

On resolving sets in the point-line incidence graph of $PG(n, q)$

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Abstract

Lower and upper bounds on the size of resolving sets and semi-resolving sets for the point-line incidence graph of the finite projective space $PG(n, q)$ are presented. It is proved that if $n > 2$ is fixed, then the metric dimension of the graph is asymptotically $2q^{n-1}$.

Keywords: Point-line incidence graph, resolving sets, finite projective spaces.

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1 Introduction

For a simple, connected, finite graph $\Gamma = (V, E)$ and $x, y \in V$ let $d(x, y)$ denote the length of a shortest path joining x and y .

Definition 1.1. Let $\Gamma = (V, E)$ be a finite, connected, simple graph. Two vertices $v_1, v_2 \in V$ are *divided* by $S = \{s_1, s_2, \dots, s_r\} \subset V$ if there exists $s_i \in S$ so that $d(v_1, s_i) \neq d(v_2, s_i)$. A vertex $v \in V$ is *resolved* by S if the ordered sequence $(d(v, s_1), d(v, s_2), \dots, d(v, s_r))$ is unique. S is a *resolving set* in Γ if it resolves all the elements of V . The *metric dimension* of Γ , denoted by $\mu(\Gamma)$, is the size of the smallest example of resolving set in it.

The study of metric dimension is an interesting problem in its own right and it is also motivated by the connection with the base size of the corresponding graph. The *base size of a permutation group* is the smallest number of points whose stabilizer is the identity. The *base size* of Γ , denoted by $b(\Gamma)$, is the base size of its automorphism group $\text{Aut}(\Gamma)$. The study of base size dates back more than 50 years, see [18]. A resolving set in Γ is obviously a base for $\text{Aut}(\Gamma)$, so the metric dimension of a graph gives an upper bound on its base size. The difference $\mu(\Gamma) - b(\Gamma)$ is called the *dimension jump* of Γ . Distance-transitive graphs whose dimension jump is large with respect to the number of vertices are rare, and hence interesting objects. For more information about general results on metric dimension and base size we refer the reader to the survey paper of *Bailey and Cameron* [2].

Resolving sets for incidence graphs of some linear spaces were investigated by several authors [4, 9, 10, 12]. In these cases much better bounds than the general ones are known. Estimates on the size of blocking sets can be used to prove lower bounds on the metric dimension, and the knowledge of geometric properties is useful for constructions and upper bounds. It was shown by *Héger and Takáts* [12] that the metric dimension of the point-line incidence graph of a projective plane of order q is $4q - 4$ if $q \geq 23$. In a recent paper *Héger et al.* [11] extended this result for small values of q , too.

There are two natural generalizations of this planar result in higher dimensional spaces: one can consider either the point-hyperplane incidence graph, or the point-line incidence graph of $\text{PG}(n, q)$. In the former case resolving sets are connected with lines in a higgledy-piggledy arrangement which were investigated by *Fancsali and Sziklai* [9]. Their results were recently improved by the authors of this paper [5]. The latter case is studied in the present paper. We assume that the reader is familiar with finite projective geometries. For a detailed description of these spaces we refer to [14, 16].

Let $\Gamma_{n,q}$ denote the point-line incidence graph of the finite projective space $\text{PG}(n, q)$. The two sets of vertices of this bipartite graph correspond to points and lines of $\text{PG}(n, q)$, respectively, and there is an edge between two vertices if and only if the corresponding point is incident with the corresponding line. In $\Gamma_{n,q}$ the distance of two different lines is 2 if they intersect each other and 4 if they are skew. The distance of a point P and a line ℓ is 1 if P is on ℓ , and it is 3 if P is not on ℓ . Finally, the distance of two different points is always 2. Hence, points cannot be resolved by other points. Considering these properties, the following definitions are natural.

Definition 1.2. A set \mathcal{S} of points and lines of $\text{PG}(n, q)$ is a *semi-resolving set* for points (lines) in $\Gamma_{n,q}$ if it resolves all the vertices of $\Gamma_{n,q}$ corresponding to points (lines).

Definition 1.3. Let \mathcal{S} be a (semi-)resolving set in $\Gamma_{n,q}$. A point or a line is called *inner* (*outer*) if it is (not) in \mathcal{S} . An outer point is called *t-covered* if it is incident with exactly t

lines of \mathcal{S} . A point or a line is called *uncovered* if it has empty intersection with all elements of \mathcal{S} .

The paper is organized as follows. First, in Section 2 we provide a pure combinatorial proof of a lower bound for resolving sets in $\Gamma_{n,q}$. In the second part of the section interesting constructions are presented for $n \geq 3$ which yield examples asymptotically close to the lower bound. These resolving sets are related with regular spreads of lines in projective spaces. We prove that the metric dimension of $\Gamma_{n,q}$ is asymptotically $2q^{n-1}$ and its dimension jump is roughly $2\sqrt{v}$ where v denotes the number of its vertices. In Section 3 algebraic curves and blocking sets are applied. We consider a different type of line spread in $\text{PG}(3, q)$ and we obtain examples of resolving sets in $\Gamma_{n,q}$ of smaller size when $q = p^h$, p prime and $h > 1$. Finally, in Section 4 computer aided results for small values of q are given.

2 General bounds

In this section we present lower and upper bounds on $\mu(\Gamma_{n,q})$ for all q and $n \geq 3$. Particular attention is paid to the case $n = 3$, since general upper bounds in any dimension depend on the 3-dimensional upper bound, see Proposition 2.15 and Theorem 2.11.

Theorem 2.1. *The size of any semi-resolving set for points in $\Gamma_{n,q}$ is at least*

$$2 \frac{q^{n+1} - q}{q^2 + q - 2}.$$

Proof. Let \mathcal{S} be a semi-resolving set for points in $\Gamma_{n,q}$ which consists of k lines and m points.

Count in two different ways the number of incident point-line pairs (P, ℓ) with $\ell \in \mathcal{S}$. On the one hand, this number is exactly $k(q + 1)$. On the other hand, there is at most one uncovered point, the number of 1-covered points is at most k and any other outer point must be covered by at least two lines of \mathcal{S} . Hence

$$k(q + 1) \geq k + 2 \left(\frac{q^{n+1} - 1}{q - 1} - 1 - k - m \right),$$

so

$$k + m \geq \frac{2(q^{n+1} - q)}{(q - 1)(q + 2)} + \frac{qm}{q + 2}.$$

This gives the required inequality at once. □

Corollary 2.2. *The size of any resolving set in $\Gamma_{3,q}$ is at least $2 \left(q^2 - q + 3 - \frac{6}{q+2} \right)$.*

Corollary 2.3. *The metric dimension of $\Gamma_{3,q}$ is at least $2(q^2 - q + 3)$ for $q > 10$.*

From now on we focus on upper bounds which will be given by constructions. For our examples we need the notion of spreads, in particular line spreads.

Definition 2.4. A k -spread \mathcal{S}^k of $\text{PG}(n, q)$ is a set of k -dimensional subspaces with the property that each point of $\text{PG}(n, q)$ is incident with exactly one element of \mathcal{S}^k .

By definition, a k -spread of $\text{PG}(n, q)$ consists of $\frac{q^{n+1}-1}{q^{k+1}-1}$ elements. The following theorem about the existence of spreads was proved independently by several authors, see [1, 7, 17].

Theorem 2.5. *The projective space $\text{PG}(n, q)$ has a k -spread if and only if $(k+1) \mid (n+1)$.*

Hence there exists a line spread in any odd dimensional projective space. Two line spreads are said to be *disjoint*, if they do not share any common line. Our first construction for a semi-resolving set for points in $\Gamma_{n,q}$ is based on disjoint line spreads. We use the following theorem of Etzion [8] about the existence of disjoint line spreads.

Theorem 2.6 (Etzion). *If $n \geq 3$ is odd, then there exist at least two disjoint line spreads in $\text{PG}(n, q)$.*

Theorem 2.7. *If $n \geq 3$ is odd, then there exists a semi-resolving set for points in $\Gamma_{n,q}$ of size*

$$r_{\mathcal{P}}(n, q) = 2q^2 \frac{q^{n-1} - 1}{q^2 - 1}. \tag{2.1}$$

Proof. Let \mathcal{L}_1 and \mathcal{L}_2 be two disjoint line spreads in $\text{PG}(n, q)$, and $\ell_i \in \mathcal{L}_i$ be arbitrary lines. We claim that $\mathcal{S} = \mathcal{L}_1 \cup \mathcal{L}_2 \setminus \{\ell_1, \ell_2\}$ is a semi-resolving set for points in $\Gamma_{n,q}$.

Each point not in ℓ_i is contained in a unique pair of lines $(r_1, r_2) \in \mathcal{L}_1 \times \mathcal{L}_2$. Each point of $\ell_1 \setminus \ell_2$ is contained in a unique line of $\mathcal{L}_1 \cup \mathcal{L}_2 \setminus \{\ell_1, \ell_2\}$ and each point of $\ell_2 \setminus \ell_1$ is contained in a unique line of $\mathcal{L}_1 \cup \mathcal{L}_2 \setminus \{\ell_1, \ell_2\}$. The (possible) unique point $\ell_1 \cap \ell_2$ is the only point of $\text{PG}(n, q)$ not contained in any line of $\mathcal{L}_1 \cup \mathcal{L}_2 \setminus \{\ell_1, \ell_2\}$. The size of \mathcal{S} is $2 \frac{q^{n+1}-1}{q^2-1} - 2$, hence the statement follows. \square

Proposition 2.8. *Let Σ be a hyperplane and \mathcal{L} be a line spread in $\text{PG}(n, q)$, $n \geq 3$ odd. Then Σ contains exactly $\frac{q^{n-1}-1}{q^2-1}$ elements of \mathcal{L} .*

Proof. Any element of \mathcal{L} is either fully contained in Σ , or intersects it in exactly 1 point. The elements of \mathcal{L} partition the set of points of Σ . Hence, if x denotes the number of fully contained lines, then

$$\frac{q^n - 1}{q - 1} = (q + 1)x + \left(\frac{q^{n+1} - 1}{q^2 - 1} - x \right).$$

The claim follows from this equation at once. \square

Theorem 2.9. *Let \mathcal{L}_1 be a line spread in $\text{PG}(3, q)$. Then there exists another line spread \mathcal{L}_2 in $\text{PG}(3, q)$ such that \mathcal{L}_1 and \mathcal{L}_2 do not share any common line.*

Proof. Let $f(X, Y)$ be an irreducible homogeneous quadratic polynomial and \mathcal{H}'_i denote the hyperbolic quadric in $\text{PG}(3, q)$ with equation

$$f(X_0, X_1) + if(X_2, X_3) = 0$$

for $i = 1, 2, \dots, q - 1$. Apply a suitable linear transformation so that the images ℓ_1 and ℓ_2 of the lines $\ell'_1: X_0 = X_1 = 0$ and $\ell'_2: X_2 = X_3 = 0$ do not belong to \mathcal{L}_1 . Let \mathcal{H}_i denote the image of \mathcal{H}'_i , and let \mathcal{E}_i and \mathcal{F}_i denote the two reguli of lines on \mathcal{H}_i . Then for each i at most one of \mathcal{E}_i and \mathcal{F}_i contains some elements of \mathcal{L}_1 , because any line of \mathcal{E}_i intersects

any line of \mathcal{F}_i and no two elements of \mathcal{L}_1 intersect each other. Hence we can choose the notation so that \mathcal{E}_i does not contain any element of \mathcal{L}_1 for all i . This implies that the spread

$$\mathcal{L}_2 = \bigcup_{i=1}^{q-1} \mathcal{E}_i \cup \{\ell_1, \ell_2\}$$

does not share any common line with \mathcal{L}_1 . □

The next proposition gives a useful recursive construction method.

Proposition 2.10. *Let \mathcal{S} be a semi-resolving set for points in $\Gamma_{d,q}$ of size k . Suppose that m elements of \mathcal{S} are contained in a hyperplane Σ_{d-1} of $\text{PG}(d, q)$, and Σ_{d-1} also contains the (at most one) uncovered point. Then $\Gamma_{d+1,q}$ has a semi-resolving set for points of size $(q + 1)k - qm$.*

Moreover, if \mathcal{S} is a resolving set in $\Gamma_{d,q}$ and Σ_{d-1} also contains the (at most one) uncovered line, then $\Gamma_{d+1,q}$ admits a resolving set of size $(q + 1)k - qm + \frac{q^{d-1}-1}{q-1}$.

Proof. Embed $\Sigma_{d-1} \subset \text{PG}(d, q)$ into $\text{PG}(d+1, q)$, and consider in $\text{PG}(d+1, q)$ the pencil of hyperplanes with carrier Σ_{d-1} . These hyperplanes, $\Sigma_d^1, \Sigma_d^2, \dots, \Sigma_d^{q+1}$, are isomorphic to $\text{PG}(d, q)$. Take a copy of \mathcal{S} in Σ_d^i and denote it by \mathcal{S}^i for $i = 1, 2, \dots, q + 1$. Finally, let

$$\overline{\mathcal{S}} = \bigcup_{i=1}^{q+1} \mathcal{S}^i.$$

We claim that $\overline{\mathcal{S}}$ is a semi-resolving set for points in $\Gamma_{n+1,q}$. Inner points are resolved by definition. If two outer points, P_1 and P_2 , are in the same Σ_d^i , then they are already divided by \mathcal{S}^i . If P_1 is in \mathcal{S}^i and P_2 is in \mathcal{S}^j with $i \neq j$, then, as none of P_1 and P_2 is uncovered and none of them is in Σ_{d-1} , there exist distinct lines $\ell^i \in \mathcal{S}^i$ through P_1 and $\ell^j \in \mathcal{S}^j$ through P_2 . Hence ℓ^i does not contain P_2 , so $d(P_1, \ell^i) \neq d(P_2, \ell^i)$. Since the size of $\overline{\mathcal{S}}$ is $m + (q + 1)(k - m)$, the first part of the statement is proved.

Now suppose that \mathcal{S} is a resolving set in $\Gamma_{d,q}$. Then the elements of any point-line pair are obviously divided by $\overline{\mathcal{S}}$. Let ℓ_1 and ℓ_2 be two lines. If at least one of them is an element of $\overline{\mathcal{S}}$, then they are divided by definition. From now on we assume that none of the two lines is an element of $\overline{\mathcal{S}}$. We distinguish three main cases and some subcases.

1. If both of them are entirely contained in the same Σ_d^i , then they are divided by \mathcal{S}^i .
2. If there is no Σ_d^i that contains both ℓ_1 and ℓ_2 , but each of the lines is entirely contained in some Σ_d^i , say $\ell_1 \subset \Sigma_d^{i_1}$ and $\ell_2 \subset \Sigma_d^{i_2}$, then none of the lines is in Σ_{d-1} . Let P_j denote the unique point $\ell_j \cap \Sigma_{d-1}$ for $j = 1, 2$.
 - If $P_1 = P_2$, then let $P_3 \neq P_1$ be a point on ℓ_1 . Since \mathcal{S}^{i_1} is a semi-resolving set for points in $\Sigma_d^{i_1}$ and P_3 is not an uncovered point, either $P_3 \in \mathcal{S}^{i_1}$ or there exists at least one line $\ell \in \mathcal{S}^{i_1}$ which contains P_3 but does not contain P_1 . In the former case $d(\ell_1, P_3) = 1 \neq 3 = d(\ell_2, P_3)$. In the latter case $d(\ell_1, \ell) = 2 \neq 4 = d(\ell_2, \ell)$, so we are done.
 - If $P_1 \neq P_2$, then we may assume that P_1 is not an uncovered point, because there is at most one uncovered point. Again, either $P_1 \in \mathcal{S}^{i_1}$ or there exists at least one line $\ell \in \mathcal{S}^{i_1}$ which contains P_1 but does not contain P_2 . In the former case $d(\ell_1, P_1) = 1 \neq 3 = d(\ell_2, P_1)$, while in the latter case $d(\ell_1, \ell) = 2 \neq 4 = d(\ell_2, \ell)$, so ℓ_1 and ℓ_2 are divided by \mathcal{S}^{i_1} .

3. If ℓ_1 is not contained in any Σ_d^i , then it cannot meet Σ_{d-1} , so there exists a unique point $P_1^i = \ell_1 \cap \Sigma_d^i$ for all $i = 1, 2, \dots, q + 1$.

- If ℓ_2 is not contained in any Σ_d^i , then it cannot meet Σ_{d-1} , so there exists a unique point $P_2^i = \ell_2 \cap \Sigma_d^i$ for all $i = 1, 2, \dots, q + 1$. The two lines have at most one point of intersection, hence there exist at least q superscripts so that $P_1^i \neq P_2^i$. Since \mathcal{S}^i is a semi-resolving set for points in Σ_d^i , there exists at least one element $s \in \mathcal{S}^i$ so that $d(P_1^i, s) \neq d(P_2^i, s)$. Hence

$$d(\ell_1, s) = d(P_1^i, s) + 1 \neq d(P_2^i, s) + 1 = d(\ell_2, s),$$

so the lines are divided by \mathcal{S}^i .

- If ℓ_2 is contained in a unique Σ_d^i , then it is not contained in Σ_{d-1} , so there exists a unique point $P_2 = \ell_2 \cap \Sigma_{d-1}$. Let $j \neq i$ and consider Σ_d^j , which contains both P_1^j and P_2 . Since \mathcal{S}^j is a semi-resolving set for points in Σ_d^j , there exists at least one element $s \in \mathcal{S}^j$ so that $d(P_1^j, s) \neq d(P_2, s)$. Hence

$$d(\ell_1, s) = d(P_1^j, s) + 1 \neq d(P_2, s) + 1 = d(\ell_2, s),$$

the claim is proved.

- Finally, suppose that ℓ_2 is contained in Σ_{d-1} . Then ℓ_1 and ℓ_2 are not necessarily divided by $\bar{\mathcal{S}}$. Suppose that $\bar{\mathcal{S}}$ consists of lines only. Then ℓ_2 and P_1^i are divided by \mathcal{S}^i , but it could happen that a line of \mathcal{S}^i intersects ℓ_2 if and only if it contains P_1^i . If it holds for all i , then ℓ_1 and ℓ_2 have the same distance sequence with respect to $\bar{\mathcal{S}}$. We can handle this problem by extending $\bar{\mathcal{S}}$ with all the $\frac{q^{d-1}-1}{q-1}$ points of a hyperplane in Σ_{d-1} . Then ℓ_2 contains at least one of these points and ℓ_1 does not contain any of them. Hence the two lines are divided.

The size of the constructed resolving set is $(q + 1)k - qm + \frac{q^{d-1}-1}{q-1}$, the statement is proved. □

Theorem 2.11. *If $n \geq 4$ is even, then there exists a semi-resolving set for points in $\Gamma_{n,q}$ of size*

$$r_{\mathcal{P}}(n, q) = 2q^{n-1} + 2q^{n-2} + 2(q^{n-4} + q^{n-6} + \dots + q^2). \tag{2.2}$$

Proof. We apply Proposition 2.10 for $d = n - 1$. Let \mathcal{S} be the semi-resolving set for points in $\Gamma_{n-1,q}$ which was constructed in Theorem 2.7. Its size is

$$k = 2 \frac{q^n - q^2}{q^2 - 1}.$$

By Proposition 2.8, we can choose the hyperplane Σ_{n-2} so that it contains

$$m = 2 \left(\frac{q^{n-2} - 1}{q^2 - 1} - 1 \right) = 2 \frac{q^{n-2} - q^2}{q^2 - 1}$$

elements of \mathcal{S} . Thus we get from Proposition 2.10 that there exists a semi-resolving set for points in $\Gamma_{n,q}$ of size

$$\begin{aligned} r_{\mathcal{P}}(n, q) &= 2(q + 1) \frac{q^n - q^2}{q^2 - 1} - 2q \frac{q^{n-2} - q^2}{q^2 - 1} \\ &= 2q^{n-1} + 2q^{n-2} + 2(q^{n-4} + q^{n-6} + \dots + q^2). \end{aligned} \tag{□}$$

Now we turn to semi-resolving sets for lines. Let us start with a simple, but very useful observation.

Lemma 2.12. *Let Σ be a hyperplane in $\text{PG}(n, q)$, \mathcal{S} be a semi-resolving set for points in Σ and ℓ_1 and ℓ_2 be two distinct lines in $\text{PG}(n, q)$. Suppose that none of the lines is contained in Σ and the points $P_1 = \Sigma \cap \ell_1$ and $P_2 = \Sigma \cap \ell_2$ are distinct. Then the lines ℓ_1 and ℓ_2 are divided by \mathcal{S} in $\Gamma_{n,q}$.*

Proof. Since \mathcal{S} is a semi-resolving set for points in Σ , there exists at least one element $s \in \mathcal{S}$ so that $d(P_1^1, s) \neq d(P_2^1, s)$. Hence

$$d(\ell_1, s) = d(P_1^1, s) + 1 \neq d(P_2^1, s) + 1 = d(\ell_2, s),$$

the statement follows. □

Theorem 2.13. *For all $n > 3$ and $q \geq 2n - 1$ there exists a semi-resolving set for lines in $\Gamma_{n,q}$ of size $r_{\mathcal{L}}(n, q) = 2nr_{\mathcal{P}}(n - 1, q)$, where $r_{\mathcal{P}}$ follows (2.1) or (2.2) depending on the parity of $n - 1$.*

Proof. Let $\mathcal{H} = \{\Sigma_1, \Sigma_2, \dots, \Sigma_{2n}\}$ be a subset of $2n$ hyperplanes of the $(q + 1)$ -element set formed by the dual hyperplanes of points on a normal rational curve. Then these hyperplanes are in general position, no $n + 1$ of them have a point in common. Let \mathcal{S}^i be a semi-resolving set for points in Σ_i . We claim that $\mathcal{S} = \bigcup_{i=1}^{2n} \mathcal{S}^i$ is a semi-resolving set for lines in $\Gamma_{n,q}$.

Let ℓ_1 and ℓ_2 be two distinct lines in $\text{PG}(n, q)$. We may assume that ℓ_j is contained in the intersection of m_j elements of \mathcal{H} for $j = 1, 2$, and $m_1 \geq m_2$. The elements of \mathcal{H} are in general position, so $n - 1 \geq m_j$, hence $2n - m_1 - m_2 \geq 2$. We may assume without loss of generality that ℓ_j intersects Σ_i in a single point, denoted by P_j^i , for $i = 1, 2, \dots, 2n - m_1 - m_2$ and $j = 1, 2$. It could happen, that $P_1^{i_1} = P_1^{i_2} = \dots = P_1^{i_k}$ for some indices, but $k \leq n - m_2$, otherwise the point would be a common point of at least $m_1 + (n - m_2 + 1) > n$ elements of \mathcal{H} . So we may assume that $P_1^1 \neq P_1^2$. As ℓ_2 contains at most one point of ℓ_1 , we may also assume that P_1^1 is not on ℓ_2 . Then, by Lemma 2.12, ℓ_1 and ℓ_2 are divided by \mathcal{S}^1 .

By Theorems 2.7 and 2.11, the size of \mathcal{S} is at most $2nr_{\mathcal{P}}(n - 1, q)$ for $n > 3$, thus the theorem is proved. □

The union of a semi-resolving set for points and a semi-resolving set for lines is a resolving set. Thus Theorems 2.7, 2.11 and 2.13 give our first general upper bound.

Corollary 2.14. *For all $n > 3$ and $q \geq 2n - 1$ there exists a resolving set in $\Gamma_{n,q}$ of size*

$$r(n, q) = 2q^{n-1} + (4n + 1 + (-1)^n)q^{n-2} + g_n(q),$$

where g_n is a polynomial of degree $n - 3$ whose coefficients depend only on n .

In this bound the coefficient of the second highest degree term depends on the dimension. In the next part, by a more sophisticated construction, we prove an upper bound in which the coefficient of the second highest degree term is a constant.

Proposition 2.15. *Let $q = p^h$, p prime. Suppose that there exists a resolving set \mathcal{S}_3 in $\Gamma_{3,q}$ of size $2q^2 + aq + g_3(p)$, where $a \in \mathbb{R}$, g_3 is a polynomial of degree $s \leq h - 1$, and \mathcal{S}_3 contains the $2q^2 + 2$ elements of two disjoint line spreads. Then there exists a resolving set \mathcal{S}_{4m+3} in $\Gamma_{4m+3,q}$ of size*

$$2q^{n-1} + aq^{n-2} + g_{4m+3}(p)$$

where g_{4m+3} is a polynomial of degree at most $(n - 3)h + s$.

Proof. As $(4m + 3) + 1$ is divisible by $3 + 1$, there exists a 3-spread in $\text{PG}(4m + 3, q)$. This 3-spread contains $t = \frac{q^{4(m+1)} - 1}{q^4 - 1}$ elements, say $\Sigma_3^1, \Sigma_3^2, \dots, \Sigma_3^t$, each of them is isomorphic to $\text{PG}(3, q)$. By the assumption of the theorem, in each Σ_3^i there exists a resolving set \mathcal{S}_3^i of size $2q^2 + aq + g_3(p)$. We claim that

$$\mathcal{S} = \bigcup_{i=1}^t \mathcal{S}_3^i$$

is a resolving set in $\Gamma_{4m+3,q}$. The elements of any pair of points and any point-line pair are obviously divided by \mathcal{S} . Let ℓ_1 and ℓ_2 be two lines. If at least one of them is contained in a Σ_3^i , then they are divided by \mathcal{S}_3^i . If none of them is contained in any Σ_3^i , then we may assume without loss of generality that $\ell_1 \cap \Sigma_3^1$ is a point P which is not on ℓ_2 . Let s_1 and s_2 be the two elements of the disjoint line spreads in \mathcal{S}_3^1 which are incident with P . Then $d(\ell_1, s_1) = d(\ell_1, s_2) = 2$. As ℓ_2 is not contained in \mathcal{S}_3^1 , it cannot intersect both s_1 and s_2 . Hence at least one of the distances $d(\ell_2, s_1)$ and $d(\ell_2, s_2)$ is 4. Thus ℓ_1 and ℓ_2 are divided by $\mathcal{S}_3^1 \subset \mathcal{S}$.

The size of \mathcal{S} is

$$(2q^2 + aq + g_3(p)) \frac{q^{n+1} - 1}{q^4 - 1} = 2q^{n-1} + aq^{n-2} + g_{4m+3}(p),$$

where the degree of g_{4m+3} is $(n - 3)h + \deg g_{4m+3} = (n - 3)h + s \leq (n - 2)h - 1$, so we are done. □

Theorem 2.16. *Let $q = p^h$, p prime. Suppose that there exists a resolving set in $\Gamma_{3,q}$ of size $2q^2 + aq + g_3^3(p)$ where g_3^3 is a polynomial of degree $s \leq h - 1$. Then for $n \geq 3$ there exists a resolving set in $\Gamma_{n,q}$ of size*

$$r(n, q) = \begin{cases} 2q^{n-1} + (a + 2)q^{n-2} + g_{n,0}(p), & \text{if } n \equiv 0 \pmod{4}, \\ 2q^{n-1} + (a + 2)q^{n-2} + g_{n,1}(p), & \text{if } n \equiv 1 \pmod{4}, \\ 2q^{n-1} + (a + 4)q^{n-2} + g_{n,2}(p), & \text{if } n \equiv 2 \pmod{4}, \\ 2q^{n-1} + aq^{n-2} + g_{n,3}(p), & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

where $g_{n,i}$ ($i = 0, 1, 2, 3$) is a polynomial of degree $(n - 3)h + s$ whose coefficients depend only on n .

Proof. We prove it by induction on the dimension modulo 4. For $n \equiv 3 \pmod{4}$ the statement follows from Proposition 2.15.

If $n \equiv 0 \pmod{4}$, then we apply Proposition 2.10 for $d = n - 1$ with $k = r_{\mathcal{P}}(n - 1, q)$ and $m = 0$. Therefore by the induction hypothesis

$$\begin{aligned} r_{\mathcal{P}}(n, q) &\leq (q + 1)r_{\mathcal{P}}(n - 1, q) + \frac{q^{n-2} - 1}{q - 1} \\ &= 2q^{n-1} + (a + 2)q^{n-2} + (q + 1)g_{n-1,3}(p) + (a + 1)q^{n-3} + q^{n-4} + \dots + 1. \end{aligned}$$

Thus $g_{n,0}(p) = (q + 1)g_{n-1,3}(p) + (a + 1)q^{n-3} + q^{n-4} + \dots + 1$, hence its degree is $(n - 3)h + s \leq (n - 2)h - 1$.

If $n \equiv 1 \pmod{4}$, then $n - 2 \equiv 3 \pmod{4}$, hence we can apply Proposition 2.10 for $d = n - 1$ so that Σ_{d-1} contains a resolving set constructed in Proposition 2.15. Then $k = 2q^{n-2} + (a + 2)q^{n-3} + g_{n-1,0}(p)$ and $m = 2q^{n-3} + aq^{n-4} + g_{n-2,3}(p)$. Hence

$$r_{\mathcal{P}}(n, q) = (q + 1)k - qm + \frac{q^{n-2} - 1}{q - 1} = 2q^{n-1} + (a + 2)q^{n-2} + g_{n,1}(p),$$

where

$$g_{n,1}(p) = (q + 1)g_{n-1,0}(p) - qg_{n-2,3}(p) + 3q^{n-3} + q^{n-4} + \dots + 1,$$

so its degree is $(n - 3)h + s \leq (n - 2)h - 1$.

Finally, if $n \equiv 2 \pmod{4}$, then $n - 3 \equiv 3 \pmod{4}$. Hence we cannot do better than apply Proposition 2.10 for $d = n - 1$ so that Σ_{d-1} contains entirely only elements of a $(d - 2)$ -dimensional resolving set constructed in Proposition 2.15. Now $k = 2q^{n-2} + (a + 2)q^{n-3} + g_{n-1,1}(p)$ and $m = 2q^{n-4} + aq^{n-5} + g_{n-3,3}(p)$. This gives

$$r_{\mathcal{P}}(n, q) = (q + 1)k - qm + \frac{q^{n-2} - 1}{q - 1} = 2q^{n-1} + (a + 4)q^{n-2} + g_{n,2}(p),$$

where

$$g_{n,2}(p) = (q + 1)g_{n-1,1}(p) - qg_{n-3,3}(p) + (a + 1)q^{n-3} + q^{n-4} + \dots + 1,$$

thus its degree is $(n - 3)h + s \leq (n - 2)h - 1$ again. The theorem is proved. \square

Let us remark that the polynomials $g_{n,i}$ can be determined exactly. We omit the long, but straightforward calculations, because their coefficients do not play any role in the rest of the paper.

In the next part of the section semi-resolving sets for lines in $\Gamma_{n,q}$ are investigated. In their constructions double blocking sets and their duals play an important role. For the relevant definitions and estimates on their sizes we refer to the paper of Ball and Blokhuis [3].

Theorem 2.17. *For all $q > 3$ there exists a semi-resolving set for lines in $\Gamma_{3,q}$ of size*

$$r_{\mathcal{L}}(3, q) = \min\{12q - 22, 4\tau_2(q) - 10\},$$

where $\tau_2(q)$ denotes the size of the smallest minimal double blocking set in $\text{PG}(2, q)$.

Proof. First, we construct two sets of lines in $\text{PG}(2, q)$ which are semi-resolving sets for points.

1. Let E_1, E_2 , and E_3 be the vertices of a triangle, ℓ_i denote the line E_jE_k and \mathcal{P}_i be the pencil of lines with carrier E_i . Let

$$\mathcal{S} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \setminus \{\ell_1, \ell_2, \ell_3, \ell\},$$

where $\ell \in \mathcal{P}_1, \ell_2 \neq \ell \neq \ell_3$. Then \mathcal{S} is a semi-resolving set for points in $\Gamma_{2,q}$, because $U = \ell \cap \ell_1$ is a unique uncovered point, every point in the set $\ell_1 \cup \ell_2 \cup \ell_3 \setminus \{E_1, E_2, E_3, U\}$ is 1-covered and all other points are at least 2-covered, hence resolved. The size of \mathcal{S} is $3q - 4$.

2. Let \mathcal{D} be a dual double blocking set in $\text{PG}(2, q)$. Then, by definition, each point is incident with at least two lines of \mathcal{D} . Thus if we delete an arbitrary line ℓ from \mathcal{D} , then the set of lines $\mathcal{D} \setminus \{\ell\}$ is still a semi-resolving set for points and, by the Principle of Duality, its size is at most $\tau_2(q) - 1$.

Hence, for all $q > 3$ there is a set of lines in $\text{PG}(2, q)$ of size $\min\{3q - 4, \tau_2(q) - 1\}$ which is a semi-resolving set for points. Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3,$ and \mathcal{H}_4 be the faces of a tetrahedron \mathcal{K} in $\text{PG}(3, q)$. Let \mathcal{T}^i be a semi-resolving set for points in \mathcal{H}_i which consists of lines only. We can choose \mathcal{T}^i so that each edge of \mathcal{K} belongs to both corresponding semi-resolving sets, because the full collineation group of $\text{PG}(2, q)$ acts transitively on triangles. We claim that $\mathcal{S} = \cup_{i=1}^4 \mathcal{T}^i$ is a semi-resolving set for lines in $\Gamma_{3,q}$.

The edges of \mathcal{K} belong to \mathcal{S} , thus they are resolved by definition. Let ℓ_1 and ℓ_2 be lines such that none of them is an edge of \mathcal{K} . Then each of them is contained in at most one face of \mathcal{K} , so we may assume without loss of generality that ℓ_1 intersects \mathcal{H}_i in a single point, denoted by P_1^i , for $i = 1, 2, 3$. We distinguish two main cases.

1. If $P_1^1 = P_1^2 = P_1^3$, then this point is a vertex K of \mathcal{K} .
 - If ℓ_2 also contains K , then $\mathcal{H}_4 \cap \ell_1 \neq \mathcal{H}_4 \cap \ell_2$, hence, by Lemma 2.12, the two lines are divided by \mathcal{T}^4 .
 - If ℓ_2 does not contain K , then we may assume that $\mathcal{H}_2 \cap \ell_2$ is a single point P_2^2 . Since $P_2^2 \neq K$, by Lemma 2.12, the two lines are divided by \mathcal{T}^2 .
2. If none of ℓ_1 and ℓ_2 contains any vertex, then we may assume that $P_1^1 \neq P_1^2$.
 - If ℓ_2 is not contained in neither \mathcal{H}_1 nor \mathcal{H}_2 , then it intersects \mathcal{H}_i in a single point, denoted by P_2^i , for $i = 1, 2$. Since $\ell_1 \cap \ell_2$ contains at most one point, we may assume that $P_1^1 \neq P_2^1$. Then, by Lemma 2.12, the two lines are divided by \mathcal{T}^1 .
 - Finally, if ℓ_2 is contained in one of \mathcal{H}_1 and \mathcal{H}_2 , then we may assume that $\ell_2 \subset \mathcal{H}_1$ and $\ell_2 \cap \mathcal{H}_2$ in a single point P_2^2 . Then P_2^2 is in \mathcal{H}_1 , so $P_2^2 \neq P_1^1$, because otherwise $\ell_1 \subset \mathcal{H}_1$. Hence, by Lemma 2.12, the two lines are divided by \mathcal{T}^2 .

Since \mathcal{S} has the required size, we are done. □

Remark 2.18. Let us remark that if the double blocking set \mathcal{D} in the proof of Theorem 2.17 is the disjoint union of two dual blocking sets, then not only one, but two lines can be deleted without violating the semi-resolving set property. We will consider this case in Section 3, Theorem 3.1.

Unfortunately, the exact value of $\tau_2(q)$ is not known in general. It is known that $\tau_2(q) = 2q + 2\sqrt{q} + 2$ for q is a square and $q > 16$ [3, Theorem 3.1], and for some small values of q . In the latter case for the known values $\tau_2(q) > 3q - 3$ always holds. Combining the semi-resolving set for points constructed in Corollary 2.7 and the semi-resolving set for lines of Theorem 2.17, we get the following upper bound on $\mu(\Gamma_{3,q})$.

Theorem 2.19. *The metric dimension of $\Gamma_{3,q}$ satisfies the inequality*

$$\mu(\Gamma_{3,q}) \leq 2q^2 + 12q - 24$$

for all $q > 3$.

Proof. For $q > 3$ let $\mathcal{S}_{\mathcal{L}}$ be a semi-resolving set for lines of size $12q - 22$ constructed in Theorem 2.17. Let \mathcal{L}_1 be a regular line spread of $\text{PG}(3, q)$ which contains two skew (non-intersecting) elements of $\mathcal{S}_{\mathcal{L}}$. Such spread exists, because the collineation group of $\text{PG}(3, q)$ acts transitively on the pairs of skew lines. Create a semi-resolving set for points $\mathcal{S}_{\mathcal{P}}$ which contains \mathcal{L}_1 as we did it in Corollary 2.7. Then $\mathcal{S} = \mathcal{S}_{\mathcal{L}} \cup \mathcal{S}_{\mathcal{P}}$ is a resolving set in $\Gamma_{3,q}$ and its size is $2q^2 + 12q - 24$. This proves the inequality. \square

By combining Theorem 2.16, with $s = 0$, and Theorem 2.19, we get the following bounds.

Corollary 2.20. *Let $n \geq 3$ and $q > 3$. Then the metric dimension of $\Gamma_{n,q}$ satisfies the inequality*

$$\mu(\Gamma_{n,q}) \leq \begin{cases} 2q^{n-1} + 14q^{n-2} + h_{n,2}(q), & \text{if } n \equiv 0 \text{ or } n \equiv 1 \pmod{4}, \\ 2q^{n-1} + 16q^{n-2} + h_{n,3}(q), & \text{if } n \equiv 2 \pmod{4}, \\ 2q^{n-1} + 12q^{n-2} + h_{n,1}(q), & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

where $h_{n,i}$ ($i = 1, 2, 3$) is a polynomial of degree at most $n - 3$ whose coefficients depend only on n .

The metric dimension of $\Gamma_{2,q}$ for $q \geq 23$ was determined by Héger and Takáts [12]. For higher dimensions we do not know the exact value, but Theorems 2.1, 2.19, and Corollary 2.20 imply the following result.

Corollary 2.21. *For all $n > 2$ and $q > 3$*

$$|\mu(\Gamma_{n,q}) - 2q^{n-1}| = O(q^{n-2}).$$

This means that $\mu(\Gamma_{n,q})$ is asymptotically $2q^{n-1}$. The number of vertices in $\Gamma_{n,q}$ is $v = \frac{q^{n+1}-1}{q-1} + \frac{(q^{n+1}-1)(q^n-1)}{(q+1)(q-1)^2}$, so its metric dimension is roughly $2\sqrt{v}$. The automorphism group of $\text{PG}(n, q)$ is $\text{P}\Gamma\text{L}(n + 1, q)$ and it is well-known that its base size is $n + 1$ if q is a prime, and it is $n + 2$ if $q = p^h$ with $h > 1$. Hence the dimension jump of $\Gamma_{n,q}$ is roughly $2\sqrt{v}$.

3 Bounds for $q = p^h, h \geq 2$

In this section we consider the case $q = p^h, h > 1$. In the case h even, we will present a better bound on the size of a semi-resolving set for points in $\Gamma_{3,q}$ using small dual double blocking sets in $\text{PG}(2, q)$. When $h > 2$, then we will show that a particular type of spread of lines in $\text{PG}(3, q)$ can be used to resolve the lines. In fact, for a regular spread, there exist many pairs of lines of the spaces intersecting the same set of elements of the spread. We now investigate a different type of spread, called aregular, and we determine all the lines of the space intersecting the same set of elements of the spread; see Theorem 3.6. The main goal is to construct a set of lines of $\text{PG}(3, q)$ which resolves all the lines of the spaces; see Theorem 3.7.

Theorem 3.1. *If q is a square, then the metric dimension of $\Gamma_{3,q}$ satisfies the inequality*

$$\mu(\Gamma_{3,q}) \leq 2q^2 + 8q + 8\sqrt{q} - 8.$$

Proof. The union of the sets of lines of two disjoint Baer subplanes is a dual double blocking set in $\text{PG}(2, q)$ and its size is $2q + 2\sqrt{q} + 2$. This set is the disjoint union of two dual blocking sets. Hence, by a result of Héger and Takáts [12, Proposition 22], we can delete two of its lines so that the remaining set is still a semi-resolving set for points in $\text{PG}(2, q)$; see also Remark 2.18.

Thus we can construct a semi-resolving set for lines $\mathcal{S}_{\mathcal{L}}$ of size $8q + 8\sqrt{q} - 6$ by the method applied in the proof of Theorem 2.17. Finally, we can extend it to a resolving set of size $2q^2 + 8q + 8\sqrt{q} - 8$ in the same way as we did in the proof of Theorem 2.19. \square

By combining Theorem 2.16, with $s = 0$, and Theorem 2.19, we get the following bounds.

Corollary 3.2. *If q is a square and $n \geq 3$, then the metric dimension of $\Gamma_{n,q}$ satisfies the inequality*

$$\mu(\Gamma_{n,q}) \leq \begin{cases} 2q^{n-1} + 10q^{n-2} + h_{n,2}(q), & \text{if } n \equiv 0 \text{ or } 1 \pmod{4}, \\ 2q^{n-1} + 12q^{n-2} + h_{n,3}(q), & \text{if } n \equiv 2 \pmod{4}, \\ 2q^{n-1} + 8q^{n-2} + h_{n,1}(q), & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

where $h_{n,i}$ ($i = 1, 2, 3$) is a polynomial of degree at most $n - 3$ whose coefficients depend only on n .

Theorem 3.3 ([13, Theorem 17.3.3]). *Let $q = p^h$, $h > 1$, and choose $b, c \in \mathbb{F}_q^*$ such that the polynomial $t^{p+1} - tb + c$ has no roots in \mathbb{F}_q . Let*

$$\mathcal{A}_{b,c} = \{t_{\alpha,\beta} : \alpha, \beta \in \mathbb{F}_q\} \cup \{Z = T = 0\},$$

where $t_{\alpha,\beta}$ is the line through the points $(\alpha : \beta : 1 : 0)$ and $(c\beta^p : \alpha^p + b\beta^p : 0 : 1)$. Then $\mathcal{A}_{b,c}$ is a spread, called the aregular spread.

In what follows, we will associate to each line r of the space an algebraic curve $\mathcal{C}_r : F_r(X, Y, T) = 0$ such that $t_{\alpha,\beta}$ intersects r if and only if $F_r(\alpha, \beta, 1) = 0$. We distinguish four types of lines.

1. Lines $r_{x,y,\ell,m}$ through the points $(x : y : 0 : 1)$ and $(\ell : m : 1 : 0)$. Note that if $x = cm^p$ and $y = \ell^p + bm^p$ then $r_{x,y,\ell,m}$ coincides with $t_{\ell,m} \in \mathcal{A}_{b,c}$. From now on we consider $(x, y) \neq (cm^p, \ell^p + bm^p)$. A line $t_{\alpha,\beta}$ intersects $r = r_{x,y,\ell,m}$ if and only if for some $\lambda \in \mathbb{F}_q$ the points

$$(x + \lambda\ell : y + \lambda m : \lambda : 1), \quad (\alpha : \beta : 1 : 0), \quad (c\beta^p : \alpha^p + b\beta^p : 0 : 1)$$

are collinear, that is

$$x + \lambda\ell - c\beta^p = \lambda\alpha, \quad y + \lambda m - (\alpha^p + b\beta^p) = \lambda\beta.$$

This implies

$$\lambda(\ell - \alpha) = c\beta^p - x, \quad \lambda(m - \beta) = \alpha^p + b\beta^p - y,$$

and therefore

$$(c\beta^p - x)(m - \beta) = (\alpha^p + b\beta^p - y)(\ell - \alpha).$$

In this case, $F_r(X, Y, T) = (cY^p - xT^p)(mT - Y) - (X^p + bY^p - yT^p)(\ell T - X)$.

2. Lines $s = s_{x,y,z,\ell}$ through the points $(x : y : z : 1)$ and $(\ell : 1 : 0 : 0)$. A line $t_{\alpha,\beta}$ intersects $s_{x,y,z,\ell}$ if and only if for some $\lambda \in \mathbb{F}_q$ the points

$$(x + \lambda\ell : y + \lambda : z : 1), \quad (\alpha : \beta : 1 : 0), \quad (c\beta^p : \alpha^p + b\beta^p : 0 : 1)$$

are collinear, that is

$$x + \lambda\ell - c\beta^p = z\alpha, \quad y + \lambda - (\alpha^p + b\beta^p) = z\beta.$$

This implies

$$\lambda\ell = -x + c\beta^p + z\alpha, \quad \lambda = -y + z\beta + \alpha^p + b\beta^p$$

and therefore

$$-x + c\beta^p + z\alpha = \ell(-y + z\beta + \alpha^p + b\beta^p).$$

So, $F_s(X, Y, T) = -\ell X^p + (c - \ell b)Y^p + zXT^{p-1} - \ell zYT^{p-1} + (\ell y - x)T^p$.

3. Lines $u = u_{x,y,z}$ through the points $(x : y : z : 1)$ and $(1 : 0 : 0 : 0)$. In this case, $F_u(X, Y, T) = X^p + bY^p + zYT^{p-1} - yT^p$.
4. Lines $v = v_{x,y,z}$ contained in the planes $T = 0$ and $xX + yY + zZ = 0$. Then, $F_v(X, Y, T) = xX + yY + zT$.

Such a curve is absolutely irreducible if $z \neq 0$, otherwise it collapses into a single line.

Proposition 3.4. *Consider the curves $C_r, C_s, C_u,$ and C_v . Then*

1. C_r is absolutely irreducible;
2. C_s is either absolutely irreducible or a line repeated p times;
3. C_u is either absolutely irreducible or a line repeated p times.

Proof.

1. Now we prove that C_r is absolutely irreducible. Let $\varphi(X, Y, T) = (X + x_0T, Y + y_0T, T)$ with $x_0^p = y - bx/c, y_0^p = x/c$. Then

$$\begin{aligned} F_r(\varphi(X, Y, T)) &= (c(Y + y_0T)^p - xT^p)(mT - Y - y_0T) \\ &\quad - ((X + x_0T)^p + b(Y + y_0T)^p - yT^p)(\ell T - X - x_0T) \\ &= (cY^p + cy_0^pT^p - xT^p)(mT - Y - y_0T) \\ &\quad - (X^p + x_0^pT^p + bY^p + by_0^pT^p - yT^p)(\ell T - X - x_0T) \\ &= (cY^p + xT^p - xT^p)(mT - Y - y_0T) \\ &\quad - (X^p + yT^p - bx/cT^p + bY^p + bx/cT^p - yT^p)(\ell T - X - x_0T) \\ &= cY^p(mT - Y - y_0T) - (X^p + bY^p)(\ell T - X - x_0T) \\ &= G_r(X, Y, T). \end{aligned}$$

Finally

$$G_r(X, 1, Y) = c(mY - 1 - y_0Y) - (X^p + b)(\ell Y - X - x_0Y),$$

that is the curve \mathcal{C}_r is \mathbb{F}_q -isomorphic to

$$\mathcal{C}' : Y = \frac{c - X(X^p + b)}{cm - cy_0 + (x_0 - \ell)(X^p + b)},$$

which is an irreducible rational curve with $q + 1$ \mathbb{F}_q -rational points (note that $(x, y) \neq (cm^p, \ell^p + bm^p)$ yields $m \neq y_0$ or $\ell \neq x_0$). This means that the curve \mathcal{C}_r is absolutely irreducible.

2. First, note that the homogeneous term $-\ell X^p + (c - \ell b)Y^p$ cannot vanish otherwise $c = 0$, a contradiction.

- If $(\ell, z) = (0, 0)$, \mathcal{C}_s is a line of type $b_0Y + c_0T = 0$ repeated p times.
- If $\ell = 0$ and $z \neq 0$, then $F_s(X, Y, T)$ reads $cY^p + zXT^{p-1} - xT^p$ and \mathcal{C}_s is absolutely irreducible.
- If $\ell \neq 0$ and $z = 0$ then \mathcal{C}_s is a (repeated) line $a_0X + b_0Y + c_0T = 0$, where $a_0^p = -\ell$, $b_0^p = (c - \ell b)$, $c_0^p = (\ell y - x)$.
- If $\ell \neq 0$ and $z \neq 0$ then consider $\varphi(X, Y, T) = (X + \sqrt[p]{(c - \ell b)/\ell} Y, Y, T)$ and so

$$G_s(X, Y, T) = F_s(\varphi(X, Y, T)) = -\ell X^p + zXT^{p-1} + z(\sqrt[p]{(c - \ell b)/\ell} - \ell)YT^{p-1} + (\ell y - x)T^p.$$

By our assumption of b, c , there is no $\ell \in \mathbb{F}_q$ such that $\sqrt[p]{(c - \ell b)/\ell} - \ell = 0$. The curve $G_s(X, Y, T) = 0$ is rational and irreducible and it is \mathbb{F}_q -isomorphic to \mathcal{C}_s .

3. Clear. □

Proposition 3.5. *Let $q = p^h$, $h > 2$. Two lines of the same type (r, s, u, v) do not intersect the same set of lines of the regular spread $\mathcal{A}_{b,c}$.*

Proof. The assumption $h > 2$ implies $q + 1 > (p + 1)^2$. The curves $\mathcal{C}_r, \mathcal{C}_s, \mathcal{C}_u, \mathcal{C}_v$ have degree at most p and they have $q + 1$ \mathbb{F}_q -rational points (corresponding to the lines of the spread intersecting them). By Proposition 3.4, such curves are either absolutely irreducible or they consist of a repeated line. Thus, if two curves attached to the lines w_1 and w_2 of the same type share $q + 1$ \mathbb{F}_q -rational points, the corresponding polynomials must be proportional. By direct computations, this yields $w_1 = w_2$. □

Theorem 3.6. *Let $q = p^h$, $h > 2$. If two lines in $\text{PG}(3, q)$ intersect the same set of lines of the spread $\mathcal{A}_{b,c}$ then one of them lies on the plane $Z = 0$ and the other on the plane $T = 0$.*

Proof. The reduced (absolutely irreducible) curves associated with the different types of lines (r, s, u, v) have degree $p + 1$, degree p or 1, degree p or 1, and degree 1, respectively. They can share $q + 1$ \mathbb{F}_q -rational points only in the following cases:

- both \mathcal{C}_s and \mathcal{C}_u have degree p ;
- both \mathcal{C}_s and \mathcal{C}_u have degree 1;
- both \mathcal{C}_s and \mathcal{C}_v have degree 1;

- both \mathcal{C}_u and \mathcal{C}_v have degree 1.

The first case is not possible. The second case would imply $c = 0$, a contradiction. Recall that the lines $v = v_{x,y,z}$ are contained in the plane $T = 0$ of $\text{PG}(3, q)$. The claim follows observing that if \mathcal{C}_s or \mathcal{C}_u have degree 1, then $s = s_{x,y,0,\ell}$ or $u = u_{x,y,0}$. So, both s and u are contained in the plane $Z = 0$. \square

Theorem 3.7. *Let $q = p^h$, $h > 2$. Then there exists a set of $q^2 + 3$ lines resolving all the lines of $\text{PG}(3, q)$.*

Proof. Consider the aregular spread $\mathcal{A}_{b,c}$ with $b, c \in \mathbb{F}_q^*$ and such that the polynomial $t^{p+1} - tb + c$ has no roots in \mathbb{F}_q . We already know by Theorem 3.6 that lines of $\text{PG}(3, q)$ intersecting the same set of elements of $\mathcal{A}_{b,c}$ are contained in the planes $Z = 0$ or $T = 0$. Note that two lines in a fixed plane cannot intersect the same elements of $\mathcal{A}_{b,c}$. Consider two distinct extra lines w_1 and w_2 contained in $Z = 0$ intersecting the line $Z = T = 0$ at two distinct points. It is readily seen that $\mathcal{A}_{b,c} \cup \{w_1, w_2\}$ resolves all the lines of $\text{PG}(3, q)$. \square

Corollary 3.8. *If $q = p^h$, $h > 2$, then there exists a resolving set in $\Gamma_{3,q}$ of size $2q^2 + 2$.*

Proof. Consider the set of $q^2 + 3$ lines from Theorem 3.7 and use the argument of Theorem 2.9. One of the two extra lines could be an element of the other spread. Finally, delete one line from the modified regular spread. \square

Finally, the following bounds are obtained combining again Theorem 2.16, with $s = 0$, and Theorem 2.19.

Corollary 3.9. *If $q = p^h$, $h > 2$, then the metric dimension of $\Gamma_{n,q}$ satisfies the inequality*

$$\mu(\Gamma_{n,q}) \leq \begin{cases} 2q^{n-1} + 2q^{n-2} + h_{n,2}(q), & \text{if } n \equiv 0 \text{ or } 1 \pmod{4}, \\ 2q^{n-1} + 4q^{n-2} + h_{n,3}(q), & \text{if } n \equiv 2 \pmod{4}, \\ 2q^{n-1} + h_{n,1}(q), & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

where $h_{n,i}$ ($i = 1, 2, 3$) is a polynomial of degree at most $n - 3$ whose coefficients depend only on n .

4 Resolving sets for small q

We performed a computer search to obtain sets of lines that are semi-resolving sets for lines in $\text{PG}(3, q)$ for small q . We used MAGMA, a computer algebra system for symbolic computation developed at the University of Sydney; see [6]. We started classifying all set of lines of a certain size k . Then we extended the non-equivalent sets of size k using a backtracking algorithm.

In $\text{PG}(3, 2)$ there are 35 lines, so a semi-resolving set for lines must contain at least six elements. We found that there are 165 non-equivalent sets of lines of size six. Forty-eight of them are semi-resolving sets for lines in $\text{PG}(3, 2)$. An example is the following set of six lines:

$$\begin{aligned} & \{ \langle (1 : 0 : 0 : 0), (0 : 1 : 0 : 0) \rangle, \langle (0 : 1 : 0 : 0), (0 : 0 : 0 : 1) \rangle, \\ & \langle (0 : 0 : 1 : 0), (0 : 0 : 0 : 1) \rangle, \langle (0 : 0 : 0 : 1), (1 : 1 : 0 : 0) \rangle, \\ & \langle (0 : 1 : 0 : 0), (1 : 1 : 1 : 1) \rangle, \langle (1 : 1 : 0 : 0), (0 : 0 : 1 : 1) \rangle \}. \end{aligned}$$

In $\text{PG}(3, 3)$ there are 130 lines, so a semi-resolving set for lines must contain at least eight elements. We found that there are 10681 non-equivalent sets of lines of size seven. An exhaustive search by backtracking has proved that no set of lines of size eight or nine is a semi-resolving set for lines in $\text{PG}(3, 3)$. There exist semi-resolving sets for lines of size ten. An example is the following set of ten lines:

$$\begin{aligned} & \{ \langle (1 : 0 : 0 : 0), (0 : 1 : 0 : 0) \rangle, \langle (1 : 0 : 0 : 0), (0 : 0 : 0 : 1) \rangle, \\ & \langle (0 : 1 : 0 : 0), (0 : 0 : 1 : 0) \rangle, \langle (0 : 1 : 0 : 0), (1 : 0 : 0 : 1) \rangle, \\ & \langle (1 : 0 : 1 : 2), (0 : 1 : 1 : 0) \rangle, \langle (1 : 0 : 0 : 2), (0 : 0 : 1 : 1) \rangle, \\ & \langle (1 : 0 : 0 : 0), (0 : 1 : 1 : 2) \rangle, \langle (1 : 0 : 0 : 0), (0 : 1 : 1 : 0) \rangle, \\ & \langle (0 : 1 : 1 : 1), (1 : 2 : 0 : 0) \rangle, \langle (1 : 1 : 1 : 0), (0 : 1 : 2 : 0) \rangle \}. \end{aligned}$$

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