A new family of maximum scattered linear sets in $\text{PG}(1, q^6)$

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Abstract

We generalize the example of linear set presented by the last two authors in “Vertex properties of maximum scattered linear sets of $\text{PG}(1, q^n)$” (2019) to a more general family, proving that such linear sets are maximum scattered when $q$ is odd and, apart from a special case, they are new. This solves an open problem posed in “Vertex properties of maximum scattered linear sets of $\text{PG}(1, q^n)$” (2019). As a consequence of Sheekey’s results in “A new family of linear maximum rank distance codes” (2016), this family yields to new MRD-codes with parameters $(6, 6, q; 5)$.

Keywords: Scattered linear set, MRD-code, linearized polynomial.


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1 Introduction

Let $\Lambda = \text{PG}(V,\mathbb{F}_{q^n}) = \text{PG}(1, q^n)$, where $V$ is a vector space of dimension 2 over $\mathbb{F}_{q^n}$. If $U$ is a $k$-dimensional $\mathbb{F}_q$-subspace of $V$, then the $\mathbb{F}_q$-linear set $L_U$ is defined as

$$L_U = \{ \langle u \rangle_{\mathbb{F}_{q^n}} : u \in U \setminus \{0\} \},$$

and we say that $L_U$ has rank $k$. Two linear sets $L_U$ and $L_W$ of $\text{PG}(1, q^n)$ are said to be PT-linear-equivalent if there is an element $\phi$ in $\text{PGL}(2, q^n)$ such that $L_U^\phi = L_W$. It may happen that two $\mathbb{F}_q$-linear sets $L_U$ and $L_W$ of $\text{PG}(1, q^n)$ are PT-linear-equivalent even if the $\mathbb{F}_q$-vector subspaces $U$ and $W$ are not in the same orbit of $\Gamma L(2, q^n)$ (see [5, 12] for further details).

In this paper we focus on maximum scattered $\mathbb{F}_q$-linear sets of $\text{PG}(1, q^n)$, that is, $\mathbb{F}_q$-linear sets of rank $n$ in $\text{PG}(1, q^n)$ of size $(q^n - 1)/(q - 1)$.

If $\langle (0, 1) \rangle_{\mathbb{F}_{q^n}}$ is not contained in the linear set $L_U$ of rank $n$ of $\text{PG}(1, q^n)$ (which we can always assume after a suitable projectivity), then $U = U_f := \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$ for some linearized polynomial (or $q$-polynomial) $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$. In this case we will denote the associated linear set by $L_f$. If $L_f$ is scattered, then $f(x)$ is called a scattered $q$-polynomial; see [24].

The first examples of scattered linear sets were found by Blokhuis and Lavrauw in [3] and by Lunardon and Polverino in [18] (recently generalized by Sheekey in [24]). Apart from these, very few examples are known, see Section 3.

In [24, Section 5], Sheekey established a connection between maximum scattered linear sets of $\text{PG}(1, q^n)$ and MRD-codes, which are interesting because of their applications to random linear network coding and cryptography. We point out his construction in the last section. By the results of [1] and [2], it seems that examples of maximum scattered linear sets are rare.

In this paper we will prove that any

$$f_h(x) = h^{q-1}x^{q} - h^{q-2}x^{q^2} + x^{q^4} + x^{q^5}, \quad h \in \mathbb{F}_{q^n}, \quad h^{q^3+1} = -1, \quad q \text{ odd} \quad (1.1)$$

is a scattered $q$-polynomial. This will be done by considering two cases:

**Case 1:** $h \in \mathbb{F}_q$, that is, $f_h(x) = x^{q} - x^{q^2} + x^{q^4} + x^{q^5}$; the condition $h^{q^3+1} = -1$ implies $q \equiv 1 \pmod{4}$.

**Case 2:** $h \not\in \mathbb{F}_q$. In this case $h \neq \pm \sqrt{-1}$, otherwise $h \in \mathbb{F}_{q^2}$ and then we have $h^{q^3+1} = 1$, a contradiction to $h^{q^3+1} = -1$.

Note that in Case 1, this example coincides with the one introduced in [27], where it has been proved that $f_h$ is scattered for $q \equiv 1 \pmod{4}$ and $q \leq 29$. In Corollary 3.11 we will prove that the linear set $\mathcal{L}_h$ associated with $f_h(x)$ is new, apart from the case of $q$ a power of 5 and $h \in \mathbb{F}_q$. This solves an open problem posed in [27].

Finally, in Section 4 we prove that the $\mathbb{F}_q$-linear MRD-codes with parameters $(6, 6, q; 5)$ arising from linear sets $\mathcal{L}_h$ are not equivalent to any previously known MRD-code, apart from the case $h \in \mathbb{F}_q$ and $q$ a power of 5; see Theorem 4.1.

2 $\mathcal{L}_h$ is scattered

A $q$-polynomial (or linearized polynomial) over $\mathbb{F}_{q^n}$ is a polynomial of the form

$$f(x) = \sum_{i=0}^{t} a_i x^{q^i},$$
where $a_i \in \mathbb{F}_{q^n}$ and $t$ is a positive integer. We will work with linearized polynomials of degree less than or equal to $q^n - 1$. For such a kind of polynomial, the Dickson matrix$^1$ $M(f)$ is defined as

$$
M(f) := \begin{pmatrix}
a_0 & a_1 & \cdots & a_{n-1} \\
a_{n-1}^q & a_0 & \cdots & a_{n-2}^q \\
\vdots & \vdots & \ddots & \vdots \\
a_1^{q^{n-1}} & a_2^{q^{n-1}} & \cdots & a_0^{q^{n-1}}
\end{pmatrix} \in \mathbb{F}_{q^n}^{n \times n},
$$

where $a_i = 0$ for $i > t$.

Recently, different results regarding the number of roots of linearized polynomials have been presented, see [4, 9, 22, 23, 26]. In order to prove that a certain polynomial is scattered, we make use of the following result; see [4, Corollary 3.5].

**Theorem 2.1.** Consider the $q$-polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ over $\mathbb{F}_{q^n}$ and, with $m$ as a variable, consider the matrix

$$
M(m) := \begin{pmatrix}
m & a_1 & \cdots & a_{n-1} \\
a_{n-1}^q & m^q & \cdots & a_{n-2}^q \\
\vdots & \vdots & \ddots & \vdots \\
a_1^{q^{n-1}} & a_2^{q^{n-1}} & \cdots & m^{q^{n-1}}
\end{pmatrix}.
$$

The determinant of the $(n - i) \times (n - i)$ matrix obtained by $M(m)$ after removing the first $i$ columns and the last $i$ rows of $M(m)$ is a polynomial $M_{n-i}(m) \in \mathbb{F}_{q^n}[m]$. Then the polynomial $f(x)$ is scattered if and only if $M_0(m)$ and $M_1(m)$ have no common roots.

### 2.1 Case 1

Let

$$
f(x) = x^q - x^{q^2} + x^{q^4} + x^{q^5} \in \mathbb{F}_{q^6}[x].$$

By Theorem 2.1, $f(x)$ is scattered if and only if for each $m \in \mathbb{F}_{q^6}$ the determinants of the following two matrices do not vanish at the same time

$$M_5(m) = \begin{pmatrix}
1 & -1 & 0 & 1 & 1 \\
m^q & 1 & -1 & 0 & 1 \\
1 & m^q & 1 & -1 & 0 \\
1 & 1 & m^q & 1 & -1 \\
0 & 1 & 1 & m^q & 1
\end{pmatrix},
$$

$$M_6(m) = \begin{pmatrix}
m & 1 & -1 & 0 & 1 & 1 \\
1 & m^q & 1 & -1 & 0 & 1 \\
1 & 1 & m^q & 1 & -1 & 0 \\
0 & 1 & 1 & m^q & 1 & -1 \\
-1 & 0 & 1 & 1 & m^q & 1 \\
1 & -1 & 0 & 1 & 1 & m^q
\end{pmatrix}.$$
Theorem 2.2. The polynomial $f(x)$ is scattered if and only if $q \equiv 1 \pmod{4}$.

Proof. If $q$ is even, then for $m = 0$ the matrix $M_6(0)$ has rank two and $f(x)$ is not scattered.

Suppose now $q \equiv 3 \pmod{4}$. Then let $\overline{m} \in F_{q^2} \setminus F_q$ such that $\overline{m}^2 = -4$. So $\overline{m} = \overline{m}^q = \overline{m}^q = -\overline{m}^q = -\overline{m}^q$ and, by direct checking,

$$\det(M_5(\overline{m})) = (\overline{m}^2 + 4)^2 = 0, \quad \det(M_6(\overline{m})) = -(\overline{m}^2 + 4)^3 = 0$$

and $f(x)$ is not scattered.

Assume $q \equiv 1 \pmod{4}$ and suppose that $f(x)$ is not scattered. Then there exists $m_0 \in F_{q^6}$ such that

$$(\det(M_5(m_0)))^{q^i} = 0, \quad (\det(M_6(m_0)))^{q^i} = 0, \quad s, t = 0, 1, 2, 3, 4, 5. \quad (2.1)$$

Consider

$$P_1 = \det \begin{pmatrix} 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 & 1 \\ 1 & Z & 1 & -1 & 0 \\ 1 & 1 & U & 1 & -1 \\ 0 & 1 & 1 & V & 1 \end{pmatrix}, \quad P_2 = \det \begin{pmatrix} X & 1 & -1 & 0 & 1 & 1 \\ 1 & Y & 1 & -1 & 0 & 1 \\ 1 & 1 & Z & 1 & -1 & 0 \\ 0 & 1 & U & 1 & -1 & 0 \\ -1 & 0 & 1 & 1 & V & 1 \\ 1 & -1 & 0 & 1 & 1 & W \end{pmatrix}. \quad (2.2)$$

Therefore,

$$X = m_0, \quad Y = m_0^q, \quad \ldots, \quad W = m_0^{q^5} \quad (2.3)$$

is a root of $P_1 =: P_1^{(0)}, P_2 =: P_2^{(0)}$ and of the polynomials inductively defined by

$$P_i^{(j)}(X, Y, Z, U, V, W) = P_i^{(j-1)}(Y, Z, U, V, W, X), \quad j = 1, 2, 3, 4, 5, \quad i = 1, 2,$$

which arise from Equation 2.1. These polynomials satisfy

$$\left( P_i^{(j-1)}(m_0, m_0^q, m_0^{q^2}, m_0^{q^3}, m_0^{q^4}, m_0^{q^5}) \right)^q = P_i^{(j)}(m_0, m_0^q, m_0^{q^2}, m_0^{q^3}, m_0^{q^4}, m_0^{q^5}).$$

One obtains a set $S$ of twelve equations in $X, Y, Z, U, V, W$ having a nonempty zero set. The following arguments are based on the fact that taking the resultant $R$ of two polynomials in $S$ with respect to any variable, the equations $S \cup \{ R \}$ admit the same solutions.

We have

$$P_1 = YZUV - YZU - 2YZ + 2YU + 4Y - ZUV + 2ZV - 2UV + 4V + 16 = 0. \quad (2.4)$$

Consider the following resultants:

$$Q_1 := Res_\nu(P_1^{(3)}, P_1) = 2(XY^2ZU - XY^2ZW + XY^2UW + 2XY^2W$$

$$- 2XYU + 2XYZW - 2XYUW + 8XYW + 8XY - 8YW + 16X$$

$$- Y^2ZUW - 2Y^2ZU + 2YZUW - 8YZU - 8YZ + 8YW - 8YW$$

$$+ 8ZU - 16Z + 16U - 16W),$$

$$Q_2 := Res_\nu(P_1^{(4)}, P_1) = XYZW - XYZ - XYW + 2XZ$$

$$- 2XW - 2YZ + 2YW + 4Z + 4W + 16,$$

$$Q_3 := Res_\nu(P_1^{(5)}, P_1) = XYZU - XYZ - 2XY + 2XZ$$

$$+ 4X - YZU + 2YU - 2ZU + 4U + 16.$$
They all must be zero, as well as
\[ \text{Res}_W(\text{Res}_U(Q_1, Q_3), Q_2) = 8(YZ - 4)(Y^2 + 4)(X - Z)(XZ + 4)(XY - 4). \] (2.5)

We distinguish a number of cases.

1. Suppose that \( Y^2 = -4 \). Since \( q \equiv 1 \pmod{4} \), \( X = Y = Z = U = V = W \). So
\[ P_1 = X^4 - 2X^3 + 8X + 16 \]
and the resultant between \( X^2 + 4 \) and \( P_1 \) with respect to \( X \) is \( 2^{27} \neq 0 \) and then (2.3) is not a root of \( P_1 \), a contradiction.

2. Condition \( YZ = 4 \) is clearly equivalent to \( XY = 4 \). This means that \( Y = U = W = 4/X, Z = V = X \). Therefore, by (2.4) we get \( X^2 + 4 = 0 \) and we proceed as above.

3. Case \( XZ = -4 \). In this case \( Z = -4/X, U = -4/Y, V = -4/Z = X, W = Y, X = Z \) and therefore \( X^2 = -4 \) and we can proceed as above.

4. Condition \( X = Z \) implies \( X \in \mathbb{F}_{q^2} \) and so \( X = Z = V \) and \( Y = U = W \). By substituting in \( P_1 \) and \( P_2 \),
\[ X^3Y^3 + 3X^3Y - 6X^2Y^2 - 12X^2 + 3XY^3 + 24XY - 12Y^2 - 64 = 0, \]
\[ X^2Y^2 - X^2Y + 2X^2 - XY^2 - 4XY + 4X + 2Y^2 + 4Y + 16 = 0. \]
Eliminating \( Y \) from these two equations one gets
\[ 8(X^2 + 4)^6 = 0, \]
and so \( X^2 + 4 = 0 \). We proceed as in the previous cases.

This proves that such \( m_0 \in \mathbb{F}_{q^6} \) does not exist and the assertion follows. \( \square \)

2.2 Case 2

We apply the same methods as in Section 2.1. In the following preparatory lemmas (and in the rest of the paper) \( q \) is a power of an arbitrary prime \( p \).

**Lemma 2.3.** Let \( h \in \mathbb{F}_{q^6} \) be such that \( h^{q^3+1} = -1, h^4 \neq 1 \). Then

1. \( h^q \neq -h \);
2. \( h^{q^2+1} \neq 1 \);
3. \( h^{q^2+1} \neq \pm h^q \), if \( q \) is odd;
4. \( h^{4q^2+4} + 14h^{2q^2+2q+2} + h^{4q} = 0 \) implies \( p = 2 \) and \( h^{q^2-q+1} = 1 \) or \( q = 3^{2s}, s \in \mathbb{N}^* \), \( h^{q^2-q+1} = \pm \sqrt{-1}. \)

**Proof.** The first three are easy computations. Consider now
\[ h^{4q^2+4} + 14h^{2q^2+2q+2} + h^{4q} = 0. \]
For \( p = 2 \) the equation above implies \( h^{q^2-q+1} = 1. \)
Assume now $p \neq 2$. Since $h \neq 0$, it is equivalent to

$$(h^{q^2-q+1})^4 + 14(h^{q^2-q+1})^2 + 1 = 0,$$

that is $(h^{q^2-q+1})^2 = -7 \pm 4\sqrt{3} = (\sqrt{-3} \pm 2\sqrt{-1})^2$. Let $z = -7 \pm 4\sqrt{3}$. Note that $h^{q^2-q+1} = \pm\sqrt{z}$ belongs to $\mathbb{F}_{q^2}$. We distinguish two cases.

- $\sqrt{z} \in \mathbb{F}_q$. Then
  
  
  $$-1 = h^{q^3+1} = (h^{q^2-q+1})^{q+1} = (\pm\sqrt{z})^{q+1} = z = -7 \pm 4\sqrt{3},$$

  a contradiction if $p \neq 3$. Also, $z = -1$, $q$ is an even power of 3, and $h^{q^2-q+1} = \pm\sqrt{-1}$.

- $\sqrt{z} \notin \mathbb{F}_q$. Then
  
  
  $$-1 = h^{q^3+1} = (h^{q^2-q+1})^{q+1} = (\pm\sqrt{z})^{q+1} = -z = 7 \mp 4\sqrt{3},$$

  a contradiction if $p \neq 2$. \hfill \qed

**Lemma 2.4.** Let $h \in \mathbb{F}_{q^6}$ be such that $h^{q^3+1} = -1$, $h^4 \neq 1$. If a root $\sigma$ of the polynomial

$$h^{q+1}T^{q+1} + (h^{q^2+q+2} + h^{2q^2+2})T^q + (h^{2q^2+2} - h^{q^2+1})T$$

$$+ h^{q^2+2q+1} + h^{2q^2+q+1} - h^q - h^{q^2+q} \in \mathbb{F}_{q^6}[T]$$

belongs to $\mathbb{F}_{q^6}$, then one of the following cases occurs:

- $p = 2$, $h^{q^2-q+1} = 1$; or
- $q = 3^{2s}$, $s > 0$, $h^{q^2-q+1} = \pm\sqrt{-1}$; or
- $\sigma = \pm(h^q + h^q)$; or
- $h \in \mathbb{F}_q$.

**Proof.** First, note that $\sigma = 0$ would imply $h^q(h^q + h)^q(h^{q^2+1} - 1) = 0$ which is impossible by Lemma 2.3. Therefore $\sigma \neq 0$ and $\sigma^q = \frac{\ell_i(X)}{m_i(X)}$, where

$$\ell_1(X) = -(h^{q^2+1} - 1)(h^{q^2+1}X + h^q + h^{q^2+q})$$

$$m_1(X) = h(h^qX + h^{q^2+q+1} + h^{2q^2+1})$$

$$\ell_2(X) = -(h^q + h)(2h^{q^2+q+1}X + h^{2q^2+q+2} + h^{3q^2+2} + h^3 + h^{q+2})$$

$$m_2(X) = h^{q+1}(h^{2q^2+2}X + h^qX + 2h^{q^2+2q+1} + 2h^{2q^2+q+1})$$

$$\ell_3(X) = (h^q + h)^q(3h^{2q^2+q+2}X + h^qX + h^{3q^2+2q+3} + h^{4q^2+3} + 3h^{q^2+3q+1} + 3h^{2q^2+2q+1})$$

$$m_3(X) = h^{q^2+q}(h^{3q^2+3}X + 3h^{q^2+2q+1}X + 3h^{2q^2+2q+2} + 3h^{3q^2+q+2} + h^{4q} + h^{q^2+3q})$$
Lemma 2.5. Let $h \in \mathbb{F}_{q^6}$ be such that $h^3 + 1 = -1$, $h^4 = 1$. If a root $\sigma$ of the polynomial 

$$
\sigma^{q+1} T \sigma^{2+1} + (h^q + h)^{q+1} \in \mathbb{F}_{q^6}[T]
$$

belongs to $\mathbb{F}_{q^6}$, then 

$$
\sigma = \pm (h^{q^2} + h^q).
$$

Proof. If $\sigma = 0$, then $h^q + h = 0$, a contradiction to Lemma 2.3. So we can suppose $\sigma \neq 0$. Then 

$$
\sigma^{q^2} = - \frac{(h^{q-1} + 1)^{q+1}}{\sigma},
$$

$$
\sigma^{q^4} = (h^{q-1} + 1)^{q^3+q^2-q-1} \sigma,
$$

$$
\sigma^{q^6} = - \frac{(h^{q-1} + 1)^{q^5+q^4-q^3-q^2+q+1}}{\sigma} = \frac{(h^q + h)^{2q}}{\sigma}.
$$

So, $\sigma = \pm (h^{q^2} + h^q)$. 

Let $h \in \mathbb{F}_{q^6}$ be such that $h^{q^3+1} = -1$, $h^4 \neq 1$. By Theorem 2.1 the polynomial 

$$
f_h(x) = h^{q-1} x^q - (h^{q^2-1}) x^{q^2} + x^{q^4} + x^{q^5}
$$
is scattered if and only if for each \( m \in \mathbb{F}_q^6 \) the determinant of the following two matrices do not vanish at the same time

\[
M_6(m) = \begin{pmatrix}
  m & h^{q-1} & -h^{q^2-1} & 0 & 1 & 1 \\
  1 & m^q & h^{q^2-q} & h^{-q-1} & 0 & 1 \\
  1 & 1 & m^{q^2} & -h^{-q^2-1} & h^{-q^2-q} & 0 \\
  0 & 1 & 1 & m^{q^3} & h^{-1-q} & -h^{-1-q^2} \\
  h^{q+1} & 0 & 1 & 1 & m^{q^4} & h^{q-q^2} \\
 -h^{q^2+1} & h^{q^2+q} & 0 & 1 & 1 & m^{q^5}
\end{pmatrix}, \quad (2.6)
\]

\[
M_5(m) = \begin{pmatrix}
  h^{q-1} & -h^{q^2-1} & 0 & 1 & 1 \\
  m^q & h^{q^2-q} & h^{-q-1} & 0 & 1 \\
  1 & m^{q^2} & -h^{-q^2-1} & h^{-q^2-q} & 0 \\
  1 & 1 & m^{q^3} & h^{-1-q} & -h^{-1-q^2} \\
  0 & 1 & 1 & m^{q^4} & h^{q^2-q^2}
\end{pmatrix}. \quad (2.7)
\]

**Theorem 2.6.** Let \( h \in \mathbb{F}_q^6, q = 2^s \), be such that \( h^{q^3+1} = 1 \). Then the polynomial

\[ f_h(x) = h^{q-1}x^q - (h^{q-1})x^{q^2} + x^{q^4} + x^{q^5} \]

is not scattered.

**Proof.** Consider \( \overline{m} = h^{q^2} + h^q \). So,

\[
\overline{m}^q = \frac{1}{h} + h^{q^2}, \quad \overline{m}^{q^2} = \frac{1}{h} + \frac{1}{h^q}, \quad \overline{m}^{q^3} = \frac{1}{h^{q^2}} + \frac{1}{h^q},
\]

\[
\overline{m}^{q^4} = h + \frac{1}{h^{q^2}}, \quad \overline{m}^{q^5} = h^q + h.
\]

By direct checking, in this case, both \( \det(M_6(\overline{m})) = \det(M_5(\overline{m})) = 0 \) and therefore \( f_h(x) \) is not scattered. \( \square \)

**Theorem 2.7.** Let \( h \in \mathbb{F}_q^6, q = p^s, p > 2 \), be such that \( h^{q^3+1} = -1 \) and \( h \notin \mathbb{F}_q \). Then the polynomial

\[ f_h(x) = h^{q-1}x^q - (h^{q-1})x^{q^2} + x^{q^4} + x^{q^5} \]

is scattered.

**Proof.** First we note that \( h^4 \neq 1 \) since \( q \) is odd, \( h \notin \mathbb{F}_q \), and \( h^{q^3+1} = -1 \). Suppose that \( f(x) \) is not scattered. Then \( \det(M_6(m_0)) = \det(M_5(m_0)) = 0 \) for some \( m_0 \in \mathbb{F}_q^6 \).

Consider

\[ X = m_0, \quad Y = m_0^{q^4}, \quad Z = m_0^{q^2}, \quad U = m_0^{q^3}, \quad V = m_0^{q^4}, \quad W = m_0^{q^5}. \]

With a procedure similar to the one in the proof of Theorem 2.2, we will compute resultants starting from the polynomials associated with \( \det(M_6(m_0)) \), \( \det(M_5(m_0))^{q^3} \), and \( \det(M_5(m_0))^{q^5} \).

Eliminating \( W \) using \( \det(M_5(m_0))^{q^3} = 0 \) and \( U \) using \( \det(M_5(m_0))^{q^5} = 0 \), one gets from \( \det(M_6(m_0)) = 0 \)

\[
h^{q^2+2q+1} \varphi_1(X,Y) \varphi_2(X,Y,Z,V) \varphi_3(X,Y,Z,V) = 0,
\]
where

\[ \varphi_1(X, Y) = h^{q+1}XY + h^{2q^2+2}X - h^{q^2+1}X + h^{q^2+q+2}Y + h^{2q^2+2}Y + h^{2q^2+1} + h^{2q^2+q+1} - h^{2q} - h^{q^2+q}; \]

\[ \varphi_2(X, Y, Z, V) = h^{q^2+q+2}XYZV - h^{q^2+q+2}XYZ - h^2XY - h^{q+1}XY \]

\[ - h^{2q^2+q+1}XZV - h^{2q^2+2}XV - h^{2q^2+q+1}XV - h^{q^2+q+3}YZ - h^{2q^2+q+3}YZ - h^{2q^2+q+2}Y - h^{q^2+q+1}Y \]

\[ - h^{2q^2+q+1}YZ - h^{2q^2+q+1}ZV - h^{2q^2+q+1}V - h^{3q^2+1}V - h^{2q^2+q}V - h^{2q^2+q+3} + h^{3q^2+3} + h^{2q^2+q+2} + h^{3q^2+q+2} - 2h^{q^2+q+2} - 2h^{q^2+q+1} - 2h^{2q^2+q+1} + h^{q^2+1} + h^{2q} + h^{q^2+q}; \]

\[ \varphi_3(X, Y, Z, V) = h^{q^2+q+2}XYZV + h^{q^2+q+2}XYZ - h^2XY - h^{q+1}XY \]

\[ + h^{2q^2+q+1}XZV - h^{2q^2+2}XV - h^{2q^2+q+1}XV - h^{q^2+q+3}YZ - h^{2q^2+q+3}YZ + h^{2q^2+q+2}Y + h^{q^2+q+1}Y \]

\[ + h^{2q^2+q+1}YZ - h^{q^2+q+1}ZV - h^{2q^2+q+1}V + h^{2q^2+q+3} + h^{3q^2+3} + h^{2q^2+q+2} + h^{3q^2+q+2} - 2h^{q^2+q+2} - 2h^{q^2+q+1} - 2h^{2q^2+q+1} + h^{q^2+1} + h^{2q} + h^{q^2+q}. \]

- If \( \varphi_1(X, Y) = 0 \), then by Lemma 2.4 either \( q = 3^{2s} \) and \( h^{q^2+q+1} = \pm \sqrt{-1} \), or \( X = \pm (h^{q^2} + h^q) \).

In this last case,

\[ Y = \pm (-h^{-1} + h^q), \quad Z = \pm (-h^q - h^{-1}), \quad U = \pm (-h^q - h^{-q}) \]

\[ V = \pm (h - h^{-q^2}), \quad W = \pm (h^q + h). \]

By substituting in \( \det(M_5(m_0)) \) one obtains

\[ 4(h + h^q)^{q+1}(h^{q^2+1} - 1)(h^{q^2+1} - h^q) = 0 \]

and

\[ 4(h + h^q)^{q+1}(h^{q^2+1} - 1)(h^{q^2+1} + h^q) = 0, \]

respectively. Both are not possible due to Lemma 2.3.

Consider now the case \( q = 3^{2s}, h^{q^2+q+1} = \pm \sqrt{-1} \) and \( X \neq \pm (h^{q^2} + h^q) \). So, using \( \varphi_1(X, Y) = 0 \) and \( h^{q^2+q+1} = \pm \sqrt{-1} \),

\[ \det(M_5(m_0)) = 0 \implies h^{q^2+2q+1}(h^{q^2} + h^q)(h^q + h)(h^{q^2+1} - 1)(h^{q^2+q} + h^q)^3(h^{q^2+q} - h^q)^3 \cdot (h^{2q^2+2} - h^{q^2+1} + h^{2q})(X + h^q + h^q)(X - h^q - h^{q^2})^2 = 0. \]

By Lemma 2.3 we get

\[ h^{2q^2+2} - h^{q^2+1} + h^{2q} = 0, \]

which yields to a contradiction.
• If \( \varphi_2(X, Y, Z, V) = 0 \) and \( \varphi_1(X, Y) \neq 0 \), eliminating \( V \) in \( \det(M_5(m_0)) = 0 \) one gets

\[
2h^{3q^2+2q+1}(h^{q+2}YZ - h^{q^2+2} - h^{q^2+q+1} + h^q + h) \cdot
\]

\[
\cdot (hXY + h^{q^2+q+1} + h^2q^2+1 - h^q - h) \cdot
\]

\[
\cdot (h^{q+1}XZ + h^{q+1} + h^q + h^q + h^2 + h^q+q) \cdot
\]

\[
\cdot (h^{q+2}YZ + hY + h^qY - h^{q^2+q+1}Z + h^qZ - h^{q^2+2} - h^{q^2+q+1} + h^q + h) = 0.
\]

- If \( h^{q+2}YZ - h^{q^2+2} - h^{q^2+q+1} + h^q + h = 0 \) then, from

\[
Z = \frac{h^{q^2+2} + h^{q^2+q+1} - h^q - h}{h^{q+2}Y},
\]

\( \det(M_5) = 0 \) gives

\[
(h^q + h)^{q+1}(hY - h^{q^2+1} + 1)(hY + h^{q^2+1} - 1) = 0.
\]

So, (2.8) holds and as in the case \( \varphi_1(X, Y) = 0 \) a contradiction arises.

- If \( hXY + h^{q^2+q+1} + h^2q^2+1 - h^q - h = 0 \) then, from

\[
Y = \frac{-h^{q^2+q+1} - h^2q^2+1 + h^q + h^q}{hX},
\]

the equation \( \det(M_5(m_0)) = 0 \) yields

\[
(h^q + h)(h^{q^2+1} - 1)(X - h^{q^2} - h^q)(X + h^{q^2} + h^q) = 0.
\]

So, (2.8) holds and as in the case \( \varphi_1(X, Y) = 0 \), a contradiction.

- If \( h^{q+1}XZ + h^{q+1} + h^q + h^q + h^2 + h^q+q = 0 \) then by Lemma 2.5

\[
(X - h^q - h^q)(X + h^q^2 + h^q) = 0,
\]

again a contradiction as before.

- If \( h^{q+2}YZ + hY + h^qY - h^{q^2+q+1}Z + h^qZ - h^{q^2+2} - h^{q^2+q+1} + h^q + h = 0 \) then

\[
Z = -\frac{(h^q + h)Y - h^{q^2+2} - h^{q^2+q+1} + h^q + h}{h^{q+2}Y - h^{q^2+q+1} + h^q}.
\]

So, substituting \( U = Z^q, V = Z^{q^2}, W = Z^{q^3}, X = Z^{q^4} \) in \( \det(M_5(m_0)) = 0 \) we get

\[
(h - 1)^{q+1}(h + 1)^{q+1}(h^q + h)^{q+1}(h^{q^2+1} - 1) \cdot
\]

\[
\cdot (hY - h^{q^2+1} + 1)^2(hY + h^{q^2+1} - 1)^2 = 0.
\]

By Lemma 2.3, \( (hY - h^{q^2+1} + 1)(hY + h^{q^2+1} - 1) = 0 \). Since \( Y = \pm(h^{q^2} - 1/h) \) then (2.8) holds and a contradiction arises as in the case \( \varphi_1(X, Y) = 0 \).
• If \( \varphi_3(X, Y, Z, V) = 0 \) and \( \varphi_1(X, Y) \neq 0 \), eliminating \( U \) from \( \det(M_5(m_0)) = 0 = \det(M_5(m_0))^q \) and then eliminating \( V \) using \( \varphi_3(X, Y, Z, V) = 0 \) one gets

\[
2h^{3q^3+q+1}(h^q + h)^q(h^{q^2+2} + h^{q^2+q+1} + h^q + h)^2.
\]

\[
\cdot (hXY + h^{q^2+q+1} + h^{2q^2+1} - h^q - h^q).
\]

\[
\cdot (h^{q^2+1}XZ + h^{q^2+1} + h^{2q^2+1} + h^{2q} + h^{q^2+q}) = 0.
\]

A contradiction follows as in the case \( \varphi_2(X, Y, Z, V) = 0 \) and \( \varphi_1(X, Y) \neq 0. \)

\[\blacksquare\]

3 The equivalence issue

We will deal with the linear sets \( \mathcal{L}_h = L_{f_h} \) associated with the polynomials defined in (1.1). Note that when \( h \in \mathbb{F}_q \), such a linear set coincide with the one introduced in [27, Section 5].

3.1 Preliminary results

We start by listing the non-equivalent (under the action of \( \Gamma L(2, q^6) \)) maximum scattered subspaces of \( \mathbb{F}_q^2 \), i.e. subspaces defining maximum scattered linear sets.

Example 3.1.

1. \( U^1 := \{(x, x^q) : x \in \mathbb{F}_{q^6}\} \), defining the linear set of pseudoregulus type, see [3, 11];

2. \( U^2_\delta := \{(x, \delta x^q + x^{q^3}) : x \in \mathbb{F}_{q^6}\}, N_{q^6/q}(\delta) \notin \{0, 1\} \), defining the linear set of LP-type, see [16, 18, 20, 24];

3. \( U^3_\delta := \{(x, x^q + \delta x^{q^4}) : x \in \mathbb{F}_{q^6}\}, N_{q^6/q^3}(\delta) \notin \{0, 1\} \), satisfying further conditions on \( \delta \) and \( q \), see [6, Theorems 7.1 and 7.2] and [23] \(^2\);

4. \( U^4_\delta := \{(x, x^q + x^{q^3} + \delta x^{q^5}) : x \in \mathbb{F}_{q^6}\}, q \) odd and \( \delta^2 + \delta = 1 \), see [10, 21].

In order to simplify the notation, we will denote by \( L^1 \) and \( L^1_\delta \) the \( \mathbb{F}_q \)-linear set defined by \( U^1 \) and \( U^1_\delta \), respectively. We will also use the following notation:

\[
\mathcal{U}_h := U_{h^{q-1}x^q - h^{q^2-1}x^{q^2} + x^{q^3} + x^{q^5}}.
\]

Remark 3.2. Consider the non-degenerate symmetric bilinear form of \( \mathbb{F}_{q^6} \) over \( \mathbb{F}_q \) defined by

\[
\langle x, y \rangle = \text{Tr}_{q^6/q}(xy),
\]

for each \( x, y \in \mathbb{F}_{q^6} \). Then the adjoint \( \hat{f} \) of the linearized polynomial \( f(x) = \sum_{i=0}^{5} a_i x^{q^i} \in \mathcal{L}_{6,q} \) with respect to the bilinear form \( \langle , \rangle \) is

\[
\hat{f}(x) = \sum_{i=0}^{5} a_i^{q^6-i} x^{q^6-i},
\]

i.e.

\[
\text{Tr}_{q^6/q}(xf(y)) = \text{Tr}_{q^6/q}(y\hat{f}(x)),
\]

for any \( x, y \in \mathbb{F}_{q^6} \).

\(^2\)Here \( q > 2 \), otherwise it is not scattered.
In [10, Propositions 3.1, 4.1 and 5.5] the following result has been proved.

**Lemma 3.3.** Let $L_f$ be one of the maximum scattered of $\text{PG}(1, q^6)$ listed before. Then a linear set $L_U$ of $\text{PG}(1, q^6)$ is $\text{PGL}$-equivalent to $L_f$ if and only if $U$ is $\Gamma L$-equivalent either to $U_f$ or to $U^\delta_f$. Furthermore, $L_U$ is $\text{PGL}$-equivalent to $L^3$ if and only if $U$ is $\Gamma L$-equivalent to $U^\delta$.

We will work in the following framework. Let $x_0, \ldots, x_5$ be the homogeneous coordinates of $\text{PG}(5, q^6)$ and let

$$
\Sigma = \{(x, x^q, \ldots, x^{q^5})_{\mathbb{F}_{q^6}} : x \in \mathbb{F}_{q^6}\}
$$

be a fixed canonical subgeometry of $\text{PG}(5, q^6)$. The collineation $\hat{\sigma}$ of $\text{PG}(5, q^6)$ defined by $\langle(x_0, \ldots, x_5)\rangle_{\mathbb{F}_{q^6}} = \langle(x^q_0, x^q_0, \ldots, x^q_4)\rangle_{\mathbb{F}_{q^6}}$ fixes precisely the points of $\Sigma$. Note that if $\sigma$ is a collineation of $\text{PG}(5, q^6)$ such that $\text{Fix}(\sigma) = \Sigma$, then $\sigma = \hat{\sigma}^s$, with $s \in \{1, 5\}$.

Let $\Gamma$ be a subspace of $\text{PG}(5, q^6)$ of dimension $k \geq 0$ such that $\Gamma \cap \Sigma = \emptyset$, and $\dim(\Gamma \cap \Gamma^\sigma) \geq k - 2$. Let $r$ be the least positive integer satisfying the condition

$$
\dim(\Gamma \cap \Gamma^\sigma \cap \Gamma^{\sigma^2} \cap \cdots \cap \Gamma^{\sigma^r}) > k - 2r. \tag{3.1}
$$

Then we will call the integer $r$ the intersection number of $\Gamma$ w.r.t. $\sigma$ and we will denote it by $\text{int}_{\sigma}(\Gamma)$; see [27].

Note that if $\hat{\sigma}$ is as above, then $\text{int}_{\hat{\sigma}}(\Gamma) = \text{int}_{\hat{\sigma}^s}(\Gamma)$ for any $\Gamma$.

As a consequence of the results of [11, 27] we have the following result.

**Result 3.4.** Let $L$ be a scattered linear set of $\Lambda = \text{PG}(1, q^6)$ which can be realized in $\text{PG}(5, q^6)$ as the projection of $\Sigma = \text{Fix}(\sigma)$ from $\Gamma \simeq \text{PG}(3, q^6)$ over $\Lambda$. If $\text{int}_{\sigma}(\Gamma) \neq 1, 2$, then $L$ is not equivalent to any linear set neither of pseudoregulus type nor of LP-type.

### 3.2 $L_h$ is new in most of the cases

The linear set $L_h$ can be obtained by projecting the canonical subgeometry

$$
\Sigma = \{(x, x^q, x^{q^2}, x^{q^3}, x^{q^4}, x^{q^5})_{\mathbb{F}_{q^6}} : x \in \mathbb{F}_{q^6}\}
$$

from

$$
\Gamma: \begin{cases}
x_0 = 0 \\
h^{q-1}x_1 - h^{q^2-1}x_2 + x_4 + x_5 = 0
\end{cases}
$$

to

$$
\Lambda: \begin{cases}
x_1 = 0 \\
x_2 = 0 \\
x_3 = 0 \\
x_4 = 0.
\end{cases}
$$

Then

$$
\Gamma^\sigma: \begin{cases}
x_1 = 0 \\
h^{q^2-q}x_2 + h^{-q-1}x_3 + x_5 + x_0 = 0
\end{cases}
$$
and
\[
\Gamma^\sigma: \begin{cases} 
    x_2 = 0 \\
    -h^{-1-q^2}x_3 + h^{-q^2-q}x_4 + x_0 + x_1 = 0.
\end{cases}
\]

Therefore,
\[
\Gamma \cap \Gamma^\sigma: \begin{cases} 
    x_0 = 0 \\
    x_1 = 0 \\
    -h^{q^2-1}x_2 + x_4 + x_5 = 0 \\
    h^{q^2-q}x_2 + h^{-q-1}x_3 + x_5 = 0
\end{cases}
\]

and
\[
\Gamma \cap \Gamma^\sigma \cap \Gamma^{\sigma^2}: \begin{cases} 
    x_0 = 0 \\
    x_1 = 0 \\
    x_2 = 0 \\
    x_4 + x_5 = 0 \\
    h^{-q-1}x_3 + x_5 = 0 \\
    -h^{q^2-1}x_3 + h^{-q^2-q}x_4 = 0.
\end{cases}
\]

Hence, \( \dim_{\mathbb{F}_q}(\Gamma \cap \Gamma^\sigma) = 1 \) and \( \dim_{\mathbb{F}_q}(\Gamma \cap \Gamma^\sigma \cap \Gamma^{\sigma^2}) = -1 \), since \( q \) is odd and \( h^{q^3+1} \neq 1 \). So, \( \int \Gamma_n(\Gamma) = 3 \) and hence, by Result 3.4 it follows that \( L_h \) is not equivalent neither to \( L^1 \) nor to \( L_3^6 \).

Generalizing [27, Propositions 5.4 and 5.5] we have the following two propositions.

**Proposition 3.5.** The linear set \( L_h \) is not \( \Gamma \Gamma \)-equivalent to \( L_3^6 \).

**Proof.** By Lemma 3.3, we have to check whether \( U_h \) and \( U_3^6 \) are \( \Gamma \Gamma \)-equivalent, with \( N_{q^6/q^3}(\delta) \notin \{0, 1\} \). Suppose that there exist \( \rho \in \text{Aut}(\mathbb{F}_{q^6}) \) and an invertible matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that for each \( x \in \mathbb{F}_{q^6} \) there exists \( z \in \mathbb{F}_{q^6} \) satisfying
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\rho \\ z^q + \delta z^{q^4} \end{pmatrix} = 0.
\]

Equivalently, for each \( x \in \mathbb{F}_{q^6} \) we have\(^3\)
\[
cx^\rho + d(h^{q-1}x^{\rho q} - h^{q^2-q}x^{\rho q^2} + x^{\rho q^4} + x^{\rho q^5}) = \]
\[
\alpha^q x^{\rho q} + b\gamma(h^{q^2-q}x^{\rho q^2} + h^{-q-1}x^{\rho q^3} + x^{\rho q^5} + x^\rho) + dh^{q-1} = h^{q^2-q}b^q + \delta bq^4
\]
\[
= h^{-1-q}b^q + \delta b q^4 \\
d = \delta a q^4 \\
d = b^q + \delta h^{q^2-q}b q^4.
\]

This is a polynomial identity in \( x^\rho \) and hence we have the following relations:
\[
\begin{cases} 
    c = b^q + \delta h^{q+1}b q^4 \\
    d h^{q-1} = \alpha^q \\
    -dh^{q^2-1} = h^{q^2-q}b g^q + \delta b q^4 \\
    0 = h^{-1-q}b^q + \delta b q^4 \\
    d = \delta a q^4 \\
    d = b^q + \delta h^{q-2}b q^4.
\end{cases}
\]

\(^3\)We may replace \( h^\rho \) by \( h \), since \( h^{q^3+1} = -1 \) if and only if \( (h^\rho)^{q^3+1} = -1 \).
From the second and the fifth equations, if \( a \neq 0 \) then \( \delta h^{q-1} = a^{q-1}q^2 \) and \( N_{q^6/q^5}(\delta) = 1 \), which is not possible and so \( a = d = 0 \) and \( b, c \neq 0 \). By the last equation, we would get \( N_{q^6/q^5}(\delta) = 1 \), a contradiction. \( \square \)

**Proposition 3.6.** The linear set \( L_h \) is PGL-equivalent to \( L_\delta^4 \) (with \( \delta^2 + \delta = 1 \)) if and only if there exist \( a, b, c, d \in \mathbb{F}_{q^6} \) and \( \rho \in \text{Aut}(\mathbb{F}_{q^6}) \) such that \( ad - bc \neq 0 \) and either

\[
\begin{align*}
\begin{cases}
  c = b^q - \delta k q^2 + 1 b^q \\
  a = -k^{-q+1} b^q - \delta q b^2 \\
  d = k^{-q+1} b^q + \delta b^5 \\
  b^3 + (k^{-q-1} + \delta q b^5) b^q = 0 \\
  k q^2 - a b^2 + (k^2 - q) b^3 + \delta k q^2 - 1 b^5 = 0 \\
  -\delta b^q + (k^{-q+1} + \delta^2 k^{-1} q^2) b^3 + \delta b^5 = 0
\end{cases}
\end{align*}
\]

or

\[
\begin{align*}
\begin{cases}
  c = \delta b^q - k q^2 + 1 b^q \\
  a = -\delta q k q^2 + 1 b^q - b^2 \\
  d = k^{-q+1} b^q + b^5 \\
  \delta b^q + (k^{-q-1} - \delta k q^2 + q) b^q = 0 \\
  \delta k q^2 - a b^2 + (k^2 - q + 1) b^3 + k q^2 - 1 b^5 = 0 \\
  \delta^2 b^q + (k^{-q+1} + \delta^2 k^{-1} q^2 + 1) b^3 + b^5 = 0
\end{cases}
\end{align*}
\]

where \( k = h^\rho \).

**Proof.** By Lemma 3.3 we have to check whether \( U_h \) is equivalent either to \( U_\delta^4 \) or to \( (U_\delta^4)^\perp \). Suppose that there exist \( \rho \in \text{Aut}(\mathbb{F}_{q^6}) \) and an invertible matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that for each \( x \in \mathbb{F}_{q^6} \) there exists \( z \in \mathbb{F}_{q^6} \) satisfying

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} h^\rho(q-1) x^\rho \\ h^\rho(q^2 - 1) x^\rho q^2 + x^\rho q^4 + x^\rho q^5 \end{pmatrix} = \begin{pmatrix} z^q + z q^3 + \delta z q^5 \end{pmatrix}.
\]

Equivalently, for each \( x \in \mathbb{F}_{q^6} \) we have

\[
\begin{align*}
&cx^\rho + d(k q^{-1} x^\rho q - k q^2 - 1 x^\rho q^2 + x^\rho q^4 + x^\rho q^5) = \\
&\quad \begin{pmatrix} a^q x^\rho q + b^q (k^2 q^{-2} - q x^\rho q^2 + k^{-1} q x^\rho q^3 + x^\rho q^5 + x^\rho) \\
&+ a^q x^\rho q^3 + b^3 (k^{-q+1} x^\rho q^2 - k^{-q+1} q x^\rho q + x^\rho q + x^\rho q^2) \\
&+ \delta [a^q x^\rho q^5 + b^3 (-k^{-1} q^2 x^\rho + k q^2 + q x^\rho q + x^\rho q^3 + x^\rho q^4)]
\end{pmatrix}.
\end{align*}
\]

This is a polynomial identity in \( x^\rho \) which yields to the following equations

\[
\begin{align*}
\begin{cases}
  c = b^q - \delta k q^2 + 1 b^q \\
  d k q^{-1} = a^q + b^3 + \delta k q^2 + b q^5 \\
  -d k q^{2-1} = k q^2 - a^q + b^3 q^2 + \delta b^5 \\
  0 = k^{-q+1} b + a^q + \delta b^5 \\
  d = k^{-q+1} b^3 + \delta b^5 \\
  d = b^q - k q^2 + 1 b^q + \delta a q^5
\end{cases}
\end{align*}
\]
which can be written as (3.3).

Now, suppose that there exist $\rho \in \text{Aut}(\mathbb{F}_{q^6})$ and an invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that for each $x \in \mathbb{F}_{q^6}$ there exists $z \in \mathbb{F}_{q^6}$ satisfying

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left( h^\rho(q-1)x^\rho q - h^\rho(q^2-1)x^\rho q^2 + x^\rho q^4 + x^\rho q^5 \right) = \left( \delta z^q + z^{q^3} + z^{q^5} \right).
$$

Equivalently, for each $x \in \mathbb{F}_{q^6}$ we have

$$
cx^\rho + d(k^q-1x^\rho q - k^q -1x^\rho q^2 + x^\rho q^4 + x^\rho q^5) = \\
\delta[a^q x^\rho q + b^q(k^q-1x^\rho q^2 + k^q -1x^\rho q^3 + x^\rho q^5 + x^\rho q)] \\
+ a^q x^\rho q^3 + b^q(k^q-1x^\rho q^4 - k^q x^\rho q^5 + x^\rho q + x^\rho q^2) \\
+ a^q x^\rho q^5 + b^q(- k^q+1x^\rho q + k^q x^\rho q^4 + x^\rho q^3 + x^\rho q^4),
$$

This is a polynomial identity in $x^\rho$ which yields to the following equations

$$
\begin{cases}
    c = \delta b^q - k^q+1b^q \\
    dk^q-1 = \delta a^q + b^q + k^q + q^2b^q \\
    -dk^q-1 = \delta k^q+q^2b^q + b^q \\
    0 = \delta k^q-1b^q + a^q + b^q \\
    d = k^q+q^2b^q + b^q \\
    d = \delta b^q - k^q+1b^q + a^q
\end{cases}
$$

which can be written as (3.4).

We are now ready to prove that when $h \notin \mathbb{F}_{q^2}$, $L_h$ is new.

**Proposition 3.7.** If $h \notin \mathbb{F}_{q^2}$, then $L_h$ is not PGL-equivalent to $L_h^1$ (with $\delta^2 + \delta = 1$).

**Proof.** By Proposition 3.6 we have to show that there are no $a, b, c$ and $d$ in $\mathbb{F}_{q^6}$ such that $ad - bc \neq 0$ and (3.3) or (3.4) are satisfied. Note that $b = 0$ in (3.3) and (3.4) yields $a = c = d = 0$, a contradiction. So, suppose $b \neq 0$. Since $h \notin \mathbb{F}_{q^2}$ then $k \notin \mathbb{F}_{q^2}$. We start by proving that the last three equations of (3.3), i.e.

$$
\begin{align*}
\text{Eq}_1 : b^{q^3} + (k^{q-1} + \delta k^{q+q^2})b^q & = 0 \\
\text{Eq}_2 : k^{q^2} - q^2bq + (1 + k^q - q)b^q + \delta k^{q-1}b_2 & = 0 \\
\text{Eq}_3 : -\delta b^q + (k^{q+1} + \delta k^{1-q^2})b^q + \delta b^q & = 0,
\end{align*}
$$

yield a contradiction. As in the above section, we will consider the $q$\textsuperscript{th} powers of $\text{Eq}_1$, $\text{Eq}_2$ and $\text{Eq}_3$, replacing $b^{q^i}$, $k^{q^j}$, and $\delta^{q^\ell}$ (respectively) by $X_i$, $Y_j$, and $Z_\ell$ with $i, j \in \{0, 1, 2, 3, 4, 5\}$ and $\ell \in \{0, 1\}$. Consider the set $S$ of polynomials in the variables $X_i$, $Y_j$, and $Z_\ell$

$$
S := \{ \text{Eq}_1^{q^\alpha}, \text{Eq}_2^{q^\beta}, \text{Eq}_3^{q^\gamma} : \alpha, \beta, \gamma \in \{0, 1, 2, 3, 4, 5\} \}.
$$

By eliminating from $S$ the variables $X_5$, $X_4$, $X_3$, and $X_2$ using $\text{Eq}_1$, $\text{Eq}_1^q$, $\text{Eq}_1^{q^2}$, and $\text{Eq}_1^{q^3}$ respectively we obtain

$$
X_0Y_1(Z_1Y_0^2Y_2 - Z_1Y_0Y_2^2 - Z_1Y_0 + Z_1Y_2 - Z_0^2Z_2 - Z_2) = 0.
$$
By the conditions on $b$ and $k$, $X_0 Y_1 \neq 0$ and therefore
\[ P := Z_1 Y_0^2 Y_2 - Z_1 Y_0 Y_2^2 - Z_1 Y_0 + Z_1 Y_2 - Z_0^2 Z_2 - Z_2 = 0. \]
We eliminate $Z_1$ in $S$ using $P$, obtaining, w.r.t. $b, k$, and $\delta$,
\[ bkq^2+1 (k-k^q)(k+k^q)(kq^2+1) - 1)(kq^2+1) = 0, \]
a contradiction to $k \notin \mathbb{F}_{q^2}$.

Consider now the last three equations of (3.4), i.e.
\[ \begin{align*}
\text{Eq}_1 : & \quad \delta b q^3 + (kq^2-1 - \delta k q^2+q) b q^5 = 0 \\
\text{Eq}_2 : & \quad \delta k q^2-a b q + (kq^2-q+1) b q^3 + kq^2-1 b q^5 = 0 \\
\text{Eq}_3 : & \quad \delta^2 b q^3 + (k-q^2+1 + \delta^2 k q^2+1) b q^5 + b q^5 = 0.
\end{align*} \]
As before, we will consider the $q$-th powers of $\text{Eq}_1$, $\text{Eq}_2$, and $\text{Eq}_3$ replacing $b q^i$, $k q^j$, and $\delta q^k$ (respectively) by $X_i, Y_j$, and $Z_\ell$ with $i, j \in \{0, 1, 2, 3, 4, 5\}$ and $\ell \in \{0, 1\}$. Consider the set $S$ of polynomials in the variables $X_i, Y_j$ and $Z_\ell$
\[ S := \{ \text{Eq}_1^q, \text{Eq}_2^q, \text{Eq}_3^q : \alpha, \beta, \gamma \in \{0, 1, 2, 3, 4, 5\} \}. \]
We eliminate in $S$ the variables $X_5, X_4, X_3$, and $X_2$ using $\text{Eq}_1, \text{Eq}_1^q, \text{Eq}_1^q$, and $\text{Eq}_1^q$ respectively, and we get
\[ Y_0 X_0 (Z_1 Y_0^2 Y_2^2 + 2 Z_1 Y_0 Y_2 Y_2 + 2 Z_1 Y_0 Y_2 + Z_1 Y_1^2 - Y_0^2 Y_2 - Y_0 Y_2 - Y_1^2) = 0. \]
Since $b \neq 0$ and $k \notin \mathbb{F}_{q^2}$, $X_0 Y_0 \neq 0$ and therefore
\[ P := Z_1 Y_0^2 Y_2^2 + 2 Z_1 Y_0 Y_1^2 Y_2 + 2 Z_1 Y_0 Y_1^2 + Z_1 Y_1^2 - Y_0^2 Y_2 - Y_0 Y_2 - Y_1^2 = 0. \]
Once again we consider the resultants of the polynomials in $S$ and $P$ w.r.t. $Z_1$ and we obtain
\[ bkq^2+2q (k-k^q)(k+k^q)(kq^2+1) - 1)(kq^2+1) = 0, \]
a contradiction to $k \notin \mathbb{F}_{q^2}$. \hfill \square

As a consequence of the above considerations and Propositions 3.5 and 3.7, we have the following.

**Corollary 3.8.** If $h \notin \mathbb{F}_{q^2}$, then $\mathcal{L}_h$ is not PGL-equivalent to any known scattered linear set in $\text{PG}(1, q^6)$.

### 3.3 $\mathcal{L}_h$ may be defined by a trinomial

Suppose that $h \in \mathbb{F}_{q^2}$, then the condition on $h$ becomes $h^{q+1} = -1$. For such $h$ we can prove that the linear set $\mathcal{L}_h$ can be defined by the $q$-polynomial $(h^{-1} - 1)x^q + x^{q^3} + (h-1)x^{q^5}$.

**Proposition 3.9.** If $h \in \mathbb{F}_{q^2}$, then the linear set $\mathcal{L}_h$ is PGL-equivalent to
\[ L_{\text{tri}} := \{ ((x, (h^{-1} - 1)x^q + x^{q^3} + (h-1)x^{q^5})\}_{x \in \mathbb{F}_{q^6}}. \]
Proof. Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \GL(2, q^6) \) with \( a = -h + h^{-1}, b = 1, c = h^{-1} - 1 - h^3 + h^2 \) and \( d = h - h^2 - 1 \). Straightforward computations show that the subspaces \( U_h \) and \( U_{(h^{-1}-1)x^9+x^3+(h^{-1})x^{q_5^5}} \) are \( \Gamma L(2, q^6) \)-equivalent under the action of the matrix \( A \). Hence, the linear sets \( \mathcal{L}_h \) and \( \mathcal{L}_{\text{tr}} \) are \( \Gamma L \)-equivalent.

The fact that \( \mathcal{L}_h \) can also be defined by a trinomial will help us to completely close the equivalence issue for \( \mathcal{L}_h \) when \( h \in \mathbb{F}_{q^2} \). Indeed, we can prove the following:

**Proposition 3.10.** If \( h \in \mathbb{F}_{q^2} \), then the linear set \( \mathcal{L}_h \) is \( \Gamma L \)-equivalent to some \( L_\delta^4 (\delta^2 + \delta = 1) \) if and only if \( h \in \mathbb{F}_q \) and \( q \) is a power of 5.

Proof. Recall that by [27, Proposition 5.5] if \( h \in \mathbb{F}_q \) and \( q \) is a power of 5, then \( \mathcal{L}_h \) is \( \Gamma L \)-equivalent to some \( L_\delta^4 \). As in the proof of Proposition 3.6, by Lemma 3.3 we have to check whether \( U_{(h^{-1}-1)x^9+x^3+(h^{-1})x^{q_5^5}} \) is \( \Gamma L \)-equivalent either to \( U_\delta^3 \) or to \( (U_\delta^3)^\perp \). Suppose that there exist \( \rho \in \text{Aut}(\mathbb{F}_{q^6}) \) and an invertible matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that for each \( x \in \mathbb{F}_{q^6} \) there exists \( z \in \mathbb{F}_{q^6} \) satisfying

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} (h^{-\rho} - 1)x^\rho q + x^{\rho q_3} + (h^\rho - 1)x^\rho q_5 = (z^q + z^{q_3} + \delta z^{q_5}).
\]

Let \( k = h^\rho \), for which \( k^{q+1} = -1 \). As in Proposition 3.5, we obtain a polynomial identity, whence

\[
\begin{align*}
\begin{cases}
c &= b^5 (k^q - 1) + b^3 q + b^{\delta^5} (k^{-q} - 1) \\
d &= a^q \\
0 &= b^4 (k^q - 1) + b^3 (k^q - 1) + b^5 \delta \\
d &= a^q \\
0 &= b^3 (k^q - 1) + b^{\delta^5} (k^q - 1) \delta \\
d &= a^q \\
0 &= b^3 (k^{-q} - 1) + b^{\delta^5} (k^q - 1) \delta \\
d &= a^q \\
0 &= b^3 (k^q - 1) + b^{\delta^5} (k^q - 1) \delta
\end{cases} \tag{3.5}
\end{align*}
\]

By subtracting the fifth equation from the third equation raised to \( q^2 \), we get

\[
b^q = b^{\delta^5} (k^q - 1),
\]

i.e. either \( b = 0 \) or \( k^q - 1 = (b^q)^{q^4-1} \), whence we get either \( b = 0 \) or \( N_{q^6/q^2} (k^q - 1) = 1 \).

If \( b \neq 0 \), since \( k - 1 \in \mathbb{F}_{q^2} \) and \( N_{q^6/q^2} (k - 1) = (k - 1)^3 = 1 \), then

\[
k^3 - 3k^2 + 3k - 2 = 0
\]

and, since \( N_{q^6/q^2} (k^q - 1) = 1 \) and \( k^q = -1/k \),

\[
2k^3 + 3k^2 + 3k + 1 = 0,
\]

from which we get

\[
9k^2 - 3k + 5 = 0. \tag{3.6}
\]

- If \( k \notin \mathbb{F}_q \) then \( k \) and \( k^q \) are the solutions of (3.6) and

\[-1 = k^{q+1} = \frac{5}{9},
\]

which holds if and only if \( q \) is a power of 7. By (3.6) it follows that \( k \in \mathbb{F}_q \), a contradiction.
If \( k \in \mathbb{F}_q \), then \( k^2 = -1 \) and by (3.6) we have \( k = -4/3 \), which is possible if and only if \( q \) is a power of 5.

Hence, if either \( k \notin \mathbb{F}_q \) or \( k \in \mathbb{F}_q \) with \( q \) not a power of 5, we have that \( b = 0 \) and hence \( c = 0 \), \( a \neq 0 \) and \( d \neq 0 \).

By combining the second and the fourth equation of (3.5), we get \( N_{q^6/q^2}(k^{-1} - 1) = 1 \) and, since \( k^q = -1/k \), \( N_{q^6/q^2}(k^q + 1) = -1 \). Arguing as above, we get a contradiction whenever \( k \notin \mathbb{F}_q \) or \( k \in \mathbb{F}_q \) with \( q \) not a power of 5.

Now, suppose that there exist \( \rho \in \text{Aut}(\mathbb{F}_{q^6}) \) and an invertible matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that for each \( x \in \mathbb{F}_{q^6} \) there exists \( z \in \mathbb{F}_{q^6} \) satisfying
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\rho \\ (h^{-\rho} - 1) x^q + x^{q^3} + (h^\rho - 1) x^{q^5} \end{pmatrix} = \begin{pmatrix} z \\ \delta z^q + z^{q^3} + z^{q^5} \end{pmatrix}.
\]

Let \( k = h^\rho \). As before, we get the following equations
\[
\begin{align*}
  c &= \delta b^q (k^q - 1) + b^q + b^{q^5} (k^{-q} - 1) \\
  d(k^{-1} - 1) &= \delta a^q \\
  0 &= \delta b^q (k^{-q} - 1) + b^q (k^q - 1) + b^{q^5} \\
  d &= a^{q^3} \\
  0 &= \delta b^q + b^q (k^{-q} - 1) + b^{q^5} (k^q - 1) \\
  d(k - 1) &= a^{q^5}.
\end{align*}
\]

By subtracting the fifth equation from the third raised to \( q^2 \) of the above system we get
\[
b^q = b^{q^3} (k^{-q} - 1).
\]

If \( b \neq 0 \), then \( N_{q^6/q^2}(k^{-q} - 1) = 1 \). Hence, arguing as above, we get that \( b = 0 \) and hence \( c = 0 \), \( a, d \neq 0 \). By combining the fourth equation with the second and the fifth equation of (3.7) we get \( N_{q^6/q^2}(k - 1) = 1 \), which yields again to a contradiction when \( k \notin \mathbb{F}_q \) or \( k \in \mathbb{F}_q \) with \( q \) not a power of 5.

So, as a consequence of Corollary 3.8 and of the above proposition, we have the following result.

**Corollary 3.11.** Apart from the case \( h \in \mathbb{F}_q \) and \( q \) a power of 5, the linear set \( \mathcal{L}_h \) is not PTL-equivalent to any known scattered linear set in \( \text{PG}(1, q^6) \).

By Proposition 3.9, when \( h \in \mathbb{F}_{q^2} \), \( \mathcal{L}_h \) is a linear set of the family presented in [23, Section 7]. Also, we get an extension of [21, Table 1], where it is shown examples of scattered linear sets which could generalize the family presented in [10]. We do not know whether the linear set \( \mathcal{L}_h \), for each \( h \in \mathbb{F}_{q^6} \setminus \mathbb{F}_{q^2} \) with \( h^{q^2+1} = -1 \), may be defined by a trinomial or not.

### 4 New MRD-codes

Delsarte in [13] (see also [14]) introduced in 1978 rank metric codes as follows. A **rank metric code** (or RM-code for short) \( \mathcal{C} \) is a subset of the set of \( m \times n \) matrices \( \mathbb{F}_q^{m \times n} \) over \( \mathbb{F}_q \) equipped with the distance function
\[
d(A, B) = \text{rk} (A - B)
\]
for \( A, B \in \mathbb{F}_q^{m \times n} \). The **minimum distance** of \( C \) is
\[
d = \min \{ d(A, B) : A, B \in C, \ A \neq B \}.
\]

We will say that a rank metric code of \( \mathbb{F}_q^{m \times n} \) with minimum distance \( d \) has parameters \((m, n, q; d)\). When \( C \) is an \( \mathbb{F}_q \)-subspace of \( \mathbb{F}_q^{m \times n} \), we say that \( C \) is \( \mathbb{F}_q \)-linear. In the same paper, Delsarte also showed that the parameters of these codes fulfill a Singleton-like bound, i.e.
\[
|C| \leq q^{\max\{m,n\}(\min\{m,n\}-d+1)}.
\]

When the equality holds, we call \( C \) a **maximum rank distance** (MRD for short) code. We will consider only the case \( m = n \) and we will use the following equivalence definition for codes of \( \mathbb{F}_q^{n \times m} \). Two \( \mathbb{F}_q \)-linear RM-codes \( C \) and \( C' \) are **equivalent** if and only if there exist two invertible matrices \( A, B \in \mathbb{F}_q^{m \times m} \) and a field automorphism \( \sigma \) such that \( \{ AC^\sigma B : C \in C\} = C' \), or \( \{ AC^{T\sigma} B : C \in C\} = C' \), where \( T \) denotes transposition. Also, the **left and right idealisers** of \( C \) are \( L(C) = \{ A \in \text{GL}(m, q) : AC \subseteq C\} \) and \( R(C) = \{ B \in \text{GL}(m, q) : CB \subseteq C\} \) [17, 19]. They are important invariants for linear rank metric codes, see also [15] for further invariants.

In [24, Section 5] Sheekey showed that scattered \( \mathbb{F}_q \)-linear sets of \( \text{PG}(1, q^n) \) of rank \( n \) yield \( \mathbb{F}_q \)-linear MRD-codes with parameters \((n, n, q; n-1)\) with left idealiser isomorphic to \( \mathbb{F}_q^n \); see [7, 8, 25] for further details on such kind of connections. We briefly recall here the construction from [24]. Let \( U_f = \{(x, f(x)) : x \in \mathbb{F}_q^n\} \) for some scattered \( q \)-polynomial \( f(x) \). After fixing an \( \mathbb{F}_q \)-basis for \( \mathbb{F}_q^n \), we can define an isomorphism between the rings \( \text{End}(\mathbb{F}_q^n, \mathbb{F}_q) \) and \( \mathbb{F}_q^{n \times n} \). In this way the set
\[
C_f := \{ x \mapsto af(x) + bx : a, b \in \mathbb{F}_q^n \}
\]
corresponds to a set of \( n \times n \) matrices over \( \mathbb{F}_q \) forming an \( \mathbb{F}_q \)-linear MRD-code with parameters \((n, n, q; n-1)\). Also, since \( C_f \) is an \( \mathbb{F}_q^n \)-subspace of \( \text{End}(\mathbb{F}_q^n, \mathbb{F}_q) \) its left idealiser \( L(C_f) \) is isomorphic to \( \mathbb{F}_q^n \). For further details see [6, Section 6].

Let \( C_f \) and \( C_h \) be two MRD-codes arising from maximum scattered subspaces \( U_f \) and \( U_h \) of \( \mathbb{F}_q^n \times \mathbb{F}_q^n \). In [24, Theorem 8] the author showed that there exist invertible matrices \( A, B \) and \( \sigma \in \text{Aut}(\mathbb{F}_q) \) such that \( AC_f^\sigma B = C_h \) if and only if \( U_f \) and \( U_h \) are \( \Gamma L(2, q^n) \)-equivalent.

Therefore, we have the following.

**Theorem 4.1.** The \( \mathbb{F}_q \)-linear MRD-code \( C_{f_h} \), arising from the \( \mathbb{F}_q \)-subspace \( U_{f_h} \), has parameters \((6, 6, q; 5)\) and left idealiser isomorphic to \( \mathbb{F}_{q^6} \), and is not equivalent to any previously known \( \mathbb{F}_q \)-linear, MRD-code, apart from the case \( h \in \mathbb{F}_q \) and \( q \) a power of 5.

**Proof.** From [6, Section 6], the previously known \( \mathbb{F}_q \)-linear MRD-codes with parameters \((6, 6, q; 5)\) and with left idealiser isomorphic to \( \mathbb{F}_{q^6} \) arise, up to equivalence, from one of the maximum scattered subspaces of \( \mathbb{F}_{q^6} \times \mathbb{F}_{q^6} \) described in Section 3. From Corollaries 3.8 and 3.11 the result then follows.

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