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# Imprimitivity of locally finite, 1-ended, planar graphs

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**Abstract**

Using results from group theory, we offer a concise proof of the imprimitivity of locally finite, vertex-transitive, 1-ended planar graphs, a result previously established by J .E. Graver and M. E. Watkins (2004) using graph-theoretical methods.

*Keywords: Planar graph, end, primitive permutation group, residually finite group, co-compact group.*

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## 1 Introduction

Locally finite planar graphs whose automorphism group is primitive were characterized in [14] as follows (by a *lobe*, we mean a maximal 2-connected subgraph):

> **1.1** *The automorphism group of an infinite, locally finite, planar graph $\Gamma$ is primitive if and only if, for some integer $m \geq 2$, every vertex of $\Gamma$ is incident with exactly $m$ lobes and no separating edge. Moreover, either all of the lobes are isomorphic to $K_4$ or all are circuits of length $p$ for some fixed odd prime $p$.*

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The sufficiency of the condition, that the graphs described above are indeed primitive, had been previously done in [10], and so the authors' task in [14] was to prove that no other infinite planar graphs with primitive automorphism group exist. The argument in [14] that no 1-ended examples exist is particularly long, with a variety of cases and subcases wherein nontrivial systems of imprimitivity were constructed by brute force. In this note, using group-theoretical methods, we offer a concise proof of the imprimitivity of 1-ended planar graphs.

## 2 Group-theoretic preliminaries

Let $X$ be a nonempty set and let $G$ be a group of permutations on $X$. We denote by $\iota$ the identity of $G$. If $W \subseteq X$, and $\alpha \in G$, then $\alpha(W)$ is the subset $\{\alpha(w) : w \in W\}$. We say that $G$ *acts transitively* on $X$ if for all $x, y \in X$, there exists $\alpha \in G$ such that $\alpha(x) = y$. For $x \in X$, the *stabilizer* of $x$ is the subgroup $G_x = \{\alpha \in G : \alpha(x) = x\}$.

A subset $B \subseteq X$ is called a *block* (of imprimitivity) with respect to $G$ when, for all $\alpha \in G$, either $\alpha(B) = B$ or $\alpha(B) \cap B = \varnothing$. Clearly $\varnothing$, $X$, and the singleton subsets of $X$ are blocks. If $G$ acts transitively on $X$ and admits only these so-called *trivial* blocks, then $G$ is *primitive* on $X$; if $G$ acts transitively on $X$ but admits nontrivial blocks, then $G$ is *imprimitive* on $X$. A basic fact concerning imprimitive permutation groups is the following (see, for example, [2], Theorem 1.7).

**Proposition 2.1.** *A permutation group $G$ acting transitively on a set $X$ is imprimitive if and only if, for any element $x \in X$, there exists a subgroup $M$ such that $G_x < M < G$.*

Note that the subgroup containment in Proposition 2.1 is proper.

**Definition 2.2.** An infinite group $G$ is *residually finite* when for any $\alpha \in G \backslash \{\iota\}$ there exists a subgroup $N$ of $G$ such that $\alpha \notin N$ and $[G : N] < \infty$.

The following characterization of residual finiteness will be used.

**Lemma 2.3.** An infinite group $G$ is residually finite if and only if, for any finite set $S \subset G$, there exists a normal subgroup $N$ of $G$ such that $S \cap (N \backslash \{\iota\}) = \varnothing$ and $|S| < [G : N] < \infty$.

*Proof.* Suppose that $G$ is residually finite, and let $S = \{\sigma_1, \dots, \sigma_m\} \subset G$. By Definition 2.2, for each $i = 1, \dots, m$, there exists a subgroup $K_i$ of $G$ such that $\sigma_i \notin K_i \setminus \{\iota\}$ and $[G : K_i] < \infty$. Then $K = \bigcap_{i=1}^{m} K_i$ is a subgroup of $G$ of finite index disjoint from $S \setminus \{\iota\}$. If $[G : K] > |S|$, we let $L = K$; otherwise we proceed as follows.

Let $\alpha_1 \in K \setminus \{\iota\}$. By Definition 2.2, there exists a subgroup $L_1$ of $G$ of finite index that excludes $\alpha_1$, and therefore $[G : K \cap L_1] > [G : K]$. If $[G : K \cap L_1] > |S|$, then we let $L = K \cap L_1$. Otherwise, choose $\alpha_2 \in (K \cap L_1) \setminus \{\iota\}$ and repeat this argument. In finitely many steps, one arrives at a subgroup $L$ such that $S \cap (L \setminus \{\iota\}) = \varnothing$ and $|S| < [G : L] < \infty$.

Finally, let $N$ be the core of $L$, that is, let $N$ be the intersection of all conjugates of $L$ in $G$. Since $[G : L]$ is finite, there are only finitely many such conjugates. Hence $[G : N]$ is finite as well, with $[G : N] \geq [G : L]$. Obviously, $N$ is a normal subgroup of $G$ with all of the required properties.

The converse is immediate. $\square$

**Definition 2.4.** A group $G$ acting on the plane $\mathcal{P}$ is a *planar discontinuous group* if each point $p$ in the plane lies in a neighborhood $O(p)$ such that, for all $\alpha \in G$, either $\alpha(p) = p$ or $\alpha(p) \notin O(p)$. Such a group $G$ is *co-compact* if the quotient space $\mathcal{P}/G$ is compact.

**Lemma 2.5.** Every co-compact planar discontinuous group is residually finite.

*Proof.* By Theorem 4.10.1 of [16], every co-compact planar discontinuous group has a surface group of finite index, where a surface group is simply a fundamental group of some compact orientable surface. Such groups are known to be residually finite (see e.g. [5]). Since residual finiteness is a property that is hereditary upward to supergroups of finite index, the result follows. □

## 3 Graph-theoretic preliminaries

In this note, graphs and their subgraphs are denoted by capital Greek letters. We let Aut($\Gamma$) denote the automorphism group of the graph $\Gamma$. The graphs that we consider are *locally finite*, i.e., all vertices have finite valence.

An *end* in an infinite graph is an equivalence class of rays (1-way infinite paths) whereby two rays are in the same end if there exist infinitely many pairwise-disjoint finite paths joining them. If $\Gamma$ is locally finite and Aut($\Gamma$) has finitely many orbits, then, by results in [6] and [9], the set of ends of $\Gamma$ has cardinality 1, 2 or $2^{\aleph_0}$. For a (locally finite) graph $\Gamma$ with exactly one end, the following simple characterization holds: for any finite subgraph $\Phi$ of $\Gamma$, the subgraph $\Gamma - \Phi$ has exactly one infinite component. In this note, our interest is confined principally to 1-ended graphs.

**Proposition 3.1.** *([1], Lemma 2.2) If a 1-ended graph is vertex-transitive, then it is 3-connected.*

Let $M$ be the planar map induced by an embedding of a locally finite graph, and suppose that every face of $M$ has finite *covalence*, that is, has finitely many edges in its boundary. For each face of $M$, an interior point of the surface may be designated as its *center*. As usual in the theory of maps, we regard $M$ as being composed of *flags*, which are (closed) topological triangles whose three corners are a vertex $v$, the midpoint of an edge $e$ incident with $v$, and the center of a face incident with both $v$ and $e$.

The following result is crucial for the present argument.

**Lemma 3.2.** The automorphism group of every infinite, locally finite, 1-ended, vertex-transitive, planar graph is residually finite, and the stabilizer of every vertex is finite.

*Proof.* Let $\Gamma$ be an infinite, locally finite, 1-ended, vertex-transitive, planar graph, and let $\mathcal{P}$ denote the plane. Planarity, local finiteness, connectivity at least three (by Proposition 3.1) and 1-endedness imply, by Theorems 13 and 14 of [12], that there is a (topologically as well as combinatorially) unique embedding $\theta : \Gamma \to \mathcal{P}$ such that every connected component of $\mathcal{P}\backslash\theta(\Gamma)$ is homeomorphic to an open disc forming the interior of some face of the embedding. Let $M$ be the planar map induced by the embedding $\theta$. The vertex-transitivity and 1-endedness of $\Gamma$ imply that every face of $M$ has finite covalence. The two cited theorems from [12] then imply the important fact that *every point of $\mathcal{P}$ is contained in a flag of $M$.*

Invoking an extension of H. Whitney's result [15] for finite planar graphs to infinite graphs (see [7] or [13]), every automorphism of $\Gamma$ extends to a map automorphism of $M$. As the converse is obvious, we have $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(M)$. It is well known (cf. [4] Lemma 3.1) that the action of $\mathrm{Aut}(M)$ is semi-regular on the set of flags of $M$. Furthermore, it follows from Theorem 5.4 and Proposition 5.5 of [8] (and their extensions to orientation-reversing automorphisms) that the group $\mathrm{Aut}(M)$ is isomorphic to a group $G$ acting on $\mathcal{P}$ via an isomorphism that takes an automorphism $\alpha$ of $\mathrm{Aut}(M)$ to a unique homeomorphism $\bar{\alpha}$ of $\mathcal{P}$ with the property that $\bar{\alpha}$ maps every flag $x$ of $M$ pointwise to the flag $\alpha(x)$.

Since $M$ is locally finite with all covalences being finite, the stabilizer in $\mathrm{Aut}(M)$, and hence in $G$, of any point on the boundary of a flag of $M$ must be finite. Moreover, by the semi-regularity of $\mathrm{Aut}(M)$ on flags, the stabilizer in $G$ of any interior point of any flag is trivial. Since every point of $\mathcal{P}$ lies in some flag of $M$, it follows that $G$ is planar discontinuous by Definition 2.4.

The local finiteness and vertex-transitivity of $M$ further imply that the quotient space $\mathcal{P}/G$ is an identification space formed by equivalence classes of flags incident with a fixed vertex modulo its stabilizer in $G$. Hence $\mathcal{P}/G$ is compact and $G$ is co-compact. The conclusion now follows from Lemma 2.5. □

We emphasize that our remark in the above proof about every point of $\mathcal{P}$ being contained in some flag of $M$ is by no means automatic and becomes invalid in general if one drops the assumption either of vertex-transitivity or of one-endedness. In such cases one may have accumulation points in $\mathcal{P}$ but not on $\theta(\Gamma)$, and these points of $\mathcal{P}$ would belong to no flag of the embedding. Indeed, the primitive planar graphs described in statement 1.1 are infinitely ended. More extreme examples of such a situation are presented in [11] where it is shown that for every integer $k$ there exists an infinitely-ended $k$-connected vertex-transitive planar graph with each vertex lying on at least $k$ faces bounded by a two-way infinite ray. The union of all flags of such a map is $\mathcal{P}$ with a Cantor set removed, and arguments in our proof would not apply to the points in the Cantor set. Vertex-transitive graphs with exactly two ends have polynomial growth, and therefore are imprimitive (see [3], Theorem 4).

## 4 A short proof of the main result

**Theorem 4.1.** *Let $\Gamma$ be an infinite, locally finite, 1-ended planar graph, and let $G$ be a group of automorphisms that acts transitively on the vertex set of $\Gamma$. Then $G$ is imprimitive.*

*Proof.* Let $\Gamma$ and $G$ be as described in the hypothesis. It is sufficient to assume that $\Gamma$ is vertex-transitive and $G = \mathrm{Aut}(\Gamma)$. By Lemma 3.2, $G$ is residually finite and the stabilizer $G_v$ of any vertex $v$ of $\Gamma$ is finite. By Lemma 2.3, $G$ contains a normal subgroup $N$ such that $N \cap G_v = \{\iota\}$ and $[G : N] > |G_v|$. This implies that $G_v < NG_v < G$. By Proposition 2.1, $G$ is imprimitive. □

Normality of $N$ in $G$ gives our main result an extra flavor. Namely, if $M$ is a planar map arising from the unique embedding of $\Gamma$ in $\mathcal{P}$ as in the proof of Lemma 3.2, then the quotient $M/N$ is a finite map on a compact surface, possibly with boundary if $N$ contains orientation-reversing elements. Blocks of imprimitivity are then pre-images of individual vertices in the natural covering $M \to M/N$. Our construction also shows that such blocks of imprimitivity can be chosen in infinitely many ways.

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## References

[1] C. P. Bonnington, W. Imrich and M. E. Watkins, Separating double rays in locally finite, planar graphs, *Discrete Math.* **145** (1995), 61–72.

[2] P. J. Cameron, *Permutation Groups*, London Mathematical Society Student Texts 45, Cambridge, U. K., 1999.

[3] C. D. Godsil, W. Imrich, N. Seifter, M. E. Watkins and W. Woess, A note on bounded automorphissms of graphs, *Graphs and Combinatorics* **5** (1989), 333–338.

[4] J. E. Graver and M. E. Watkins, Locally Finite, Planar, Edge-Transitive Graphs, *Memoirs Amer. Math. Soc.*, Vol. 126, No. 601, 1997.

[5] H. B. Griffiths, A covering-space approach to residual properties of groups, *Michigan Math. J.* **14** (1967), 335–348.

[6] R. Halin, Automorphisms and endomorphisms of infinite locally finite graphs, *Abh. Math. Sem. Univ. Hamburg* (1973), 251–283.

[7] W. Imrich, On Whitney's theorem on the unique embeddability of 3-connected planar graphs, in: M. Fiedler (ed.) Recent Advances in Graphs Theory, Academia Praha, Prague, 1975, 303–306.

[8] G. A. Jones and D. Singerman, Theory of maps on orientable surfaces, *Proc. London Math. Soc.* **37** (1978), 373–397.

[9] H. A. Jung, A Note on fragments of infinite graphs, *Combinatorica* **1** (1981), 285–288.

[10] H. A. Jung and M. E. Watkins, On the structure of infinite vertex-transitive graphs, *Discrete Math.* **18** (1977), 45–53.

[11] B. Mohar, Tree amalgamation of graphs and tessellations of the Cantor sphere, *J. Combin. Theory Ser. B* **96** (2006), 740–753.

[12] R. B. Richter and C. Thomassen, 3-connected planar spaces uniquely embed in the sphere, *Trans. Amer. Math. Soc.* **354** (2002), 4585–4595.

[13] C. Thomassen, Planarity and duality in infinite graphs, *J. Combin. Theory Ser. B* **29** (1980), 244–271.

[14] M. E. Watkins and J. E. Graver, A characterization of infinite planar primitive graphs, *J. Combin. Theory Ser. B* **91** (2004), 87–104.

[15] H. Whitney, Non-separable and planar graphs, *Trans. Amer. Math. Soc.* **34** (1932), 339–362.

[16] H. Zieschang, E. Vogt and H.-D. Coldewey, Surfaces and Planar Discontinuous Groups, *Lecture Notes in Math.* 835, Springer-Verlag, 1980.