $G$, implying that $b^\alpha \in \langle b \rangle$. Since $\alpha$ has order 2, one has $b^\alpha = b$ or $b^{-1}$. By a direct computation, we see that $C_G(b) = \langle a \rangle \times \langle b \rangle$. Then

$$a^\alpha \in C_G(b^\alpha) = C_G(b) = \langle a \rangle \times \langle b \rangle.$$

It follows that $a^\alpha \in \langle a \rangle$. Since $\alpha$ has order 2 and $n = 2$, one has $a^\alpha = a$ or $a^{-1}$, and hence $(a^2)^\alpha = a^2$.

Assume that $c^\alpha = a^i b^j c^k$ for some $i \in \mathbb{Z}_{2^n}$, $j \in \mathbb{Z}_p$ and $k \in \mathbb{Z}_4^*$. Considering the image of the equality $c^{-1}bc = b^r$ under $\alpha$, we obtain that $(a^i b^j c^k)^{-1}(b^{\pm 1})a^i b^j c^k = b^{\pm r}$, and hence $b^{r^k} = b^r$. It follow that $k = 1$ in $\mathbb{Z}_4$, and so $c^\alpha = a^i b^j c$. Moreover, $(c^2)^\alpha = b^{j(1-r)}c^2$. From $H^\alpha = H$ we obtain that $(a^2c^2)^\alpha = a^2c^2$. As $(a^2)^\alpha = a^2$, one has $(c^2)^\alpha = c^2$. Thus, $j = 0$ in $\mathbb{Z}_p$, and so $c^\alpha = a^i c$.

Now by a direct computation, we have

$$ac, a^{-1}c, ba^2c, b^{-1}a^2c, a^{-1}bc, a^{-1}b^{-1}c \notin D. \tag{3.1}$$

Remember that $D^\alpha = D$. Since $c \in D$, $c^\alpha = a^i c$ implies that $a^i c \in D$. So the only possibility is either $c^\alpha = c$ or $c^\alpha = a^2 c$. If the latter happens, then $(bc)^\alpha = ba^2c$ or $b^{-1}a^2c$, and since $bc \in D$, either $ba^2c$ or $b^{-1}a^2c$ belongs to $D$, contrary to Equation (3.1). Thus, $c^\alpha = c$.

Recall that $a^\alpha = a^{-1}$ or $a$. For the former, we have $(abc)^\alpha = a^{-1}bc$ or $a^{-1}b^{-1}c$, and since $abc \in D$, either $a^{-1}bc$ or $a^{-1}b^{-1}c$ belongs to $D$. This is again impossible by Equation (3.1). Thus, $a^\alpha = a$, and hence we have that

$$\alpha \colon a \mapsto a, b \mapsto b^{-1}, c \mapsto c.$$

This implies that $\mathrm{Aut}(G, H, D)$ has exactly one involution. □

Below we shall determine the full automorphism group of $\Gamma_{2,n,p,r}$.

**Lemma 3.2.** *Let* $\Gamma = \Gamma_{2,n,p,r}$ *and let* $G = G_{2,n,p,r}$. *Then* $\mathrm{Aut}(\Gamma) = R_H(G) \cong G$.

*Proof.* Let $A = \mathrm{Aut}(\Gamma)$. It is easy to see that $H$ is a non-normal subgroup of $G$, and so $H$ is core-free in $G$. It follows that $G$ acts faithfully and transitively on $V(\Gamma)$ by right multiplication, and so we may view $G$ as a transitive subgroup of $A$. If $n = 2$ and $p = 5$, then by Magma [3], we obtain that $\mathrm{Aut}(\Gamma) = G$. In what follows, we shall always assume that either $p > 5$ or $n > 2$.

Noting that $ab = ba$, we have $\langle ab \rangle \cong C_{2^n p}$. Clearly, $\langle ab \rangle \trianglelefteq G$, so $\langle ab \rangle$ is semiregular on $V(\Gamma)$. Since $|V(\Gamma)| = |G : H| = 2^{n+1}p$, $\langle ab \rangle$ has two orbits on $V(\Gamma)$ which are listed as follows:

$$V_0 = \{Ha^ib^j \mid i \in \mathbb{Z}_{2^n}, j \in \mathbb{Z}_p\} \text{ and } V_1 = \{Ha^ib^jc \mid i \in \mathbb{Z}_{2^n}, j \in \mathbb{Z}_p\}.$$

The kernel of $G$ acting on $\{V_0, V_1\}$ is $\langle ab \rangle : \langle c^2 \rangle$. We can also easily obtain the following two observations:

$$\begin{aligned} &\forall i \in \mathbb{Z}_{2^n}, j \in \mathbb{Z}_p, && Ha^ib^jc^2 = Hc^2a^i(c^2b^jc^2) = Ha^{2^{n-1}+i}b^{-j}, \\ &\forall i_1, i_2 \in \mathbb{Z}_{2^n}, j_1, j_2 \in \mathbb{Z}_p, k \in \mathbb{Z}_2 && Ha^{i_1}b^{j_1}c^k = Ha^{i_2}b^{j_2}c^k \Leftrightarrow \begin{cases} i_1 \equiv i_2 \pmod{2^n}, \\ j_1 \equiv j_2 \pmod{p}. \end{cases} \end{aligned} \tag{3.2}$$

Set $\Delta_1 = \{Hd \mid d \in H\{(ab)^{\pm 1}\}H\}$ and $\Delta_2 = \{Hd \mid d \in H\{(abc)^{\pm 1}, (bc)^{\pm 1}, c^{\pm 1}\}H\}$. Then $\Gamma(H) = \Delta_1 \cup \Delta_2$. Furthermore, an easy computation shows that

$$\begin{aligned} \Delta_1 &= \{Hab, H(ab)^{-1}, Hab^{-1}, Ha^{-1}b\}, \\ \Delta_2 &= \{Habc, Ha^{1+2^{n-1}}b^rc, Hab^{-1}c, Ha^{1+2^{n-1}}b^{-r}c, \\ &\qquad Hbc, Ha^{2^{n-1}}b^rc, Hb^{-1}c, Ha^{2^{n-1}}b^{-r}c, Hc, Ha^{2^{n-1}}c\}. \end{aligned}$$

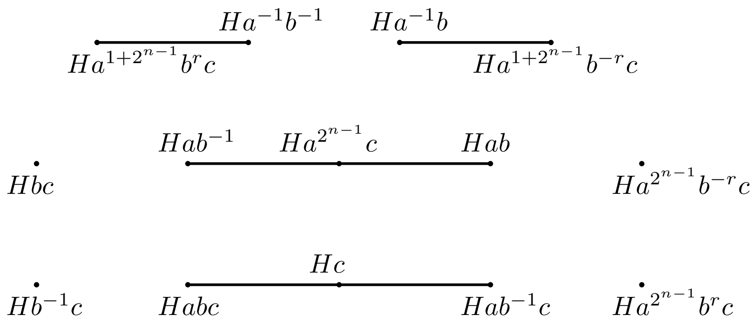

Figure 1: The subgraph induced by $\Gamma(H)$ when $p > 5$ and $n > 2$.

So for any $i \in \mathbb{Z}_{2^n}, j \in \mathbb{Z}_p$, we have

$$\begin{aligned}\Gamma(Ha^ib^j) = \{&Ha^{i+1}b^{j+1}, Ha^{i-1}b^{j-1}, Ha^{i+1}b^{j-1}, Ha^{i-1}b^{j+1}, Ha^{1-i}b^{1-jr}c,\\ &Ha^{2^{n-1}+1-i}b^{r-jr}c, Ha^{1-i}b^{-1-jr}c, Ha^{1+2^{n-1}-i}b^{-r-jr}c,\\ &Ha^{-i}b^{1-jr}c, Ha^{2^{n-1}-i}b^{r-jr}c, Ha^{-i}b^{-1-jr}c, Ha^{2^{n-1}-i}b^{-r-jr}c,\\ &Ha^{-i}b^{-jr}c, Ha^{2^{n-1}-i}b^{-jr}c\},\\ \Gamma(Ha^ib^jc) = \{&Ha^{i+1}b^{j+1}c, Ha^{i-1}b^{j-1}c, Ha^{i+1}b^{j-1}c, Ha^{i-1}b^{j+1}c, Ha^{2^{n-1}+1-i}b^{jr-1},\\ &Ha^{1-i}b^{jr-r}, Ha^{2^{n-1}+1-i}b^{1+jr}, Ha^{1-i}b^{r+jr}, Ha^{2^{n-1}-i}b^{jr-1},\\ &Ha^{-i}b^{jr-r}, Ha^{2^{n-1}-i}b^{1+jr}, Ha^{-i}b^{r+jr}, Ha^{2^{n-1}-i}b^{jr}, Ha^{-i}b^{jr}\}.\end{aligned}$$

We shall finish the proof by the following four steps.

**Step 1:** Let $\Sigma = \mathrm{Cos}(G, H, H\{(bc)^{\pm 1}, c^{\pm 1}\}H)$ and let $M = \langle bc, c, H\rangle$. Then $A \leq \mathrm{Aut}(\Sigma)$. In particular, the orbit $H^M = \{Hg \mid g \in M\}$ of $M$ on $V(\Gamma)$ containing $H$ is a block of imprimitivity of $A$ on $V(\Gamma)$.

By direct computations, we may depict the subgraph induced by $\Gamma(H)$ as in Figures 1 – 3. From these three figures one may see that the vertex-stabilizer $A_H$ fixes the following set setwise:

$$\begin{aligned}\Sigma(H) &= \{Hd \mid d \in H\{(bc)^{\pm 1}, c^{\pm 1}\}H\}\\ &= \{Hbc, Ha^{2^{n-1}}b^rc, Hb^{-1}c, Ha^{2^{n-1}}b^{-r}c, Hc, Ha^{2^{n-1}}c\}.\end{aligned}$$

From Lemma 2.3 it follows that $A \leq \mathrm{Aut}(\Sigma)$.

Since $M = \langle bc, c, H\rangle = \langle a^{2^{n-1}}\rangle \times \langle b, c\rangle \cong C_2 \times (C_p : C_4)$, the coset graph

$$\Delta = \mathrm{Cos}(M, H, H\{(bc)^{\pm 1}, c^{\pm 1}\}H\})$$

is just a component of $\Sigma$, and since $A$ is transitive on $V(\Sigma) = V(\Gamma)$, the orbit of $M$ containing $H$ is a block of imprimitivity of $A$ on $V(\Gamma)$.

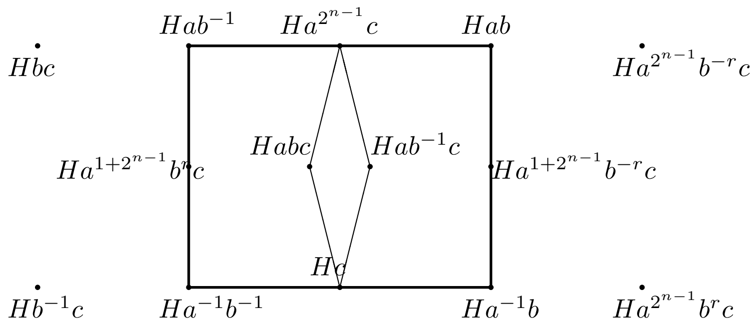


Figure 2: The subgraph induced by $\Gamma(H)$ when $p > 5$ and $n = 2$.

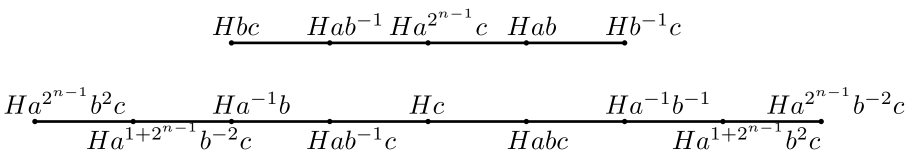


Figure 3: The subgraph induced by $\Gamma(H)$ when $p = 5$, $r = 2$ and $n > 2$.

**Step 2:** Set $N = \langle a^{2^{n-1}} \rangle \times \langle b \rangle$. Then each orbit of $N$ is a block of imprimitivity of $A$ on $V(\Gamma)$.

Let $O$ be the orbit of $M$ on $V(\Gamma)$ containing $H$. Then $O = \{Hg \mid g \in M\} = V(\Delta)$. By Step 1, $O$ is a block of imprimitivity of $A$ on $V(\Gamma)$. Let $S = \{Og \mid g \in G\}$. Then $S$ is an imprimitivity block system of $A$ on $V(\Gamma)$. Let $K$ be the kernel of $A$ acting on $S$. Note that $N \trianglelefteq G$. Since $N \leq M$, $N$ fixes every block $Og$ in $S$. It follows that $N \leq K$. Note that the subgraph $\Sigma[Og]$ of $\Sigma$ is just a component which is isomorphic to $\Delta$. It is easy to see that $N$ acts on each $Og$ semiregularly with two orbits $\{Hng \mid n \in N\}$ and $\{Htcg \mid t \in N\}$. As

$$\Sigma(Hg) = \{Hbcg, Ha^{2^{n-1}}b^{r}cg, Hb^{-1}cg, Ha^{2^{n-1}}b^{-r}cg, Hcg, Ha^{2^{n-1}}cg\},$$

the component $\Sigma[Og]$ is a bipartite graph with the two orbits of $N$ on $Og$ as its two parts. This implies that every orbit of $N$ on $V(\Gamma)$ is a block of imprimitivity of $A$ on $V(\Gamma)$.

**Step 3:** Take two adjacent orbits $B_0, B_1$ of $N$ on $V(\Gamma)$ such that $B_0 \subseteq V_0$ and $B_1 \subseteq V_1$. Then we have $A_{(B_0)} = A_{(B_1)}$. In particular, $A_H$ fixes $\Delta_1 = \{Hab, H(ab)^{-1}, Hab^{-1}, Ha^{-1}b\} = \{Hd \mid d \in H\{(ab)^{\pm 1}\}H\}$ setwise. Since $G$ acts transitively on $V(\Gamma)$, we may let $B_0 = B = \{Hn \mid n \in N\}$. Since $N \trianglelefteq G$, one has $B = NH \leq G$. Recall that

$$\begin{aligned}\Gamma(H) = \{&Hab, H(ab)^{-1}, Hab^{-1}, Ha^{-1}b,\\ &Habc, Ha^{1+2^{n-1}}b^{r}c, Hab^{-1}c, Ha^{1+2^{n-1}}b^{-r}c,\\ &Hbc, Ha^{2^{n-1}}b^{r}c, Hb^{-1}c, Ha^{2^{n-1}}b^{-r}c, Hc, Ha^{2^{n-1}}c\}.\end{aligned}$$

Each orbit of $N$ on $V(\Gamma)$ is also an independent subset of $\Gamma$. Consider the orbits $Ba, Ba^{-1}, Bc$ and $Bac$ of $N$. Since $B = NH$, one has

$$\begin{aligned}
&Hab, Hab^{-1} \in Ba,\\
&Ha^{-1}b, Ha^{-1}b^{-1} \in Ba^{-1},\\
&Habc, Ha^{1+2^{n-1}}b^r c, Hab^{-1}c, Ha^{1+2^{n-1}}b^{-r}c \in Bac,\\
&Hbc, Ha^{2^{n-1}}b^r c, Hb^{-1}c, Ha^{2^{n-1}}b^{-r}c, Hc, Ha^{2^{n-1}}c \in Bc.
\end{aligned}$$

So $Ba, Ba^{-1}, Bc$ and $Bac$ are all orbits of $N$ adjacent to $B$. Furthermore, it is easy to check that if $n > 2$, then $Ba, Ba^{-1}, Bc$ and $Bac$ are four pair-wise different orbits of $N$, and if $n = 2$, then $Ba = Ba^{-1}, Bc$ and $Bac$ are three pair-wise different orbits of $N$. Clearly, $Bc, Bac \subseteq V_1$. So $B_1 = Bc$ or $Bac$. Note that $\Gamma[B, Bc]$ has valency 6 and $\Gamma[B \cup Bac]$ has valency 4.

If $n > 2$, then both $\Gamma[B \cup Ba]$ and $\Gamma[B \cup Ba^{-1}]$ have valency 2. This implies that $A_B$ also fixes $Bc$ and $Bac$. If $n = 2$, then $\Gamma[B \cup Ba]$ has valency 4, and from Figure 2 one may see that $A_H$ fixes $\{Hab, Ha^{-1}b^{-1}, Hab^{-1}, Ha^{-1}b\}$ setwise and so $A_H$ fixes $Ba$. Again, we have $A_B$ also fixes each of $Bc$ and $Bac$.

Thus, we always have that $A_B$ also fixes each of $Bc$ and $Bac$. Clearly, the subgraph $\Gamma[B \cup B_1]$ is bipartite, where $B_1$ is either $Bc$ or $Bac$. Let $K$ be the subgroup of $\mathrm{Aut}(\Gamma[B \cup B_1])$ fixing $B$ setwise. Then $B$ and $B_1$ are two orbits of $K$. Let $K_{(B)}$ be the kernel of $K$ acting on $B$. If $K_{(B)}$ does not fix every vertex of $B_1$, then each orbit of $K_{(B)}$ on $B_1$ has length $2p$, $p$ or 2. Take two vertices $u, v$ of $B_1$ such that $u, v$ are in the same orbit of $K_{(B)}$. Then $u, v$ will share the common neighborhood in $\Gamma[B \cup B_1]$. Without loss of generality, we may assume that $H$ is one of their common neighbors.

If $B_1 = Bac$, then $u, v \in \{Habc, Ha^{1+2^{n-1}}b^r c, Hab^{-1}c, Ha^{1+2^{n-1}}b^{-r}c\}$. Note that

$$\begin{aligned}
\Gamma(Habc) \cap B &= \{Ha^{2^{n-1}}b^{r-1}, H, Ha^{2^{n-1}}b^{1+r}, Hb^{2r}\},\\
\Gamma(Ha^{1+2^{n-1}}b^r c) \cap B &= \{Hb^{-2}, Ha^{2^{n-1}}b^{-1-r}, H, Ha^{2^{n-1}}b^{r-1}\},\\
\Gamma(Hab^{-1}c) \cap B &= \{Ha^{2^{n-1}}b^{-r-1}, Hb^{-2r}, Ha^{2^{n-1}}b^{1-r}, H\},\\
\Gamma(Ha^{1+2^{n-1}}b^{-r}c) \cap B &= \{H, Ha^{2^{n-1}}b^{1-r}, Hb^2, Ha^{2^{n-1}}b^{r+1}\}.
\end{aligned}$$

It is easy to see no two of the above four sets are the same, and so no two vertices in $\{Habc, Ha^{1+2^{n-1}}b^r c, Hab^{-1}c, Ha^{1+2^{n-1}}b^{-r}c\}$ share the common neighborhood in $\Gamma[B, Bac]$, a contradiction.

If $B_1 = Bc$, then $u, v \in \{Hbc, Ha^{2^{n-1}}b^r c, Hb^{-1}c, Ha^{2^{n-1}}b^{-r}c, Hc, Ha^{2^{n-1}}c\}$. Note that

$$\begin{aligned}
\Gamma(Hbc) \cap B &= \{Ha^{2^{n-1}}b^{r-1}, H, Ha^{2^{n-1}}b^{1+r}, Hb^{2r}, Ha^{2^{n-1}}b^r, Hb^r\},\\
\Gamma(Ha^{2^{n-1}}b^r c) \cap B &= \{Hb^{-2}, Ha^{-2^{n-1}}b^{-1-r}, H, Ha^{-2^{n-1}}b^{r-1}, Hb^{-1}, Ha^{-2^{n-1}}b^{-1}\},\\
\Gamma(Hb^{-1}c) \cap B &= \{Ha^{2^{n-1}}b^{-r-1}, Hb^{-2r}, Ha^{2^{n-1}}b^{1-r}, H, Ha^{2^{n-1}}b^{-r}, Hb^{-r}\},\\
\Gamma(Ha^{2^{n-1}}b^{-r}c) \cap B &= \{H, Ha^{-2^{n-1}}b^{1-r}, Hb^2, Ha^{-2^{n-1}}b^{r+1}, Hb, Ha^{-2^{n-1}}b\},\\
\Gamma(Hc) \cap B &= \{Ha^{2^{n-1}}b^{-1}, Hb^{-r}, Ha^{2^{n-1}}b, Hb^r, Ha^{2^{n-1}}, H\},\\
\Gamma(Ha^{2^{n-1}}c) \cap B &= \{Hb^{-1}, Ha^{2^{n-1}}b^{-r}, Hb, Ha^{-2^{n-1}}b^r, H, Ha^{-2^{n-1}}\}.
\end{aligned}$$

It is easily checked that the above six sets are pair-wise different, and no two vertices in $\{Hbc, Ha^{2^{n-1}}b^rc, Hb^{-1}c, Ha^{2^{n-1}}b^{-r}c, Hc, Ha^{2^{n-1}}c\}$ share the common neighborhood in $\Gamma[B, Bc]$, a contradiction. Thus, $K_{(B)}$ also fixes every vertex of $B_1$. Consequently, we have $A_{(B_0)} = A_{(B)} = A_{(B_1)}$ with $B_1 = Bc$ or $Bac$.

**Step 4:** $A = G$.

By Step 3, $A_H$ fixes $\Delta_1$ setwise and so fixes $\Delta_2$ setwise. By Lemma 2.3, we have $A \leq \mathrm{Aut}(\Lambda)$, where $\Lambda = \mathrm{Cos}(G, H, D_2)$ with $D_2 = H\{(abc)^{\pm 1}, (bc)^{\pm 1}, c^{\pm 1}\}H$. It is easy to see that $\Lambda$ is a connected bipartite graph with $V_0$ and $V_1$ as its two parts. Let $K$ be the kernel of $A$ acting on $\{V_0, V_1\}$. Again, by Step 3, we obtain that $K$ acts faithfully on $V_0$. It is easy to see that $\Gamma[V_0] \cong \Theta = \mathrm{Cay}(\langle ab\rangle, \{ab, (ab)^{-1}, ab^{-1}, a^{-1}b\})$. By [2, Corollary 1.3], $\Theta$ is a normal Cayley graph on $\langle ab\rangle$. Then $\mathrm{Aut}(\langle ab\rangle, \{ab, (ab)^{-1}, ab^{-1}, a^{-1}b\}) \cong C_2 \times C_2$ is regular on $\{ab, (ab)^{-1}, ab^{-1}, a^{-1}b\}$. So, $|K| \leq 4|V_0|$. It follows that $|A| \leq 4|V(\Gamma)|$ and hence $|A : G| \leq 2$. Consequently, we have $G \trianglelefteq A$. By Proposition 2.2, we have $A_H = \mathrm{Aut}(G, H, D) \leq \mathrm{Aut}(\langle ab\rangle, \{ab, (ab)^{-1}, ab^{-1}, a^{-1}b\})$. If $n > 2$, then from Figures 1 and 3, we can see that $A_H$ is intransitive on the set of four neighbors of $H$ contained in $V_0$. It follows that $A_H \cong C_2$ and hence $A = G$. If $n = 2$ and $p > 5$, then by Lemma 3.1, we must have $\mathrm{Aut}(G, H, D) \cong C_2$, implying that $A = G$. □

**Corollary 3.3.** *The graph $\Gamma_{2,n,p,r}$ is non-Cayley.*

*Proof.* Let $\Gamma = \Gamma_{2,n,p,r}$, and let $A = \mathrm{Aut}(\Gamma)$. Suppose on the contrary that $\Gamma$ is a Cayley graph. Then $A$ has a regular subgroup, say $T$, and then $A = T : A_H$. Since $A$ is metacyclic, every Sylow 2-subgroup of $T$ must be cyclic. It follows that every Sylow 2-subgroup of $A$ has a cyclic maximal subgroup. However, this is impossible because from the Construction B we know that every Sylow 2-subgroup of $A$ is isomorphic to $C_{2^n} : C_4$, a contradiction. □

**Theorem 3.4.** *The graph $\Gamma_{2,n,p,r}$ is a pseudo metacirculant.*

*Proof.* Let $\Gamma = \Gamma_{2,n,p,r}$, and let $G = G_{2,n,p,r}$. Note that $G$ acts faithfully and transitively on $V(\Gamma)$ by right multiplication. Since $G = \langle ab\rangle : \langle c\rangle \cong C_{2^n \cdot p} : C_4$, $\Gamma$ is a split weak metacirculant.

Suppose that $\Gamma$ is also a metacirculant. Then by the definition of metacirculant, $\Gamma$ has two automorphisms $\sigma, \tau$ satisfying the following conditions:

(1) $\langle\sigma\rangle$ is semiregular and has $m$ orbits on $V(\Gamma)$,

(2) $\tau$ normalizes $\langle\sigma\rangle$ and cyclically permutes the $m$ orbits of $\langle\sigma\rangle$,

(3) $\tau$ has a cycle of size $m$ in its cycle decomposition.

By Lemma 3.2, we have $\mathrm{Aut}(\Gamma) = G$. By Corollary 3.3, $\Gamma$ is a non-Cayley graph, and then we have $G = \langle\sigma, \tau\rangle$. Thus, $\tau^m \neq 1$ and hence $\langle\tau^m\rangle \cong C_2$. Since $G$ is transitive on $V(\Gamma)$, we may assume that $\langle\tau^m\rangle = G_H = \langle a^{2^{n-1}}c^2\rangle$. Then there would exist an element $x$ of $G$ of order 4 such that $x^2 = \tau^m = a^{2^{n-1}}c^2$. By a direct computation, we have $C_G(a^{2^{n-1}}c^2) = \langle a, c\rangle$. Then $x \in \langle a, c\rangle$ and so $x = a^ic^j$ for some integers $i, j$. However, $x^2 = c^{2j}$ due to $c^{-1}ac = a^{-1}$. A contradiction occurs. Thus, $\Gamma$ is not a metacirculant. □

## 4 Pseudo metacirculants—Family B

**Consruction B.** *Let $m > n > 1$ be positive integers and let $p, q$ be primes such that $p^m \mid q-1$. Let $r \in \mathbb{Z}_q^*$ be of order $p^m$, and let*

$$G_{p,q,m,n,r} = \langle a, b, c \mid a^{p^n} = b^q = c^{p^m} = 1, ab = ba, ac = ca, c^{-1}bc = b^r \rangle.$$

*Let*

$$\Gamma_{p,q,m,n,r} = \mathrm{Cos}(G_{p,q,m,n,r}, H, H\{(ab)^{\pm 1}, c^{\pm 1}\}H),$$

*where $H = \langle c^{p^{m-1}} a^{p^{n-1}} \rangle$.*

We shall first give some basic properties of $G$.

**Lemma 4.1.** *Let $G = G_{p,q,m,n,r}$, and let $D = H\{(ab)^{\pm 1}, c^{\pm 1}\}H$. Then the following hold.*

(1) *$|H| = p$, $H$ is non-normal in $G$ and $C_G(H) = \langle a, c \rangle$.*

(2) *$C_G(b) = \langle a, b \rangle$.*

(3) *For any $g \in G$, if $\langle g \rangle \trianglelefteq G$, then $g \in \langle a, b \rangle$.*

(4) *$D = (\bigcup_{k \in \mathbb{Z}_p} H(ab^{r^{kp^{m-1}}})) \cup (\bigcup_{k \in \mathbb{Z}_p} H(ab^{r^{kp^{m-1}}})^{-1}) \cup Hc \cup Hc^{-1}$.*

(5) *$\mathrm{Aut}(G, H, D) \cong C_p$.*

*Proof.* Note that $G = \langle a \rangle \times (\langle b \rangle : \langle c \rangle) \cong \mathbb{Z}_{p^n} \times (\mathbb{Z}_q : \mathbb{Z}_{p^m})$. Let $P = \langle a, c \rangle$. Clearly, $P = \langle a \rangle \times \langle c \rangle \cong C_{p^n} \times C_{p^m}$, so $H = \langle c^{p^{m-1}} a^{p^{n-1}} \rangle$ has order $p$. If $H \trianglelefteq G$, then $b$ centralizes $c^{p^{m-1}} a^{p^{n-1}}$ and then centralizes $c^{p^{m-1}}$ since $a \in Z(G)$. This is impossible because $c^{-p^{m-1}} b c^{p^{m-1}} = b^{r^{p^{m-1}}} \neq b$. Thus, $H$ is non-normal in $G$. It follows that $\langle a, c \rangle \leq C_G(H) < G$. Observing that $|G : \langle a, c \rangle| = q$, we have $C_G(H) = \langle a, c \rangle$. Therefore, item (1) holds.

For (2) , it is easy to see that $\langle a, b \rangle \leq C_G(b)$ and $G = \langle a, b \rangle : \langle c \rangle$. Recall that $c^{-1}bc = b^r$ with $r$ an element of $\mathbb{Z}_q^*$ of order $p^m$. This implies that $C_G(b) = \langle a, b \rangle$.

For (3) , recall that $G/\langle a \rangle \cong \langle b \rangle : \langle c \rangle \cong \mathbb{Z}_q : \mathbb{Z}_{p^m}$, and $\langle b \rangle$ is self-centralized in $\langle b \rangle : \langle c \rangle$. This implies that $\langle b \rangle$ is the unique non-trivial normal cyclic subgroup of $\langle b \rangle : \langle c \rangle$. For any $g \in G$, if $\langle g \rangle \trianglelefteq G$, then $\langle g \rangle \langle a \rangle / \langle a \rangle$ is normal in $G/\langle a \rangle$, and then $\langle g \rangle \langle a \rangle / \langle a \rangle \leq \langle b \rangle \langle a \rangle / \langle a \rangle$. So $g \in \langle a, b \rangle$, as desired.

For (4) , we have $D = H\{(ab)^{\pm 1}, c^{\pm 1}\}H = H\{ab, (ab)^{-1}\}H \cup H\{c, c^{-1}\}H$. Since $c$ centralizes $H$, one has $H\{c, c^{-1}\}H = Hc \cup Hc^{-1}$. Clearly,

$$H\{ab, (ab)^{-1}\}H = HabH \cup H(ab)^{-1}H.$$

Then $HabH = \bigcup_{k \in \mathbb{Z}_p} Hab(c^{p^{m-1}} a^{p^{n-1}})^k = \bigcup_{k \in \mathbb{Z}_p} Habc^{kp^{m-1}} a^{kp^{n-1}}$. Since $c^{-1}bc = b^r$, one has $bc^{kp^{m-1}} = c^{kp^{m-1}} b^{r^{kp^{m-1}}}$. As $a \in Z(G)$, it follows that

$$HabH = \bigcup_{k \in \mathbb{Z}_p} Habc^{kp^{m-1}} a^{kp^{n-1}} = \bigcup_{k \in \mathbb{Z}_p} Hab^{r^{kp^{m-1}}}.$$

Similarly, $H(ab)^{-1}H = \bigcup_{k\in\mathbb{Z}_p} H(ab^{r^{kp^{m-1}}})^{-1}$. (4) is proved.

Finally, we shall prove (5) . Take $\alpha \in \mathrm{Aut}(G, H, D)$. Then $H^\alpha = H$ and $D^\alpha = D$. Observe that $\langle b\rangle$ is a normal Sylow $q$-subgroup of $G$ and $\langle a\rangle$ is just the center of $G$. It follows that $b^\alpha \in \langle b\rangle$ and $a^\alpha \in \langle a\rangle$. Since $H^\alpha = H$, one has $c^\alpha \in C_G(H) = \langle a, c\rangle$. Assume that

$$a^\alpha = a^i, b^\alpha = b^j, c^\alpha = a^s c^t, \text{ with } i \in \mathbb{Z}^*_{p^n}, j \in \mathbb{Z}^*_q, s \in \mathbb{Z}_{p^n}, t \in \mathbb{Z}^*_{p^m}.$$

Considering the image of the equality $c^{-1}bc = b^r$ under $\alpha$, we obtain that

$$(a^s c^t)^{-1} b^j (a^s c^t) = b^{jr},$$

and hence $b^{jr^t} = b^{jr}$. It follows that $r^t \equiv r \pmod q$. Since $r$ is an element of $\mathbb{Z}^*_{p^m}$ of order $p^m$ and $t \in \mathbb{Z}^*_{p^m}$, the equality $r^t \equiv r \pmod q$ implies that $t = 1$ in $\mathbb{Z}_{p^m}$, and so $c^\alpha = a^s c$. Consequently, $a^s c = c^\alpha \in D^\alpha = D$. By a direct computation, we have $a^\ell c \notin D$ for any $\ell \neq 0$ (in $\mathbb{Z}_{p^n}$). So we must have $c^\alpha = c$.

Since $ab \in D$, one has $a^i b^j = (ab)^\alpha \in D$. Since $j \in \mathbb{Z}^*_q$, one has $a^i b^j \notin Hc \cup Hc^{-1}$ because $(Hc \cup Hc^{-1}) \subseteq \langle a, c\rangle$. Then by (4) we have

$$a^i b^j \in (\bigcup_{k\in\mathbb{Z}_p} H(ab^{r^{kp^{m-1}}})) \cup (\bigcup_{k\in\mathbb{Z}_p} H(ab^{r^{kp^{m-1}}})^{-1}).$$

Note that $\langle a, b\rangle \cap H = 1$. It follows that $(ab)^\alpha = a^i b^j \in \{(ab^{r^{kp^{m-1}}})^{\pm 1} \mid k \in \mathbb{Z}_p\}$, and so $a^i b^j = ab^{r^{kp^{m-1}}}$ or $(ab^{r^{kp^{m-1}}})^{-1}$ for some $k \in \mathbb{Z}_p$. Since $\langle a, b\rangle = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_q$, one has

$$(a, b)^\alpha = (a^i, b^j) = (a, b^{r^{kp^{m-1}}}) \text{ or } (a^{-1}, b^{-r^{kp^{m-1}}}).$$

Since $H^\alpha = H$, one has $c^{p^{m-1}}(a^i)^{p^{n-1}} = (c^{p^{m-1}} a^{p^{n-1}})^\alpha \in H$, and hence $a^\alpha = a$. It follows that $(a, b)^\alpha = (a^i, b^j) = (a, b^{r^{kp^{m-1}}})$. Hence $|\mathrm{Aut}(G, H, D)| \leq p$. Note that $H \leq \mathrm{Aut}(G, H, D)$. Then $\mathrm{Aut}(G, H, D) \cong C_p$. □

Next we shall determine the full automorphism group of $\Gamma_{p,q,n,m,r}$.

**Lemma 4.2.** *Let* $\Gamma = \Gamma_{p,q,m,n,r}$ *and let* $G = G_{p,q,m,n,r}$*. Then* $\mathrm{Aut}(\Gamma) = R_H(G) \cong G$*. Moreover,* $\Gamma$ *is a non-Cayley graph.*

*Proof.* We shall first prove three claims.

**Claim 1.** *Let* $\Lambda = \mathrm{Cos}(G, H, H\{ab, (ab)^{-1}\}H)$*. Then* $\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\Lambda)$*.*

*Proof of* Claim 1. By Lemma 4.1(4), the neighborhood of $H$ in $\Gamma$ is

$$\Gamma(H) = \{H(ab^{r^{kp^{m-1}}})^{\pm 1}, Hc^{\pm 1} \mid k \in \mathbb{Z}_p\},$$

and the neighborhood of $H$ is $\Lambda$ is

$$\Lambda(H) = \{H(ab^{r^{kp^{m-1}}})^{\pm 1} \mid k \in \mathbb{Z}_p\}.$$

By direct computations, we see that for any $k \in \mathbb{Z}_p$,

$$\{H, Ha^2b^{1+r^{kp^{m-1}}} \mid k \in \mathbb{Z}_p\} \subseteq \Gamma(Hab^{r^{kp^{m-1}}}) \cap \Gamma(Hab),$$
$$\{H, H(a^2b^{1+r^{kp^{m-1}}})^{-1} \mid k \in \mathbb{Z}_p\} \subseteq \Gamma(H(ab^{r^{kp^{m-1}}})^{-1}) \cap \Gamma(H(ab)^{-1}),$$

and moreover, for any $Hx \in \{H(ab^{r^{kp^{m-1}}})^{\pm 1} \mid k \in \mathbb{Z}_p\}$ and $Hc^\ell \in \{Hc, Hc^{-1}\}$, we have

$$\Gamma(Hx) \cap \Gamma(Hc^\ell) = \{H\}.$$

It then follows that the vertex-stabilizer $\mathrm{Aut}(\Gamma)_H$ fixes the following set setwise:

$$\Lambda(H) = \{H(ab^{r^{kp^{m-1}}})^{\pm 1} \mid k \in \mathbb{Z}_p\}.$$

By Lemma 2.3, we have $\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\Lambda)$. □

**Claim 2.** *Let $V_i = \{Ha^jb^kc^i \mid j \in \mathbb{Z}_{p^n}, k \in \mathbb{Z}_q\}$ with $i \in \mathbb{Z}_{p^{m-1}}$. Then the following hold.*

(1) *$V_i$ is a block of imprimitivity of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$.*

(2) *The edges of $\Gamma$ between $V_i$ and $V_{i+1}$ form a perfect matching, where the subscripts are modulo $p^{m-1}$.*

(3) *Let $\mathcal{B} = \{V_0, V_1, \ldots, V_{p^{m-1}-1}\}$. Then the quotient graph $\Gamma_{\mathcal{B}}$ is isomorphic to $\mathrm{C}_{p^{m-1}}$.*

*Proof of* Claim 2. By Claim 1, we have $\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\Lambda)$. Recall that

$$\Lambda = \mathrm{Cos}(G, H, H\{ab, (ab)^{-1}\}H).$$

Then $\Lambda$ is disconnected, and the coset graph

$$\Delta = \mathrm{Cos}(\langle ab, H\rangle, H, H\{ab, (ab)^{-1}\}H)$$

is a component of $\Lambda$. Consequently, $V_0 = V(\Delta) = \{Hg \mid g \in \langle ab\rangle\}$ is a block of imprimitivity of $\mathrm{Aut}(\Gamma)$ on $V(\Lambda) = V(\Gamma)$. Clearly, each $V_i$ is an orbit of $\langle ab\rangle$ on $V(\Gamma)$. Since $\langle ab\rangle \trianglelefteq G$ and $G$ is transitive on $V(\Gamma)$, $V_i$ is a block of imprimitivity of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$.

For (2), observing that $V_1 \cap \Gamma(H) = \{Hc\}$, the edges between $V_0$ and $V_1$ form a perfect matching. As $c$ cyclically permutates the orbits $V_0, V_1, \ldots, V_{p^{m-1}-1}$ of $\langle ab\rangle$, for every $i \in \mathbb{Z}_{p^{m-1}}$, the subgraph of $\Gamma$ induces by the edges between $V_i$ and $V_{i+1}$ form a perfect matching, where the subscripts are modulo $p^{m-1}$.

For (3), noting that $\Gamma$ has valency $2p+2$ while $\Delta$ has valency $2p$, the quotient graph $\Gamma_{\mathcal{B}}$ must be a cycle of length $p^{m-1}$. □

**Claim 3.** *Let $\Delta = \mathrm{Cos}(\langle ab, H\rangle, H, H\{ab, (ab)^{-1}H\})$. Then $|\mathrm{Aut}(\Delta)| = 2p^{n+1}q$.*

*Proof of* Claim 3. It is easy to see that $\Delta$ is isomorphic to the following Cayley graph

$$\Theta = \mathrm{Cay}(\langle ab\rangle, \{ab^{r^{kp^{m-1}}}, (ab^{r^{kp^{m-1}}})^{-1} \mid k \in \mathbb{Z}_p\}).$$

Let $M = \langle a, b\rangle (\cong \mathbb{Z}_{p^n q})$ and $S = \{ab^{r^{kp^{m-1}}}, (ab^{r^{kp^{m-1}}})^{-1} \mid k \in \mathbb{Z}_p\}$. The maps $\alpha\colon a \mapsto a, b \mapsto b^{r^{p^{m-1}}}$ and $:a \mapsto a^{-1}, b \mapsto b^{-1}$ induce two automorphisms of $M$ which fix $S$ setwise. So, $\langle \alpha, \beta\rangle \leq \mathrm{Aut}(M, S)$. It is easy to see that $\langle \alpha, \beta\rangle$ acts transitively on $S$. Hence $\Theta$ is a connected arc-transitive Cayley graph on $M$. Suppose that $\Theta$ is not normal. It is obvious that $\Theta \not\cong \mathbf{K}_{p^n q}$. By Proposition 2.4, there exists a connected arc-transitive circulant $\Sigma$ of order $t$ such that one of the following happens:

(i) $\Theta \cong \Sigma \circ \bar{\mathbf{K}}_d$ with $p^n q = td$,

(ii) $\Theta \cong \Sigma \circ \bar{\mathbf{K}}_d - d\Sigma$, where $p^n q = td$, $d > 3$ and $\gcd(\mathrm{d}, \mathrm{t}) = 1$.

Suppose that (i) happens. Let $k$ be the valency of $\Sigma$. Then $\Sigma[\bar{\mathbf{K}}_d]$ has valency $kd$. Noting that $\Theta$ has valency $2p$, $\Theta \cong \Sigma \circ \bar{\mathbf{K}}_d$ implies that $kd = 2p$. As $p^n q = td$, one has $d = p$, and hence $k = 2$. It follows that $\Sigma$ is a cycle of length $t$. Also, $\Theta \cong \Sigma \circ \bar{\mathbf{K}}_p$ implies that there exist $t$ independent subsets, say $D_0, D_1, D_2, \ldots, D_{t-1}$ of $V(\Theta)$ of cardinality $p$ such that the subgraph induced by $D_i \cup D_{i+1}$ is isomorphic to $\mathbf{K}_{p,p}$, where the subscripts are modulo $t$. Furthermore, these $t$ subsets are also blocks of imprimitivity of $\mathrm{Aut}(\Theta)$ on $V(\Theta)$. Assume that $D_0$ contains the identity of $M$. Since $M$ acts on $V(\Theta)$ by right multiplication, $D_0$ will be a subgroup of $M$ of order $p$, and then $D_0 = \langle a^{p^{n-1}}\rangle$ and then each $D_i$ is a coset of $D_0$ in $M$. Recall that the only two blocks adjacent to $D_0$ are $D_1$ and $D_2$, and that any two adjacent blocks induce a subgraph isomorphic to $\mathbf{K}_{p,p}$. Then $D_1 \cup D_{t-1} = S$. We may assume that $ab \in D_1$. Then $D_1 = D_0 ab$ and $D_{t-1} = D_0(ab)^{-1}$. This is contrary to the fact that $S = \{ab^{r^{kp^{m-1}}}, (ab^{r^{kp^{m-1}}})^{-1} \mid k \in \mathbb{Z}_p\}$.

Suppose now that (ii) holds. Observing that the valency of $\Sigma \circ \bar{\mathbf{K}}_d - d\Sigma$ is a multiple of $d-1$, one has $d - 1 \mid 2p$, and hence $d - 1 = p$ due to $d > 3$. As $p^n q = td$ and $d = p + 1$, one has $p + 1 \mid p^n q$, implying $p + 1 = q$. This is contrary to the fact that $p, q$ are odd.

Thus, $\Theta$ is a normal Cayley graph on $M$. By Proposition 2.1, we have $\mathrm{Aut}(\Theta) = R(M) : \mathrm{Aut}(M, S)$. As $M$ is cyclic, $\mathrm{Aut}(M, S)$ is abelian, and so $\mathrm{Aut}(M, S)$ acts regularly on $S$. It follows that $|\mathrm{Aut}(M, S)| = 2p$, and so $|\mathrm{Aut}(\Theta)| = 2p^{n+1}q$, as claimed. □

*Proof of* Lemma 4.2, *continued:* Now we are ready to finish the proof. Let $A = \mathrm{Aut}(\Gamma)$. By Claim 2(1), $\mathcal{B} = \{V_0, V_1, \ldots, V_{p^{m-1}-1}\}$ is a system of blocks of $A$. Let $K$ be the kernel of $A$ acting on $\mathcal{B}$. By Claim 2(3), we have $A/K \leq \mathrm{Aut}(\Gamma_{\mathcal{B}}) \cong D_{2p^{m-1}}$. By Claim 2(2), one may see that $K$ acts faithfully on $V_0$. So $K$ can be viewed as a subgroup of the full automorphism group of the subgraph $\Delta$ of $\Gamma$ induced by $V_0$. By Claim 3, we have $|\mathrm{Aut}(\Delta)| = 2p^{n+1}q$, and so $|K| \leq 2p^{n+1}q$.

From Lemma 4.1(1) we know that $H$ is non-normal in $G$, and so $G$ acts faithfully on $V(\Gamma)$ by right multiplication. Therefore, we may identify $G$ with $R_H(G)$, and then $G$ is a vertex-transitive subgroup of $A$. Then $GK/K \cong C_{p^{m-1}}$ which is normal in $A/K \leq D_{2p^{m-1}}$. In particular, $GK \trianglelefteq A$. Furthermore, $|GK/K| = p^{m-1}$ implies that $|GK| = p^{m-1}|K| \leq 2p^{m+n}q$. So $|GK : G| \leq 2$, and hence $G$ is the unique Hall $\{p, q\}$-subgroup of $GK$. Thus, $G$ is characteristic in $GK$, and so normal in $A$ since $GK \trianglelefteq A$. By Proposition 2.2, the stabilizer of $H$ in $A$ is $A_H = \mathrm{Aut}(G, H, D)$. By Lemma 4.1(5), we have $\mathrm{Aut}(G, H, D) \cong C_p$. This implies that $G = A$.

Finally, suppose that $\Gamma$ is a Cayley graph. Then $A$ has a regular subgroup, say $T$, and then $A = T : A_H$. Since $A = G$ is metacyclic, every Sylow $p$-subgroup of $T$ must be cyclic because $A_H \cong C_p$. It follows that every Sylow $p$-subgroup of $A$ has a cyclic maximal subgroup. However, this is impossible because from the Construction B we know that every Sylow $p$-subgroup of $A$ is isomorphic to $C_{p^n} : C_{p^m}$ with $m > n > 1$. $\square$

**Theorem 4.3.** *The graph $\Gamma_{p,q,m,n,r}$ is a pseudo metacirculant.*

*Proof.* Let $\Gamma = \Gamma_{p,q,n,m,r}$, and let $G = G_{p,q,n,m,r}$. Note that $G$ acts faithfully and transitively on $V(\Gamma)$ by right multiplication. Since $G = \langle ab \rangle : \langle c \rangle \cong C_{p^n q} : C_{p^m}$, $\Gamma$ is a split weak metacirculant.

Suppose that $\Gamma$ is also a metacirculant. Then by the definition of metacirculant, $\Gamma$ has two automorphisms $\sigma, \tau$ satisfying the following conditions:

(1) $\langle \sigma \rangle$ is semiregular and has $t$ orbits on $V(\Gamma)$,

(2) $\tau$ normalizes $\langle \sigma \rangle$ and cyclically permutes the $t$ orbits of $\langle \sigma \rangle$,

(3) $\tau$ has a cycle of size $t$ in its cycle decomposition.

By Lemma 4.2, we have $A = G$ and $\Gamma$ is a non-Cayley graph. Since $|G| = |V(\Gamma)|p$, we must have $G = \langle \sigma, \tau \rangle$. Thus, $\tau^t \neq 1$ and hence $\langle \tau^t \rangle \cong C_p$. Since $G$ is transitive on $V(\Gamma)$, we may assume that $\langle \tau^t \rangle = G_H = H = \langle a^{p^{n-1}} c^{p^{m-1}} \rangle$. By Lemma 4.1(3), we have $\sigma \in \langle a, b \rangle$, and so $o(\sigma) \mid p^n q$. Consequently, $p^m \mid o(\tau)$. Let $x \in \langle \tau \rangle$ be of order $p^m$ such that $x^{p^{m-1}} = \tau^t = a^{p^{n-1}} c^{p^{m-1}}$. Then $x \in C_G(H)$, and by Lemma 4.1(1), $C_G(H) = \langle a, c \rangle$. So $x = a^i c^j$ for some integers $i, j$. However, $x^{p^{m-1}} = a^{ip^{m-1}} c^{jp^{m-1}} = c^{jp^{m-1}}$ due to $m > n$. A contradiction occurs. Thus, $\Gamma$ is not a metacirculant. $\square$

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