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Convex drawings of the complete graph: topology meets geometry*

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Abstract

In a geometric drawing of K_n, trivially each 3-cycle bounds a convex region: if two vertices are in that region, then so is the (geometric) edge between them. We define a topological drawing D of K_n in the sphere to be *convex* if each 3-cycle bounds a closed region R (either of the two sides of the 3-cycle) such that any two vertices in R have the (topological) edge between them contained in R.

While convex drawings generalize geometric drawings, they specialize topological ones. Therefore it might be surprising if all *optimal* (that is, crossing-minimal) topological drawings of K_n were convex. However, we take a first step to showing that they are convex: we show that if D has a non-convex K_5 all of whose extensions to a K_7 have no other non-convex K_5, then D is not optimal (without reference to the conjecture for the crossing number of K_n). This is the first example of non-trivial local considerations providing sufficient conditions for suboptimality. At our request, Aichholzer has computationally verified that, up to $n = 12$, every optimal drawing of K_n is convex.

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Convexity naturally lends itself to refinements, including *hereditarily convex* (h-convex) and *face convex* (f-convex). The hierarchy rectilinear $\subseteq$ f-convex $\subseteq$ h-convex $\subseteq$ convex $\subseteq$ topological provides links between geometric and topological drawings. It is known that f-convex is equivalent to pseudolinear (generalizing rectilinear) and h-convex is equivalent to pseudospherical (generalizing spherical geodesic). We characterize h-convexity by three forbidden (topological) subdrawings.

This hierarchy provides a framework to consider generalizations of other geometric questions for point sets in the plane. We provide two examples of such questions, namely numbers of empty triangles and existence of convex k-gons.

Keywords: Simple drawings, complete graphs, convex drawings.

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Konveksne risbe polnega grafa: srečanje topologije in geometrije*

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Povzetek

V geometrijski risbi grafa K_n, trivialno vsak 3-cikel omejuje konveksno območje: če sta dva vozlišča v tem območju, potem to velja tudi za (geometrijsko) povezavo med njima. Za topološko risbo D grafa K_n na sferi rečemo, da je *konveksna*, če vsak 3-cikel omejuje zaprto območje R (na katerikoli od dveh strani 3-cikla), pri tem pa imata poljubna dva vozlišča v R (topološko) povezavo med njima vsebovano v R.

Konveksne risbe po eni strani posplošujejo geometrijske risbe, po drugi strani pa so poddružina topoloških risb. Zato bi bilo lahko presenetljivo, če bi bile vse *optimalne* (to je, z minimalnim številom presečišč) topološke risbe grafa K_n konveksne. Vseeno pa storimo prvi korak k dokazu, da *so* konveksne: pokažemo, da če D vsebuje nekonveksen K_5, vse njegove razširitve do K_7 pa ne vsebujejo nobenega drugega nekonveksnega K_5, potem D ni optimalen (brez sklicevanja na domnevo o številu presečišč grafa K_n). To je prvi primer netrivialnih lokalnih argumentov, ki dajejo zadostne pogoje za suboptimalnost. Na našo

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prošnjo je Aichholzer računalniško potrdil, da je, vse do $n = 12$, vsaka optimalna risba grafa K_n konveksna.

Konveksnost naravno dopušča izpopolnitve, kot sta npr. lastnosti *hereditarno konveksen* (h-konveksen) in *lično konveksen* (f-konveksen). Hierarhija premočrten $\subseteq$ f-konveksen $\subseteq$ h-konveksen $\subseteq$ konveksen $\subseteq$ topološki opisuje relacije med geometrijskimi in topološkimi risbami. Znano je, da je f-konveksnost ekvivalentna psevdolinearnosti (ki posplošuje premočrtnost) in da je h-konveksnost ekvivalentna psevdosferičnosti (ki posplošuje sferično geodetskost). Karakteriziramo h-konveksnost s tremi prepovedanimi (topološkimi) podrisbami.

Ta hierarhija predstavlja okvir za obravnavo posplošitev tudi drugih geometrijskih problemov v zvezi z množicami točk v ravnini. Predstavimo dva primera takšnih problemov, in sicer o številu odprtih trikotnikov ter o obstoju konveksnih k-kotnikov.

Ključne besede: Enostavne risbe, polni grafi, konveksne risbe.

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