On generalized truncations of complete graphs

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Abstract

For a $k$-regular graph $\Gamma$ and a graph $\Upsilon$ of order $k$, a generalized truncation of $\Gamma$ by $\Upsilon$ is constructed by replacing each vertex of $\Gamma$ with a copy of $\Upsilon$. In [Symmetry properties of generalized graph truncations, J. Comb. Theory B 137 (2019) 291–315], E. Eiben, R. Jajcay and P. Šparl introduced a method for constructing vertex-transitive generalized truncations. For convenience, we call a graph obtained by using Eiben et al.’s method a special generalized truncation. In the above mentioned paper, the authors proposed a problem to classify special generalized truncations of a complete graph $K_n$ by a cycle of length $n - 1$. In this paper, we completely solve this problem by demonstrating that with the exception of $n = 6$, every special generalized truncation of a complete graph $K_n$ by a cycle of length $n - 1$ is a Cayley graph of $AGL(1,n)$ where $n$ is a prime power. Moreover, the full automorphism groups of all these graphs and the isomorphisms among them are determined.

Keywords: Truncation, vertex-transitive, Cayley graph, automorphism group.


1 Introduction

In [6], the symmetry properties of graphs constructed by using the generalized truncations was investigated. In particular, a method for constructing vertex-transitive generalized truncations was proposed (see [6, Construction 4.1 & Theorem 5.1]), and this method was used to construct vertex-transitive generalized truncations of a complete graph $K_n$ by a cycle of length $n - 1$ for some small values of $n$. The vertex-transitive generalized truncations of a complete graph $K_n$ by a graph $\Upsilon$ in context of [6, Theorem 5.1] can be defined as follows.

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Let $K_n$ be a complete graph of order $n$ with $n \geq 4$, and let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Let $G$ be an arc-transitive group of automorphisms of $K_n$. Then $G$ acts 2-transitively on $V(K_n)$. Let $v = v_1$, and let $O_v$ be a union of orbits of the stabilizer $G_v$ acting on $\{\{x, y\} | x \neq y, x, y \in V(K_n) \setminus \{v\}\}$. Let $\Upsilon$ be the graph with vertex set $\{V_2, V_3, \ldots, V_n\}$ and edge set $O_v$. For each $u \in V(K_n)$, let $V_u = \{(u, w) \mid w \in V(K_n) \setminus \{u\}\}$. The special generalized truncation of $K_n$ by $\Upsilon$, denoted by $T(K_n, G, \Upsilon)$, is the graph with the vertex set $\bigcup_{u \in V(K_n)} V_u$, and the adjacency relation in which a vertex $(u, w)$ is adjacent to the vertex $(w, u)$ and to all the vertices $(u, w')$ for which there exists a $g \in G$ with the property $u^g = v$ and $\{w, w'\}^g \in O_v$.

Based on the analysis of special generalized truncations of a complete graph $K_n$ by a cycle of length $n - 1$ for some small values of $n$, the authors of [6] proposed the following problem.

**Problem 1.1 ([6, Problem 5.4]).** Classify the special generalized truncations of $K_n$ ($n \geq 4$) by a cycle of length $n - 1$.

The main purpose of this paper is to give a solution of this problem. Before stating the main result of this paper, we first set some notation. For a positive integer $n$, we denote by $\mathbb{Z}_n$ the cyclic group of order $n$, and by $D_{2n}$ the dihedral group of order $2n$. Let $\mathbb{Z}_n^*$ be the multiplicative group of units mod $n$ ($\mathbb{Z}_n^*$ consists of all positive integers less than $n$ and coprime to $n$). Also we use $A_n$ and $S_n$ respectively to denote the alternating and symmetric groups of degree $n$. For two groups $M$ and $N$, $N \rtimes M$ denotes a semidirect product of $N$ by $M$. For a group $G$, the automorphism group of $G$ and the socle of $G$ will be represented by $\text{Aut}(G)$ and $\text{soc}(G)$, respectively. For a graph $\Gamma$ we denote by $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\text{Aut}(\Gamma)$ the vertex set, edge set, arc set and full automorphism group of $\Gamma$, respectively. A graph $\Gamma$ is said to be vertex-transitive (resp. arc-transitive (or symmetric)) if $\text{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$ (resp. $A(\Gamma)$). Cayley graphs form an important class of vertex-transitive graphs. Given a finite group $G$ and an inverse close subset $S \subseteq G \setminus \{1\}$, the Cayley graph $\text{Cay}(G, S)$ on $G$ with respect to $S$ is the graph with vertex set $G$ and edge set $\{(g, sg) \mid g \in G, s \in S\}$. Finally, we use $K_n$ and $C_n$ respectively to denote the complete graph and cycle with $n$ vertices.

Let $p$ be a prime and $e$ a positive integer. Let $\text{GF}(p^e)$ be the Galois field of order $p^e$ and let $x$ be a primitive root of $\text{GF}(p^e)$. Then

$$\text{AGL}(1, p^e) = \{\alpha_{x^i, z'} : z \mapsto zx^i + z', \forall z \in \text{GF}(p^e) \mid i \in \mathbb{Z}_{p^e-1}, z' \in \text{GF}(p^e)\},$$

and $\text{AGL}(1, p^e)$ is a 2-transitive permutation group on $\text{GF}(p^e)$. Let

$$H = \{\alpha_{1, z'} : z \mapsto z + z', \forall z \in \text{GF}(p^e) \mid z' \in \text{GF}(p^e)\},$$

$$K = \{\alpha_{x^i, 0} : z \mapsto zx^i, \forall z \in \text{GF}(p^e) \mid i \in \mathbb{Z}_{p^e-1}\}.$$  

Then $H$ is regular on $\text{GF}(p^e)$ and the point stabilizer $\text{AGL}(1, p^e)_0$ of the zero element $0$ of $\text{GF}(p^e)$ is $K$. So $\text{AGL}(1, p^e) = H \rtimes K$.

**Construction I.** Let $z'$ be a non-zero element of $\text{GF}(p^e)$. For each $i \in \mathbb{Z}_{p^e-1}$ with $i < \frac{p^e-1}{2}$, let

$$K^i_{p^e} = \text{Cay}(\text{AGL}(1, p^e), \{\alpha_{-1, z'}, \alpha_{x^i, 0}, \alpha_{x^{-i}, 0}\})(p > 2),$$

$$K^i_{2e} = \text{Cay}(\text{AGL}(1, 2^e), \{\alpha_{1, z'}, \alpha_{x^i, 0}, \alpha_{x^{-i}, 0}\})(p = 2).$$
Figure 1: The graph \(T(\mathbf{K}_6, A_5, C_5)\)

**Remark 1.** Let \(z', z''\) be two non-zero elements of GF\((p^e)\). There exists \(x^j \in \text{GF}(p^e) \setminus \{0\}\) such that \(z'^j = z''\). So

\[
\begin{align*}
\{z_{-1,z'}, z_{x^i,0}, z_{x^{-i},0}\}^{z_{x^j,0}} &= \{z_{-1,z''}, z_{x^i,0}, z_{x^{-i},0}\} (p > 2), \\
\{z_{1,z'}, z_{x^i,0}, z_{x^{-i},0}\}^{z_{x^j,0}} &= \{z_{1,z''}, z_{x^i,0}, z_{x^{-i},0}\} (p = 2).
\end{align*}
\]

It follows that

\[
\begin{align*}
\text{Cay}(\text{AGL}(1, p^e), \{z_{-1,z'}, z_{x^i,0}, z_{x^{-i},0}\}) &\cong \\
\text{Cay}(\text{AGL}(1, p^e), \{z_{1,z''}, z_{x^i,0}, z_{x^{-i},0}\}) (p > 2),
\end{align*}
\]

\[
\begin{align*}
\text{Cay}(\text{AGL}(1, 2^e), \{z_{-1,z'}, z_{x^i,0}, z_{x^{-i},0}\}) &\cong \\
\text{Cay}(\text{AGL}(1, 2^e), \{z_{1,z''}, z_{x^i,0}, z_{x^{-i},0}\}) (p = 2).
\end{align*}
\]

In view of this fact, up to graph isomorphism, \(K^{p^e}_{p^e}\) is independent of the choice of \(z'\).

The following is the main result of this paper.

**Theorem 1.2.** Let \(K\) be a special generalized truncation of \(K\) by \(C_{n-1}\). Then \(K\) is isomorphic to either \(T(\mathbf{K}_p, A_5, C_5)\) (see Fig. 1), or one of the graphs \(K^{p^e}_{p^e}(i \in \mathbb{Z}_{p^e-1}^\ast, i < \frac{p^e-1}{2})\). Conversely, each of the above graphs is indeed a special generalized truncation of \(K\) by a cycle of length \(n - 1\), where \(n = 6 \) or a prime power.

Furthermore, for any distinct \(i, i' \in \mathbb{Z}_{p^e-1}\) with \(i, i' < \frac{p^e-1}{2}\), \(K^{i, p^e}_{p^e} \cong K^{i', p^e}_{p^e}\) if and only if \(i' \equiv ip^j \mod p^e - 1\) for some \(1 \leq j \leq e\). Moreover, the following hold:

(i) \(\text{Aut}(T(\mathbf{K}_6, A_5, C_5)) \cong A_5\);  
(ii) \(\text{Aut}(K^1_{4}) \cong S_4\);  
(iii) \(\text{Aut}(K^1_{7}) \cong D_{42} \rtimes \mathbb{Z}_3\);  
(iv) \(\text{Aut}(K^3_{11}) \cong \text{PGL}_2(11)\);  
(v) \(\text{Aut}(K^7_{23}) \cong \text{PGL}_2(23)\);  
(vi) if \(K^{p^e}_{p^e}\) is not isomorphic to one of the graphs: \(K^1_{4}, K^1_{7}, K^3_{11} \) and \(K^7_{23}\), then \(\text{Aut}(K^{p^e}_{p^e}) \cong \text{AGL}(1, p^e)\).
2 Preliminaries

All groups considered in this paper are finite and all graphs are finite, connected, simple and undirected. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [3, 12].

Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph on a group $G$ relative to a subset $S$ of $G$. It is easy to prove that $\Gamma$ is connected if and only if $S$ is a generating subset of $G$. For any $g \in G$, $R(g)$ is the permutation of $G$ defined by $R(g) : x \mapsto xg$ for $x \in G$. Set $R(G) = \{ R(g) \mid g \in G \}$. It is well-known that $R(G)$ is a subgroup of $\text{Aut}(\Gamma)$. For briefness, we shall identify $R(G)$ with $G$ in the following. In 1981, Godsil [7] proved that the normalizer of $G$ in $\text{Aut}(\Gamma)$ is $G \rtimes \text{Aut}(G, S)$, where $\text{Aut}(G, S)$ is the group of automorphisms of $G$ fixing the set $S$ set-wise. Clearly, $\text{Aut}(G, S)$ is a subgroup of the stabilizer $\text{Aut}(\Gamma)_1$ of the identity $1$ of $G$ in $\text{Aut}(\Gamma)$. We say that the Cayley graph $\text{Cay}(G, S)$ is normal if $G$ is normal in $\text{Aut}(\text{Cay}(G, S))$ (see [13]). If $\Gamma = \text{Cay}(G, S)$ is a normal Cayley graph on $G$, then we have $\text{Aut}(G, S) = \text{Aut}(\Gamma)_1$, and if, in addition, $\Gamma$ is also arc-transitive, then $\text{Aut}(G, S)$ is transitive on $S$. From this we can easily obtain the following lemma.

**Lemma 2.1.** There does not exist an arc-transitive normal Cayley graph of odd valency at least three on a cyclic group.

A Cayley graph $\text{Cay}(G, S)$ on a group $G$ relative to a subset $S$ of $G$ is called a CI-graph of $G$, if for any Cayley graph $\text{Cay}(G, T)$, whenever $\text{Cay}(G, S) \cong \text{Cay}(G, T)$ we have $T = S^\alpha$ for some $\alpha \in \text{Aut}(G)$. The following proposition is a criterion for a Cayley graph to be a CI-graph.

**Proposition 2.2 ([1, Lemma 3.1]).** Let $\Gamma$ be a Cayley graph on a finite group $G$. Then $\Gamma$ is a CI-graph of $G$ if and only if all regular subgroups of $\text{Aut}(\Gamma)$ isomorphic to $G$ are conjugate.

Let $\Gamma$ be a connected vertex-transitive graph, and let $G \leq \text{Aut}(\Gamma)$ be vertex-transitive on $\Gamma$. For a $G$-invariant partition $B$ of $V(\Gamma)$, the quotient graph $\Gamma_B$ is defined as the graph with vertex set $B$ such that, for any two different vertices $B, C \in B$, $B$ is adjacent to $C$ if and only if there exist $u \in B$ and $v \in C$ which are adjacent in $\Gamma$. Let $N$ be a normal subgroup of $G$. Then the set $B$ of orbits of $N$ in $V(\Gamma)$ is a $G$-invariant partition of $V(\Gamma)$. In this case, the symbol $\Gamma_B$ will be replaced by $\Gamma_N$.

In view of [11, Theorem 9], we have the following proposition.

**Proposition 2.3.** Suppose that $\Gamma$ is a connected trivalent graph with an arc-transitive group $G$ of automorphisms. If $N \leq G$ has more than two orbits in $V(\Gamma)$, then $N$ is semiregular on $V(\Gamma)$, and $\Gamma_N$ is a trivalent symmetric graph with $G/N$ as an arc-transitive group of automorphisms.

3 Proof of Theorem 1.2

3.1 Special generalized truncations of $K_n$ by $C_{n-1}$

In this subsection, we shall prove the first part of Theorem 1.2 by determining all special generalized truncations of $K_n (n \geq 4)$ by $C_{n-1}$. Throughout this subsection, we shall use the following assumptions and notations.
Assumption I.

(1) Let $K_n$ be a complete graph of order $n$ with $n \geq 4$, and let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$.

(2) Let $G \leq \text{Aut}(K_n)$ be an arc-transitive group of automorphisms.

(3) Let $v = v_1$, and let $O_v$ be a union of orbits of the stabilizer $G_v$ acting on $\{\{x, y\} \mid x \neq y, x, y \in V(K_n) \setminus \{v\}\}$. Let $\Upsilon$ be the graph with vertex set $\{v_2, v_3, \ldots, v_n\}$ and edge set $O_v$.

(4) For each $u \in V(K_n)$, let $V_u = \{(u, w) \mid w \in V(K_n) \setminus \{u\}\}$.

(5) Let $\tilde{K}_n = T(K_n, G, \Upsilon)$ be the graph with the vertex set $\bigcup_{u \in V(K_n)} V_u$, and the adjacency relation in which a vertex $(u, w)$ is adjacent to the vertex $(w, u)$ and to all the vertices $(u, w')$ for which there exists a $g \in G$ with the property $w^g = v$ and $\{w, w'\} g \in O_v$.

In view of [6, Theorem 5.1], we have the following proposition.

Proposition 3.1. Use the notations in Assumption I. Then $\text{Aut}(\tilde{K}_n)$ has a vertex-transitive subgroup $\tilde{G}$ such that $\mathcal{P} = \{V_u \mid u \in V(K_n)\}$ is an imprimitivity block system for $\tilde{G}$. Furthermore, the following hold.

1. The quotient graph of $\tilde{K}_n$ relative to $\mathcal{P}$ is isomorphic to $K_n$.
2. $\tilde{G} \cong G$.
3. $\tilde{G}$ acts faithfully on $\mathcal{P}$.

For the two groups $\tilde{G}, G$ in the above proposition, we shall follow [6] to say that $\tilde{G}$ is the lift of $G$. The next lemma shows that if $\Upsilon \cong C_{n-1}$ then $\tilde{G}$ is a 2-transitive permutation group on $\mathcal{P}$ and the point stabilizer $\tilde{G}_{V_u}$ is either cyclic or dihedral.

Lemma 3.2. Use the notations in Assumption I. Let $\Upsilon \cong C_{n-1}$ and let $\tilde{G}$ be the lift of $G$. Then for each $u \in V(K_n)$, the subgraph of $\tilde{K}_n$ induced by $V_u$ is a cycle of length $n - 1$, and the subgroup $\tilde{G}_{V_u}$ of $\tilde{G}$ fixing $V_u$ set-wise acts faithfully and transitively on $V_u$. In particular, $G$ acts faithfully and 2-transitively on $\mathcal{P}$, and $\tilde{G}_{V_u} \cong \mathbb{Z}_{n-1}$, or $D_{n-1}$ (if $n$ is odd), or $D_{2(n-1)}$.

Proof. By Assumption I (3) and (5), the subgraph of $\tilde{K}_n$ induced by $V_u$ is isomorphic to $\Upsilon$. By Proposition 3.1, $\mathcal{P} = \{V_u \mid u \in V(K_n)\}$ is an imprimitivity block system for $\tilde{G}$, and so for each $u \in V(K_n)$, the subgraph of $\tilde{K}_n$ induced by $V_u$ is a cycle of length $n - 1$.

For any two vertices $u$, $w$ of $K_n$, by Assumption I (5), $\{(u, w), (w, u)\}$ is the unique edge of $\tilde{K}_n$ connecting $V_u$ and $V_w$. This implies that the subgroup $K$ of $\tilde{G}_{V_u}$ fixing $V_u$ point-wise will fix every block in $\mathcal{P}$. It then follows from Proposition 3.1 (3) that $K = 1$, and so $\tilde{G}_{V_u}$ acts faithfully on $V_u$. Since $G$ is transitive on $V(K_n)$, $\tilde{G}_{V_u}$ is transitive on $V_u$. Since the subgraph of $\tilde{K}_n$ induced by $V_u$ is a cycle of length $n - 1$, one has $\tilde{G}_{V_u} \cong \mathbb{Z}_{n-1}$, or $D_{n-1}$ (if $n$ is odd), or $D_{2(n-1)}$.

Again since $\{(u, w), (w, u)\}$ is the unique edge of $\tilde{K}_n$ connecting $V_u$ and $V_w$, it follows that $G_{V_u}$ also acts transitively on $\mathcal{P} \setminus \{V_u\}$. This implies that $G$ acts 2-transitively on $\mathcal{P}$. By Proposition 3.1 (3), $\tilde{G}$ acts faithfully on $\mathcal{P}$.
The above lemma enables us to determine the structure of $\tilde{G}$ in the case when $\Upsilon \cong C_{n-1}$.

**Lemma 3.3.** Use the notations in Assumption I. Let $\Upsilon \cong C_{n-1}$ and let $\tilde{G}$ be the lift of $G$. Then one of the following holds:

1. $n = 6$ and $\text{soc}(\tilde{G}) = A_5$;
2. $n = 4$ and $\tilde{G} \cong AGL(1,2^3)$ or $AΓL(1,2^2)$;
3. $n = p^e \neq 4$ and $\tilde{G} \cong AGL(1,p^e)$, where $p$ is a prime and $e$ is a positive integer.

**Proof.** By Lemma 3.2, $\tilde{G}$ can be viewed as a 2-transitive permutation group on $P$ with point stabilizer isomorphic to $\mathbb{Z}_{n-1}$, or $D_{n-1}$ (if $n$ is odd), or $D_{2(n-1)}$. By [5, Proposition 5.2], $\text{soc}(\tilde{G})$ is either elementary abelian or non-abelian simple, and furthermore, if $\text{soc}(\tilde{G})$ is non-abelian simple, then by checking the list of the simple groups which can occur as socles of 2-transitive groups in [5, page 8], we have $\text{soc}(\tilde{G}) = A_5$. In order to complete the proof of this lemma, it remains to deal with the case when $\text{soc}(\tilde{G})$ is elementary abelian.

In what follows, assume that $\text{soc}(\tilde{G}) \cong \mathbb{Z}_p^e$ for some prime $p$ and positive integer $e$. View $\text{soc}(\tilde{G})$ as an $e$-dimensional vector space over a field of order $p$, and let 0 denote the zero vector of $\text{soc}(\tilde{G})$. Recall that $\tilde{G}_0 \cong \mathbb{Z}_{p^e-1}$, $D_{p^e-1}$ (p odd), or $D_{2(p^e-1)}$. By checking Hering’s theorem on classification of 2-transitive affine permutation groups [8] (see also [10, Appendix 1]), we have $\tilde{G} \cong AGL(1,p^e)$ with point-stabilizer $\tilde{G}_0 \leq ΓL(1,p^e)$. As $G = \text{soc}(\tilde{G}) \rtimes \tilde{G}_0$, to determine $\tilde{G}$, we only need to determine all possible subgroups of $ΓL(1,p^e)$ which are isomorphic to $\mathbb{Z}_{p^e-1}$, $D_{p^e-1}$ (p odd), or $D_{2(p^e-1)}$, and transitive on $\text{soc}(\tilde{G}) \setminus \{0\}$.

Note that $ΓL(1,p^e)$ can be constructed in the following way. Let $GF(p^e)$ be the Galois field of order $p^e$, and view $\text{soc}(\tilde{G})$ as the additive group of $GF(p^e)$. It is well-known that the multiplicative group $GF(p^e)^*$ of $GF(p^e)$ is cyclic, and let $x$ be a generator of $GF(p^e)^*$. Then $GL(1,p^e) = \langle x \rangle$. Let $y$ be the Frobenius automorphism of $GF(p^e)$ such that $y$ maps every $g \in GF(p^e)$ to $g^p$. Then we have

$$ΓL(1,p^e) = \langle x, y \mid x^{p^e-1} = y^e = 1, y^{-1}xy = x^p \rangle.$$

In the following, we shall first determine all possible cyclic subgroups of $ΓL(1,p^e)$ of order either $p^e-1$ or $p^e-1$ (p odd) (Claim 1), and then this is used to determine all possible subgroups of $ΓL(1,p^e)$ which are isomorphic to $\mathbb{Z}_{p^e-1}$, $D_{p^e-1}$ (p odd), or $D_{2(p^e-1)}$, and transitive on $\text{soc}(\tilde{G}) \setminus \{0\}$.

**Claim 1.** Let $T$ be a cyclic subgroup of $ΓL(1,p^e)$ of order $\frac{p^e-1}{r}$ with either $r = 1$ or $r = 2$ and $p$ is odd. Then either $T = \langle x^r \rangle$, or $p^e = 3^2$, $T \cong \mathbb{Z}_{p^e-1}$ and $T = \langle xy \rangle$ or $\langle x^3y \rangle$.

Let $\ell = p^e - 1$ or $\frac{p^e-1}{2}$ (p odd). Since $T$ is a cyclic subgroup of $ΓL(1,p^e)$ of order $\ell$, we may let $T = \langle x^jy^k \rangle$ with $0 \leq j \leq p^e - 2$ and $0 \leq k \leq e - 1$. If $k = 0$, then $T \leq \langle x \rangle$ and so $T = \langle x^r \rangle$ with either $r = 1$ or $r = 2$ and $p$ is odd.

Assume now that $0 < k \leq e - 1$. Then $y^k \neq 1$. Since $y^{-1}xy = x^p$, one has $yx^py^{-1} = x$, and hence $(yxy^{-1})^p = x$. Clearly, $p^e \equiv 1 \pmod{p^e - 1}$, so $yx^py^{-1} = x^{p^e-1}$. It follows that $y^kx^jy^{-k} = x^{jp^{k(e-1)}}$, and so $y^kx^j = x^{jp^{k(e-1)}}y^k$. By this equality, we have for any positive integer $m$,

$$\langle x^jy^k \rangle^m = x^j(1+p^{k(e-1)}+p^{2k(e-1)}+\cdots+p^{(m-1)k(e-1)})^{y^{mk}} = x^{j\frac{p^{mk(e-1)}-1}{p^{e-1}-1}}y^{mk}. \quad (3.1)$$
From Eq. (3.1) it follows that \((x^j y^k)^e = x^j x^e p^{k(e-1) - 1} \). Since \(p^e - 1 \mid p^{k(e-1)} - 1\), one has
\[
(x^j y^k)^e (p^{k(e-1) - 1}) = x^j (p^{e(k(e-1)) - 1}) = 1.
\]
This implies that the order of \(x^j y^k\) divides \(e(p^{k(e-1)} - 1)\), namely, \(\ell \mid e(p^{k(e-1)} - 1)\).

Since \(\ell = p^e - 1 \) or \(\frac{p^e - 1}{2} \) (\(p\) odd), we have \(p^e - 1 \mid 2e(p^{k(e-1)} - 1)\).

Suppose that \(e \geq 3\). If \((p, e) = (2, 6)\), then \(\ell = p^e - 1 = 63\). However, it is easy to check that 63 \(\nmid 6(2^{5k} - 1)\) for any \(k \leq 5\), contrary to \(\ell \mid e(p^{k(e-1)} - 1)\). Thus, \((p, e) \neq (2, 6)\). Then by a result of Zsigmondy \[14\], there exists at least one prime \(q\) such that \(q\) divides \(p^e - 1\) but does not divide \(p^t - 1\) for any positive integer \(t < e\). Clearly, \(p \neq q\), so \(p\) is an element of \(\mathbb{Z}_q^* \cong \mathbb{Z}_{q-1}\) of order \(e\). In particular, we have \(q > e\).

Since \(q \mid p^e - 1\) and \(p^e - 1 \mid 2e(p^{k(e-1)} - 1)\), we have \(q \mid p^{k(e-1)} - 1\), implying \(k(e-1) > e\).

Since \(k \leq e - 1\), we may let \(k(e-1) = m e + t\) for some positive integers \(m\) and \(t < e\), and since \(p^m(e-1) = (p^{k(e-1)} - 1) = (p^m(e-1) - 1)\), we have \(q \mid p^t - 1\). However, this is impossible because it is assumed that \(q \nmid p^t - 1\) for any \(t < e\).

Thus, \(e < 3\). Since \(0 < k \leq e - 1\), one has \(e = 2\) and \(k = 1\), and then \(p^2 - 1 \mid 4(p-1)\). It follows that \(p + 1 \mid 4\) and hence \(p = 3\). Then \((x^j y^2)^2 = x^{4j}\) has order at most 2 since \(\langle x, y \rangle \cong \mathbb{Z}_8\), and then \(x^j y\) has order dividing 4. This implies that \(\ell = \frac{p^e - 1}{2} = 4\) and \(T = \langle x, y \rangle\) or \(\langle x^3 y \rangle\). This completes the proof of Claim 1.

By now, we have shown that Claim 1 is true. Recall that \(\tilde{G}_0 \leq \Gamma L(1, p^e)\), \(\tilde{G}_0 \cong \mathbb{Z}_{p^e - 1}\), \(\mathbb{D}_{p^e - 1}\) (\(p\) odd), or \(\mathbb{D}_{2(p^e - 1)}\) and \(\tilde{G}_0\) is transitive on \(\text{soc}(\tilde{G}) \setminus \{0\}\). We shall finish the proof by considering the following three cases.

**Case 1.** \(\tilde{G}_0 \cong \mathbb{Z}_{p^e - 1}\).

In this case, by Claim 1, we must have \(\tilde{G}_0 = \langle x \rangle = \text{GL}(1, p^e)\) and so \(\tilde{G} \cong \text{AGL}(1, p^e)\).

**Case 2.** \(\tilde{G}_0 \cong \mathbb{D}_{p^e - 1}\) (\(p\) odd).

In this case, by Claim 1, either \(x^2 \in \tilde{G}_0\), or \(p^e = 9\) and \(\tilde{G}_0\) contains \(xy\) or \(x^3 y\). For the former, we have \(\tilde{G}_0 = \langle x^2, f \rangle\), where \(f\) is an involution of \(\Gamma L(1, p^e)\) such that \(fx^2 f = x^{-2}\) and \(f \notin \langle x \rangle\). Note that \(\tilde{G}_0\) is transitive on \(\text{soc}(\tilde{G}) \setminus \{0\}\). We may let \(f = xy^k\) and \(0 < k \leq e - 1\). By Eq. (3.1), \(f^2 = (xy^k)^2 = 1\) implies that \(e\) is even and \(k = \frac{e}{2}\), and furthermore, \(x^{p^{\frac{e(e-2)}{2}} + 1} = 1\). It follows that \(p^e - 1 \mid \frac{p^e - 1}{2} + 1\). However, since \(p^e \mid \frac{p^e}{2} + 1\) \(p^{e-1} + 1\), we would have \(p^e - 1 \mid p^e + 1\), forcing that \(p = 2\), a contradiction.

For the latter, we have \(\tilde{G}_0 \cong \mathbb{D}_8\). However, it is easy to check that in \(\Gamma L(1, 9) = \langle x, y \mid x^3 = y^2 = 1, y^{-1} xy = x^{-1} \rangle\) there does not exist an involution inverting \(xy\) or \(x^3 y\), a contradiction.

**Case 3.** \(\tilde{G}_0 \cong \mathbb{D}_{2(p^e - 1)}\).

By Claim 1, we must have \(\tilde{G}_0 = \langle x \rangle \rtimes \langle y^2 \rangle\) with \(y^2 xy^2 = x^{-1}\). On the other hand, since \(y^{-1} xy = x^p\), we have \(y^2 xy^2 = x^{p^2}\) and hence \(x^p x^2 = x^{-1}\). It follows that \(p^2 \equiv -1 \pmod{p^e - 1}\) and hence \(p^e - 1\) divides \(p^2 + 1\). Consequently, we have \(p^e = 4, G_0 = \langle x, y \rangle = \Gamma L(1, 4)\), and \(G \cong \text{AGL}(1, 4) \cong S_4\).
Now we are ready to determine all possible special generalized truncations of $K_n$ by $C_{n-1}$.

**Lemma 3.4.** Use the notations in Assumption I. Let $\mathcal{Y} \cong C_{n-1}$ and let $\tilde{G}$ be the lift of $G$. Then $\tilde{K}_n = T(K_n, G, \mathcal{Y})$ is isomorphic to either $T(K_6, A_5, C_5)$ (see Fig. (1)), or one of the graphs $K^{i,e}_{p^e}(i \in \mathbb{Z}_{p^e-1}, i < \frac{p^e-1}{2})$ (see Construction I for the definition of these graphs).

**Proof.** If soc($\tilde{G}$) $\cong A_5$, then by [6, Example 5.3], we have $\tilde{G} \cong A_5$ and up to graph isomorphism, there exists a unique graph, and so we may denote this graph by $T(K_6, A_5, C_5)$ (see Fig. 1).

In what follows, we assume that soc($\tilde{G}$) $\not\cong A_5$. Then from Lemma 3.3 we see that $\tilde{G}$ has a subgroup, say $\tilde{T}$ such that $\tilde{T} \cong AGL(1, p^e)$ and $\tilde{T}$ acts regularly on $V(\tilde{K}_n)$, where $p$ is a prime and $e$ is a positive integer such that $p^e \geq 4$. It follows that $\tilde{K}_n$ is a Cayley graph on $\tilde{T}(\cong AGL(1, p^e))$ and $n = p^e$. For each $u \in V(\tilde{K}_n)$, by Lemma 3.2, the subgraph of $\tilde{K}_n$ induced by $V_u$ is a cycle of length $n - 1$, and the subgroup $\tilde{G}_{V_u}$ of $\tilde{G}$ fixing $V_u$ set-wise acts faithfully and transitively on $V_u$. Furthermore, $\tilde{G}$ acts faithfully and 2-transitively on $P$. For convenience, we may identify $P$ with $GF(p^e)$, identify $V_u$ with the zero element 0 of $GF(p^e)$ and identify $\tilde{T}$ with $AGL(1, p^e)$. We shall use the notations for $\tilde{T} = AGL(1, p^e)$ as well as its elements and subgroups $H$ and $K$ introduced in the paragraph before Construction I. Then $\tilde{T}_{V_u} = K \cong \mathbb{Z}_{p^e-1}$.

Take $(u, w) \in V_u$, and assume that $(u, w_1)$ and $(u, w_2)$ are two vertices in $V_u$ adjacent to $(u, w)$. Since $\tilde{T}_{V_u} = K \cong \mathbb{Z}_{p^e-1}$ is transitive on $V_u$, there exists a unique $\alpha_{x^{-1},0} \in \tilde{T}_{V_u}$ such that $(u, w)^{\alpha_{x^{-1},0}} = (u, w_1)$ and $(u, w)^{\alpha_{x^{-1},0}} = (u, w_2)$, and since the subgraph of $\tilde{K}_n$ induced by $V_u$ is a cycle of length $n - 1$, $i$ is coprime to $p^e - 1 (n = p^e)$. So we may let

$$\tilde{K}_n = \text{Cay}(AGL(1, p^e), \{\alpha_{x^{i},0}, \alpha_{x^{-1},0}, \alpha_{x^{j},z'}\}),$$

where $\alpha_{x^{j},z'}$ is an involution. Since $\tilde{K}_n$ is connected, if $p$ is odd, then we have $\alpha_{x^{j},z'} = \alpha_{x^{\frac{p^e-1}{2}+1-z'},z'} = \alpha_{-1,z'}$ and $z' \neq 0$, and if $p = 2$, then we have $\alpha_{x^{j},z'} = \alpha_{1,z'}$ and $z' \neq 0$, and correspondingly, we obtain the two graphs $K^{j,e}_{p^e} (p > 2)$ and $K^{j,e}_{2^e}$ (see Construction I). \qed

From Fig. (1) it is easy to see that $T(K_6, A_5, C_5)$ is a special generalized truncation of $K_6$ by a cycle of length 5. The following lemma shows that each of the Cayley graphs $K^{i,e}_{p^e}(i \in \mathbb{Z}_{p^e-1}, i < \frac{p^e-1}{2})$ is also indeed a special generalized truncation of $K_{p^e}$ by a cycle of length $p^e - 1$.

**Lemma 3.5.** Each of the graphs $K^{i,e}_{p^e}(i \in \mathbb{Z}_{p^e-1}, i < \frac{p^e-1}{2})$ (see Construction I) is a special generalized truncation of $K_{p^e}$ by a cycle of length $p^e - 1$.

**Proof.** Recall that each $K^{i,e}_{p^e}(i \in \mathbb{Z}_{p^e-1}, i < \frac{p^e-1}{2})$ is a trivalent Cayley graph on $AGL(1, p^e)$ defined as follows:

$$K^{i,e}_{p^e} = \text{Cay}(AGL(1, p^e), \{\alpha_{-1,z'}, \alpha_{x^{i},0}, \alpha_{x^{-i},0}\})(z' \neq 0, p > 2),$$
$$K^{j,e}_{2^e} = \text{Cay}(AGL(1, 2^e), \{\alpha_{1,z'}, \alpha_{x^{j},0}, \alpha_{x^{-j},0}\})(z' \neq 0).$$

(Keep in mind we use the notations for $AGL(1, p^e)$ as well as its elements and subgroups $H$ and $K$ introduced in the paragraph before Construction I.) Note that $AGL(1, p^e) = H \rtimes K$, where...
Moreover, $K$ is maximal in $\text{AGL}(1, p^e)$ since $\text{AGL}(1, p^e)$ is 2-transitive on $GF(p^e)$. As $i \in Z_{p^e - 1}^*$, one has $K = \langle \alpha_{x,i}, 0 \rangle$ and then the maximality of $K$ implies that $\langle \alpha_{z', i}, \alpha_{x, i}, 0 \rangle = \text{AGL}(1, p^e)$ for $p > 2$ and $\langle \alpha_{z, i}, \alpha_{x, i}, 0 \rangle = \text{AGL}(1, 2^e)$. Thus, every $K_{p^e}^i (i \in Z_{p^e - 1}^*, i < \frac{p^e - 1}{2})$ is connected.

It is easy to see that $\text{Cay}(K, \{\alpha_{x,i}, 0, \alpha_{x,i}, 0\}) \cong C_{p^e - 1}$ is a subgraph of $K_{p^e}^i (i \in Z_{p^e - 1}^*, i < \frac{p^e - 1}{2})$. Since $\text{AGL}(1, p^e)$ acts on $V(K_{p^e}^i)$ by right multiplication, the subgraph of $K_{p^e}^i$, induced by $Kg$ for any $g \in \text{AGL}(1, p^e)$ is a cycle of length $p^e - 1$. As $\text{AGL}(1, p^e)$ acts 2-transitively on $B = \{Kg \mid g \in \text{AGL}(1, p^e)\}$, the quotient graph of $K_{p^e}^i$ relative to $B$ is a complete graph $K_{p^e}$. So we have $K_{p^e}^i \cong T(K_{p^e}, \text{AGL}(1, p^e), \Upsilon_i)$, where $\Upsilon_i$ is the subgraph with vertex set $B - \{K\}$ and edge set $\{\{Kg, Kg\alpha_{x, i}, 0\} \mid g \in K\}$ where $\gamma = \alpha_{-1, z'}$ for $p > 2$ and $\gamma = \alpha_{1, z'}$ for $p = 2$. □

### 3.2 Automorphisms and isomorphisms

In this subsection, we shall determine the automorphism groups and isomorphisms of special generalized truncations of $K_n$ by $C_{n-1}$, and thus prove the second part of Theorem 1.2. By checking [6, Table 1], we have the following lemma.

**Lemma 3.6.** $\text{Aut}(T(K_6, A_5, C_5)) \cong A_5$.

In the following two lemmas, we shall determine the automorphisms and isomorphisms of the graphs $K_{p^e}^i (i \in Z_{p^e - 1}^*, i < \frac{p^e - 1}{2})$. We keep using the notations for $\text{AGL}(1, p^e)$ as well as its elements and subgroups $H$ and $K$ introduced in the paragraph before Construction I.

**Lemma 3.7.** Let $\Gamma$ be one of the graphs $K_{p^e}^i (i \in Z_{p^e - 1}^*, i < \frac{p^e - 1}{2})$ (see Construction I). Then Theorem 1.2(ii)–(vi) hold.

**Proof.** Recall that $\Gamma$ is a connected trivalent Cayley graph on $X = \text{AGL}(1, p^e)$. Let $A = \text{Aut}(\Gamma)$. For convenience of the statement, we view $X$ as a regular subgroup of $A$.

Suppose first that $\Gamma$ is arc-transitive. Let $N = \bigcap_{g \in A} X^g$. If $N = 1$, then by [9, Theorem 1.1], we have $\text{Aut}(\Gamma) \cong \text{PGL}_2(p^e)$ with $p^e = 11$ or 23. If $p^e = 11$, then since $i \leq 1$ and $i < 5$, we have $i = 3$ and hence $\Gamma = K_{11}^3$. If $p^e = 23$, then $i = 3, 5, 7$ or 9 as $i \in Z_{23}^*$ and $i < 11$, and by Magma [4], $\text{Aut}(K_{23}^i) \cong \text{PGL}_2(23)$ if and only if $i = 7$, and hence $\Gamma = K_{23}^7$. If $N > 1$, then $N \leq A$, and in particular, $N \leq X$. Since $\text{soc}(X) \cong Z_{p^e}$ is the unique minimal normal subgroup of $X = \text{AGL}(1, p^e)$, one has $\text{soc}(X) \leq N$. Clearly, $\text{soc}(X)$ is a Sylow $p$-subgroup of $N$ since $N \leq X$. So $\text{soc}(X)$ is characteristic in $N$ and hence normal in $A$. Consider the quotient graph $\Sigma$ of $\Gamma$ relative to $\text{soc}(X)$. Clearly, $\Sigma$ has $p^e - 1$ vertices. Since $p^e - 1 > 2$, by Proposition 2.3, $\Sigma$ would be a trivalent arc-transitive Cayley graph on $X/\text{soc}(X) \cong Z_{p^e - 1}$. Furthermore, by [2, Corollary 1.3], either $\Sigma \cong K_{3,3}$, or $\Sigma$ is a trivalent normal arc-transitive Cayley graph on $X/\text{soc}(X) \cong Z_{p^e - 1}$. However, the latter case cannot happen by Lemma 2.1. For the former, we have $p^e - 1 = 6$ and so $p = 7$ and $e = 1$. In this case, we have $i = 1$ and $\Gamma = K_{7}^1$. By Magma [4], we have $\text{Aut}(K_{7}^1) \cong D_{42} \rtimes Z_3$. 


Suppose now that $\Gamma$ is not arc-transitive. If $A > X$, then the vertex-stabilizer $A_a$ is a 2-group for any $a \in V(\Gamma)$. Then $A_a$ fixes one and only one neighbor of $a$. Assume that the neighbor of $a$ fixed by $A_a$ is $b$. Then $B = \{ (a, b)^g : g \in A \}$ is a system of blocks of imprimitivity of $A$ on $V(\Gamma)$. It follows that $\Gamma - B$ is a union of several cycles with equal lengths, and the set of vertex-sets of these cycles forms an $A$-invariant partition of $V(\Gamma)$. Let $C$ be the cycle of $\Gamma$ containing the identity $1$ of $X$. Since $\Gamma$ is a Cayley graph on $X$, $X$ acts on $V(\Gamma) = X$ by right multiplication, and since $V(C)$ is a block of imprimitivity of $A$ acting on $V(\Gamma)$, $C$ is actually a subgroup of $X$. From the definition of $\Gamma = K_{p^e}$, one may see that $V(C) = K = \{ \alpha_{x^i,0} : z \mapsto zx^i, \forall z \in GF(p^e) \mid i \in \mathbb{Z}_{p^e-1} \}$, and the vertex set of every cycle of $\Gamma - B$ is just a right coset of $K$. Let $B = \{ K \gamma g \mid g \in X \}$. Then $B$ is an $A$-invariant partition of $\Gamma$. Clearly, $X$ acts 2-transitively and faithfully on $B$, so the quotient graph of $\Gamma$ relative to $B$ is $K_{p^e}$. Now it is easy to see that $\Gamma \cong T(K_{p^e}, A, \gamma_i)$, where $\gamma_i$ is the subgraph with vertex set $B - K$ and edge set $\{ \{ K \gamma g, K \gamma \alpha_{x^i,0} g \} \mid g \in A_K \}$ where $\gamma = \alpha_{-1,z'}$ for $p > 2$ and $\gamma = \alpha_{1,z'}$ for $p = 2$. Clearly, $\gamma_i \cong C_{p^e-1}$. From Lemma 3.3 it follows that either $p^e = 4$ and $A = AGL(1,4) \cong S_4$, or $A = X \cong AGL(1,p^e)$.

**Lemma 3.8.** For any distinct $i, i' \in \mathbb{Z}_{p^e-1}$ with $i, i' < \frac{p^e-1}{2}$, $K_{p^e}^i \cong K_{p^e}^{i'}$ if and only if there exists $1 \leq j \leq e$ such that $i' \equiv ip^j$ or $-ip^j \pmod{p^e - 1}$.

**Proof.** If $p^e = 4$ or 7, then we must have $i = 1$, and so we have only one graph for each of these two cases. If $p^e = 11$ or 23, then by Magma [4], for any distinct $i, i' \in \mathbb{Z}_{p^e-1}$ with $i, i' < \frac{p^e-1}{2}$, one may check that $K_{p^e}^i \cong K_{p^e}^{i'}$ if and only if $i' \equiv ip^j$ or $-ip^j \pmod{p^e - 1}$.

Suppose that $K_{p^e}^i$ is not isomorphic to one of the graphs: $K_{4}^{1}, K_{7}^{1}, K_{11}^{3}$ and $K_{23}^{7}$. By Lemma 3.7, Aut $(K_{p^e}^i) \cong AGL(1,p^e)$ and by Proposition 2.2, $K_{p^e}^i$ is a CI-graph. Recall that

\[
K_{p^e}^i = \text{Cay}(AGL(1,p^e), \{ \alpha_{-1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0} \})(p > 2), \quad K_{p^e}^{2i} = \text{Cay}(AGL(1,2p^e), \{ \alpha_{1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0} \})(p = 2).
\]

Since $K_{p^e}^i$ is a CI-graph, for any distinct $i, i' \in \mathbb{Z}_{p^e-1}$ with $i, i' < \frac{p^e-1}{2}$, $K_{p^e}^i \cong K_{p^e}^{i'}$ if and only if there exists $\gamma \in \text{Aut}(AGL(1,p^e))$ such that $\{ \alpha_{x^i,0}, \alpha_{x^{-i},0} \}^\gamma = \{ \alpha_{x^{i'},0}, \alpha_{x^{-i'},0} \}$ and $\alpha_{-1,z'}^{\gamma} = \alpha_{1,z'}$ for $p > 2$ or $\alpha_{1,z'}^{\gamma} = \alpha_{1,z'}$ for $p = 2$.

Note that $\text{Aut}(AGL(1,p^e)) = AGL(1,p^e) \cong AGL(1,p^e) \rtimes \langle \eta \rangle$, where $\eta$ is induced by the Frobenius automorphism of $GF(p^e)$ such that $\alpha_{a,b}^\eta = \alpha_{a^p,b^p}$ for any $\alpha_{a,b} \in AGL(1,p^e)$. Suppose first that $i' \equiv ip^j$ or $-ip^j \pmod{p^e - 1}$ for some $1 \leq j \leq e$. Then one may check that $\alpha_{x^i,0}^{\eta^j \gamma} \alpha_{x^{-i},0}^{\gamma} = \alpha_{x^j,0}$ and $\{ \alpha_{x^i,0}, \alpha_{x^{-i},0} \}^{\eta^j \gamma} = \{ \alpha_{x^j,0}, \alpha_{x^{-j},0} \}$. So $K_{p^e}^i \cong K_{p^e}^{i'}$. Conversely, if $K_{p^e}^i \cong K_{p^e}^{i'}$, then there exists $\gamma \in \text{Aut}(AGL(1,p^e))$ such that $\{ \alpha_{x^i,0}, \alpha_{x^{-i},0} \}^{\gamma} = \{ \alpha_{x^{i'},0}, \alpha_{x^{-i'},0} \}$ and $\alpha_{-1,z'}^{\gamma} = \alpha_{1,z'}$ for $p > 2$ or $\alpha_{1,z'}^{\gamma} = \alpha_{1,z'}$ for $p = 2$. Since $K = \langle \alpha_{x^i,0} \rangle$, $\gamma$ normalizes $K$, and since $N_{AGL(1,p^e)}(K) = K \rtimes \langle \eta \rangle$, one has $\gamma = \alpha_{x^k,0}^{\eta^j}$, for some $k \in \mathbb{Z}_{p^e-1}$ and $1 \leq j \leq e$. Then

\[
\alpha_{x^i,0}^\gamma = \alpha_{x^i,0}^{\alpha_{x^k,0}^{\eta^j}} = \alpha_{x^i,0}^{\eta^j} = \alpha_{x^{ip^j},0}, \quad \alpha_{x^{-i},0}^\gamma = \alpha_{x^{-i},0}^{\alpha_{x^k,0}^{\eta^j}} = \alpha_{x^{-i},0}^{\eta^j} = \alpha_{x^{-ip^j},0} \in \{ \alpha_{x^{ip^j},0}, \alpha_{x^{-ip^j},0} \}.
\]

It follows that $i' \equiv ip^j$ or $-ip^j \pmod{p^e - 1}$. 

\[\Box\]
3.3 Proof of Theorem 1.2

From Lemmas 3.4 & 3.5 we can obtain the proof of the first part of Theorem 1.2, and from Lemmas 3.7 & 3.8, we obtain the proof of the second part of Theorem 1.2.

References


