The average genus for bouquets of circles and dipoles

Jinlian Zhang

*School of Mathematics and Statistics, Hunan University of Finance and Economics,
Changsha, P. R. China*

Xuhui Peng *

*MOE-LCSM, School of Mathematics and Statistics, Hunan Normal University,
Changsha, P. R. China*

Yichao Chen †

*Department of Mathematics and Physics, Suzhou University of Science and Technology,
Suzhou, P. R. China*

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Abstract

The bouquet of circles B_n and dipole graph D_n are two important classes of graphs in topological graph theory. For $n \geq 1$, we give an explicit formula for the average genus $\gamma_{\text{avg}}(B_n)$ of B_n . By this expression, one easily sees $\gamma_{\text{avg}}(B_n) = \frac{n - \ln n - \gamma + 1 - \ln 2}{2} + o(1)$, where γ is the *Euler-Mascheroni constant*. Similar results are obtained for D_n . Our method mainly depends on the technique of generating series and the knowledge in ordinary differential equations.

Keywords: Average genus, bouquet of circles, dipole, ordinary differential equation.

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E-mail addresses: jinlian916@hnu.edu.cn (Jinlian Zhang), xhpeng@hunnu.edu.cn (Xuhui Peng), ycchen@hnu.edu.cn (Yichao Chen)

1 Introduction and main results

A graph $G = (V(G), E(G))$ is permitted to have both loops and multiple edges. An *embedding* of a graph G into an orientable surface O_k is a *cellular embedding*, i.e., the interior of every face is homeomorphic to an open disc. The subscript in O_k is the genus of the orientable surface O_k , for $k \geq 0$. We denote the number of cellular embeddings of G on the surface O_k by $g_k(G)$, where, by the *number of embeddings*, we mean the number of equivalence classes under ambient isotopy. The *genus polynomial* of a graph G is given by

$$\Gamma_G(x) = \sum_{k \geq 0} g_k(G) x^k.$$

This sequence $\{g_k(G), k = 0, 1, 2, \dots\}$ is called the *genus distribution* of the graph G . For a graph G , it is well known that the total number of cellular embeddings is $\prod_{v \in V(G)} (d_G(v) - 1)!$, where $d_G(v)$ is the degree of the vertex v in G . For example, see [13, Chapter 3]. Hence,

$$\Gamma_G(1) = \sum_{k \geq 0} g_k(G) = \prod_{v \in V(G)} (d_G(v) - 1)!. \quad (1.1)$$

The *average genus* $\gamma_{\text{avg}}(G)$ of the graph G is the expected value of the genus random variable, over all labeled 2-cell orientable embeddings of G , using the uniform distribution. In other words, the average genus of G is

$$\gamma_{\text{avg}}(G) = \frac{\Gamma'_G(1)}{\Gamma_G(1)} = \sum_{k=0}^{\infty} k \cdot \frac{g_k(G)}{\Gamma_G(1)}.$$

The study of the average genus of a graph began by Gross and Furst [9], and was much further developed by Chen and Gross [1, 2, 3]. Two lower bounds were obtained in [4] for the average genus of two kinds of graphs. In [19], Stahl gave the asymptotic result for the average genus of linear graph families. The exact values for the average genus of small-order complete graphs, closed-end ladders, and cobblestone paths were derived by White [22]. More references are the following: [5, 10, 15, 17, 20] etc. For a general background in topological graph theory, we refer the reader to see Gross and Tucker [13] or White [21].

One of the purposes of the paper is to give an explicit expression of the average genus for a bouquet of circles. By a *bouquet of circles*, or more briefly, a bouquet, we mean a graph with one vertex and some self-loops. In particular, the bouquet with n self-loops is denoted by B_n . Figure 1 shows the graphs B_1, B_2, B_3 . The bouquets $\{B_n, n \geq 1\}$ are very important graphs in topological graph theory. First, since any connected graph can be reduced to a bouquet by contracting a spanning tree to a point, bouquets are fundamental building blocks of topological graph theory. Second, as shown in [8, 12], Cayley graphs and many other regular graphs are covering spaces of bouquets.

For the genus distribution of B_n , Gross, Robbins and Tucker [11] proved that the numbers $g_k(B_n)$ of embeddings of the B_n in an oriented surface of genus k satisfy the following recurrence for $n > 2$,

$$(n+1)g_k(B_n) = 4(2n-1)(2n-3)(n-1)^2(n-2)g_{k-1}(B_{n-2}) + 4(2n-1)(n-1)g_k(B_{n-1}) \quad (1.2)$$

Figure 1: The bouquets B_1, B_2 , and B_3 .

with initial conditions

$$\begin{aligned} g_k(B_0) &= 1 \text{ for } k = 0 \text{ and } g_k(B_0) = 1 \text{ for } k > 0, \\ g_k(B_1) &= 1 \text{ for } k = 0 \text{ and } g_k(B_1) = 1 \text{ for } k > 0. \end{aligned} \quad (1.3)$$

With the aid of an edge-attaching surgery technique, the total embedding polynomial of B_n was computed in [14]. Stahl [18] also did some research on the average genus of B_n . By [18, Theorem 2.5] and the definition of Euler-Mascheroni constant, one easily sees that

$$\lim_{n \rightarrow \infty} \left(\gamma_{\text{avg}}(B_n) - \left(\frac{n+1}{2} - \frac{1}{2} \sum_{k=1}^{2n} \frac{1}{k} \right) \right) = 0. \quad (1.4)$$

To achieve this, Stahl made many accurate estimates on the unsigned Stirling numbers $s(n, k)$ of the first kind. In this paper, using knowledge in ordinary differential equations and Taylor's formula, we derive an explicit expression of $\gamma_{\text{avg}}(B_n)$. By this expression, (1.4) follows immediately. Our methods are totally different from that in [18] and we do not need to make estimates on $s(n, k)$. In Section 2, we will give the computation of $\gamma_{\text{avg}}(B_n)$ in detail.

A dipole with n edges, denoted by D_n , has two vertices joined by n edges. Figure 2 shows the graphs D_1, D_2, D_3 .

Figure 2: The dipoles D_1, D_2 , and D_3 .

Another purpose of this paper is to give an explicit expression of the average genus for dipole D_n . The dipole, like the bouquet, is useful as a voltage graph. See [21] for example. Moreover, hypermaps correspond with the 2-cell embeddings of the dipole. The genus distribution of D_n is given by [14] and [16].

In Lemma 2.1 below, we obtain the following recurrence relation for $\gamma_{\text{avg}}(B_n)$

$$(n+1)\gamma_{\text{avg}}(B_n) = 2\gamma_{\text{avg}}(B_{n-1}) + (n-1)(\gamma_{\text{avg}}(B_{n-2}) + 1). \quad (1.5)$$

The most popular way to deal with sequences of numbers is to manipulate infinite series that “generate” those sequences. For instance, see [6, 7]. We apply this method to

the calculation of $\gamma_{\text{avg}}(B_n)$. Multiplying both sides of (1.5) by t^n and summing on $n \geq 1$, the generating function $u(t) = \sum_{n \geq 1} \gamma_{\text{avg}}(B_n) t^n$ will satisfy an ordinary differential equation. We solve this differential equation with the aid of a computer system and find an explicit expression for $\gamma_{\text{avg}}(B_n)$ by expanding $u(t)$ as a power series in t . The calculation of $\gamma_{\text{avg}}(D_n)$ is similar to that in $\gamma_{\text{avg}}(B_n)$. But the processes are more complicated, so we still give their details in Section 3.

2 The average genus of B_n

We begin by proving the following lemma.

Lemma 2.1. *The following recurrence relation holds for the average genus $\gamma_{\text{avg}}(B_n)$ of B_n*

$$(n+1)\gamma_{\text{avg}}(B_n) = 2\gamma_{\text{avg}}(B_{n-1}) + (n-1)(\gamma_{\text{avg}}(B_{n-2}) + 1) \quad (2.1)$$

with initial conditions $\gamma_{\text{avg}}(B_1) = 0$, $\gamma_{\text{avg}}(B_2) = \frac{1}{3}$.

Proof. Multiplying both sides of (1.2) by x^k and summing on $k \geq 0$, it holds that

$$\begin{aligned} \sum_{k \geq 0} (n+1)g_k(B_n)x^k &= \sum_{k \geq 0} 4(2n-1)(2n-3)(n-1)^2(n-2)g_{k-1}(B_{n-2})x^k \\ &\quad + \sum_{k \geq 0} 4(2n-1)(n-1)g_k(B_{n-1})x^k. \end{aligned} \quad (2.2)$$

Hence, the genus polynomial $\Gamma_{B_n}(x)$ satisfies the following recurrence

$$\begin{aligned} (n+1)\Gamma_{B_n}(x) &= 4(2n-1)(2n-3)(n-1)^2(n-2) \cdot x \cdot \Gamma_{B_{n-2}}(x) \\ &\quad + 4(2n-1)(n-1)\Gamma_{B_{n-1}}(x). \end{aligned} \quad (2.3)$$

Differentiating both sides of (2.3) and taking $x = 1$ lead to

$$\begin{aligned} (n+1)\Gamma'_{B_n}(1) &= 4(2n-1)(2n-3)(n-1)^2(n-2) \cdot \Gamma'_{B_{n-2}}(1) \\ &\quad + 4(2n-1)(2n-3)(n-1)^2(n-2) \cdot \Gamma_{B_{n-2}}(1) + 4(2n-1)(n-1)\Gamma'_{B_{n-1}}(1). \end{aligned}$$

Applying (1.1) to the graph B_n yields $\Gamma_{B_n}(1) = (2n-1)!$. Dividing both sides of the above equality by $\Gamma_{B_n}(1)$, by the definition of average genus, one arrives at

$$(n+1)\gamma_{\text{avg}}(B_n) = 2\gamma_{\text{avg}}(B_{n-1}) + (n-1)(\gamma_{\text{avg}}(B_{n-2}) + 1).$$

A direct calculation gives rise to $\gamma_{\text{avg}}(B_1) = 0$ and $\gamma_{\text{avg}}(B_2) = \frac{1}{3}$. The proof is completed. $\square$

The main purpose of this section is to prove the following theorem.

Theorem 2.2. *The average genus of B_n is given by*

$$\gamma_{\text{avg}}(B_n) = \frac{n+1}{2} - \sum_{m=0}^{n-1} \frac{1+(-1)^m}{2(m+1)} - \frac{1+(-1)^n}{4(n+1)}. \quad (2.4)$$

In particular, we have

$$\gamma_{\text{avg}}(B_n) = \frac{n - \ln n - \gamma + 1 - \ln 2}{2} + o(1),$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant.

Proof. For $n \leq 0$, we define $\gamma_{\text{avg}}(B_n) = 0$ so that (2.1) holds for any integer $n \geq 1$. For the simplicity of writing, we use a_n to denote $\gamma_{\text{avg}}(B_n)$ in the proof. Multiplying both sides of (2.1) by t^n and summing on $n \geq 1$, we obtain

$$\sum_{n \geq 1} (n+1)a_n t^n = 2 \sum_{n \geq 1} a_{n-1} t^n + \sum_{n \geq 1} (n-1)(a_{n-2} + 1)t^n. \quad (2.5)$$

Let $u(t) = \sum_{n \geq 1} a_n t^n$. Then, with the help of (2.5), we obtain

$$\begin{aligned} \left(t \cdot \sum_{n \geq 1} a_n t^n\right)' &= 2t \cdot \sum_{n \geq 1} a_{n-1} t^{n-1} + \sum_{n \geq 1} (n-2)a_{n-2} t^n + \sum_{n \geq 1} a_{n-2} t^n + \sum_{n \geq 1} (n-1)t^n \\ &= 2tu(t) + t^3 \sum_{n \geq 1} (n-2)a_{n-2} t^{n-3} + t^2 u(t) + t^2 \cdot \left(\sum_{n \geq 2} t^{n-1}\right)', \end{aligned}$$

that is

$$\begin{aligned} (tu(t))' &= 2tu(t) + t^3 \sum_{n \geq 3} (n-2)a_{n-2} t^{n-3} + t^2 u(t) + t^2 \left(\frac{t}{1-t}\right)' \\ &= 2tu(t) + t^3 \sum_{n \geq 1} na_n t^{n-1} + t^2 u(t) + t^2 \left(\frac{t}{1-t}\right)' \\ &= 2tu(t) + t^3 u'(t) + t^2 u(t) + t^2 \left(\frac{t}{1-t}\right)', \end{aligned}$$

which implies that $u(t)$ satisfies the following equation

$$(t - t^3)u'(t) + (1 - 2t - t^2)u(t) = \frac{t^2}{(1-t)^2} \quad (2.6)$$

with initial condition $u(0) = 0$. Since the above equation is a first order linear differential equation, we can solve it directly and obtain its solution:

$$u(t) = \frac{-(t^2 - 1) \ln(1-t) + (t^2 - 1) \ln(t+1) + 2t}{4(t-1)^2 t}.$$

Denote

$$u_1(t) = \frac{1}{2(t-1)^2}, \quad u_2(t) = -\frac{(t+1) \ln(1-t)}{4(t-1)t}, \quad u_3(t) = \frac{(t+1) \ln(t+1)}{4(t-1)t}.$$

Then, clearly, $u(t) = u_1(t) + u_2(t) + u_3(t)$. Using Taylor's formula, we get

$$u_1(t) = \sum_{n \geq 0} \frac{n+1}{2} t^n \quad (2.7)$$

and

$$u_2(t) = \frac{1}{4}(1+t) \cdot \frac{1}{1-t} \cdot \frac{\ln(1-t)}{t} = \frac{1}{4}(1+t) \cdot \sum_{\ell \geq 0} t^\ell \cdot \sum_{m \geq 0} \left(-\frac{1}{m+1} t^m\right)$$

$$= \frac{1}{4}(1+t) \cdot \sum_{n \geq 0} \sum_{m=0}^n \left(-\frac{1}{m+1} \right) t^n = \sum_{n \geq 0} b_n t^n, \quad (2.8)$$

where $b_0 = -\frac{1}{4}$ and $b_n = \frac{1}{4} \left[\sum_{m=0}^n \left(-\frac{1}{m+1} \right) + \sum_{m=0}^{n-1} \left(-\frac{1}{m+1} \right) \right]$, $n \geq 1$. Also by the Taylor's formula,

$$\begin{aligned} u_3(t) &= -\frac{1}{4}(1+t) \cdot \frac{1}{1-t} \cdot \frac{\ln(1+t)}{t} = -\frac{1}{4}(1+t) \cdot \sum_{\ell \geq 0} t^\ell \cdot \sum_{m \geq 0} \frac{(-1)^m}{m+1} t^m \\ &= -\frac{1}{4}(1+t) \cdot \sum_{n \geq 0} \sum_{m=0}^n \frac{(-1)^m}{m+1} t^n = \sum_{n \geq 0} c_n t^n, \end{aligned} \quad (2.9)$$

where $c_0 = -\frac{1}{4}$ and

$$c_n = -\frac{1}{4} \left[\sum_{m=0}^n \frac{(-1)^m}{m+1} + \sum_{m=0}^{n-1} \frac{(-1)^m}{m+1} \right], \quad n \geq 1.$$

It follows from (2.7)–(2.9) that

$$\begin{aligned} a_n &= \frac{n+1}{2} + b_n + c_n = \frac{n+1}{2} + \frac{1}{4} \left[\sum_{m=0}^n \left(-\frac{1}{m+1} \right) + \sum_{m=0}^{n-1} \left(-\frac{1}{m+1} \right) \right] \\ &\quad - \frac{1}{4} \left[\sum_{m=0}^n \frac{(-1)^m}{m+1} + \sum_{m=0}^{n-1} \frac{(-1)^m}{m+1} \right], \end{aligned}$$

which yields (2.4). In view of

$$\gamma = \lim_{n \rightarrow +\infty} \left[\sum_{m=0}^n \frac{1}{m+1} - \ln n \right] \quad \text{and} \quad \lim_{n \rightarrow +\infty} \sum_{m=0}^{n-1} \frac{(-1)^m}{m+1} = \ln 2, \quad (2.10)$$

we complete the proof of (2.2). $\square$

3 The average genus of D_n

Our first purpose is to show the following lemma.

Lemma 3.1. *The following recurrence relation holds for the average genus $\gamma_{\text{avg}}(D_n)$ of D_n*

$$n(n+2)\gamma_{\text{avg}}(D_{n+1}) = (2n+1)\gamma_{\text{avg}}(D_n) + (n^2-1) \cdot \gamma_{\text{avg}}(D_{n-1}) + n^2 \quad (3.1)$$

with initial conditions $\gamma_{\text{avg}}(D_1) = \gamma_{\text{avg}}(D_2) = 0$.

Proof. By [16, Theorem 5.2], we obtain

$$(n+2)g_k(D_{n+1}) = n(2n+1)g_k(D_n) + n^3(n-1)^2g_{k-1}(D_{n-1}) - n(n-1)^2g_k(D_{n-1}).$$

Applying (1.1) to the graph D_{n+1} yields $\Gamma_{D_{n+1}}(1) = (n!)^2$. Following the lines in the proof of Lemma 2.1, we derive the recurrence relation (3.1).

The initial conditions $\gamma_{\text{avg}}(D_1) = \gamma_{\text{avg}}(D_2) = 0$ are due to a direct calculation. The proof is finished. $\square$

The main purpose of this section is to prove the following theorem.

Theorem 3.2. $\gamma_{\text{avg}}(D_1) = \gamma_{\text{avg}}(D_2) = 0$ and for $n \geq 3$, we have

$$\gamma_{\text{avg}}(D_n) = n \left[\frac{1}{2} \sum_{m=4}^{n+1} \frac{(-1)^m (4m^2 - 12m + 6)}{(m-3)(m-2)(m-1)m} + \frac{1}{6} \right] - \frac{1}{2} \sum_{m=1}^{n+1} \frac{1}{m} - \sum_{m=4}^{n+1} \frac{(-1)^m (2m^2 - 6m + 3)}{(m-3)(m-1)m} + \frac{7}{12}. \quad (3.2)$$

In particular, we have

$$\gamma_{\text{avg}}(D_n) = \frac{n - \ln n - \gamma}{2} + o(1), \quad (3.3)$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant.

Proof. First, we give a proof of (3.2). For the simplicity of writing, we use a_n to denote $\gamma_{\text{avg}}(D_n)$ in the proof. Let $u(t) = \sum_{n \geq 1} a_n t^{n-3} = \sum_{n \geq 2} a_{n+1} t^{n-2}$. Multiplying both sides of (3.1) by t^{n-2} and summing on $n \geq 2$, we obtain

$$\begin{aligned} \sum_{n \geq 2} n(n+2)a_{n+1}t^{n-2} &= \sum_{n \geq 2} (2n+1)a_n t^{n-2} \\ &\quad + \sum_{n \geq 2} (n^2 - 1)a_{n-1}t^{n-2} + \sum_{n \geq 2} n^2 t^{n-2}. \end{aligned} \quad (3.4)$$

Since

$$\begin{aligned} u'(t) &= \sum_{n \geq 2} (n-2)a_{n+1}t^{n-3}, \\ u''(t) &= \sum_{n \geq 2} (n-2)(n-3)a_{n+1}t^{n-4}, \end{aligned}$$

it follows that

$$\begin{aligned} \sum_{n \geq 2} n(n+2)a_{n+1}t^{n-2} &= \sum_{n \geq 2} [(n-2)(n-3) + 7(n-2) + 8]a_{n+1}t^{n-2} \\ &= t^2 u''(t) + 7t u'(t) + 8u(t), \\ \sum_{n \geq 2} (2n+1)a_n t^{n-2} &= \sum_{n \geq 2} (2n+3)a_{n+1}t^{n-1} = \sum_{n \geq 2} (2(n-2) + 7)a_{n+1}t^{n-1} \\ &= 2t^2 u'(t) + 7tu(t), \\ \sum_{n \geq 2} (n^2 - 1)a_{n-1}t^{n-2} &= \sum_{n \geq 4} (n^2 - 1)a_{n-1}t^{n-2} = \sum_{n \geq 2} (n^2 + 4n + 3)a_{n+1}t^n \\ &= \sum_{n \geq 2} [(n-2)(n-3) + 9(n-2) + 15]a_{n+1}t^n \\ &= t^4 u''(t) + 9t^3 u'(t) + 15t^2 u(t), \\ \sum_{n \geq 2} n^2 t^{n-2} &= \sum_{n \geq 2} n(n-1)t^{n-2} + \sum_{n \geq 2} nt^{n-2} = v''(t) + \sum_{n \geq 0} nt^{n-2} - t^{-1} \\ &= v''(t) + \frac{v'(t)}{t} - t^{-1} = \frac{3t - 4 - t^2}{(t-1)^3}, \end{aligned}$$

where $v(t) = \sum_{n \geq 0} t^n$, $v'(t) = \sum_{n \geq 0} nt^{n-1}$, $v''(t) = \sum_{n \geq 0} n(n-1)t^{n-2}$. Substituting the above equalities into (3.4), $u(t)$ satisfies the following second order linear differential equation

$$(t^2 - t^4)u''(t) + (7t - 2t^2 - 9t^3)u'(t) + (8 - 7t - 15t^2)u(t) = \frac{3t - 4 - t^2}{(t-1)^3}$$

with initial conditions $u(0) = a_3 = \gamma_{\text{avg}}(D_3) = \frac{1}{2}$, $u'(0) = a_4 = \gamma_{\text{avg}}(D_4) = \frac{5}{6}$.

With the help of a computer algebra systems, the solution of the above equation is

$$u(t) = \frac{1}{4(t-1)t^2} + \frac{w(t)}{4(t-1)^2t^4}, \quad (3.5)$$

where

$$\begin{aligned} w(t) = & -t^3 + 2t^3 \ln(t+1) + 3t^2 - 2t^2 \ln(t+1) \\ & - 2t \ln(1-t) - 2t \ln(t+1) + 2 \ln(1-t) + 2 \ln(t+1). \end{aligned}$$

By Taylor's formula, we get

$$\begin{aligned} \frac{1}{4(t-1)t^2} &= \sum_{m \geq -2} \left(-\frac{1}{4}\right)t^m, \\ w(t) &= t^2 - t^3 \\ &\quad + \sum_{m \geq 4} \frac{2(4(-1)^m m^2 + m^2 - 12(-1)^m m - 5m + 6(-1)^m + 6)}{(m-3)(m-2)(m-1)m} t^m, \\ \frac{1}{4(t-1)^2t^4} &= \sum_{m \geq -4} \frac{m+5}{4} t^m. \end{aligned}$$

Therefore, comparing the coefficients of t^{n-3} of the both sides of (3.5) gives

$$\begin{aligned} a_n &= -\frac{1}{4} + \frac{n}{4} - \frac{n-1}{4} \\ &\quad + \sum_{m=4}^{n+1} \frac{2(4(-1)^m m^2 + m^2 - 12(-1)^m m - 5m + 6(-1)^m + 6)}{(m-3)(m-2)(m-1)m} \cdot \frac{n-m+2}{4} \\ &= \frac{n}{2} \sum_{m=4}^{n+1} \left[\frac{(-1)^m (4m^2 - 12m + 6)}{(m-3)(m-2)(m-1)m} + \frac{(m^2 - 5m + 6)}{(m-3)(m-2)(m-1)m} \right] \\ &\quad - \sum_{m=4}^{n+1} \frac{(-1)^m (4m^2 - 12m + 6) + (m^2 - 3m) + (-2m + 6)}{(m-3)(m-2)(m-1)m} \cdot \frac{m-2}{2} \\ &= \frac{n}{2} \sum_{m=4}^{n+1} \frac{(-1)^m (4m^2 - 12m + 6)}{(m-3)(m-2)(m-1)m} + \frac{n}{2} \sum_{m=4}^{n+1} \frac{1}{(m-1)m} \\ &\quad - \frac{1}{2} \sum_{m=4}^{n+1} \frac{(-1)^m (4m^2 - 12m + 6)}{(m-3)(m-1)m} - \frac{1}{2} \sum_{m=4}^{n+1} \frac{1}{m-1} + \sum_{m=4}^{n+1} \frac{1}{m(m-1)} \end{aligned}$$

$$\begin{aligned}
&= \frac{n}{2} \sum_{m=4}^{n+1} \frac{(-1)^m (4m^2 - 12m + 6)}{(m-3)(m-2)(m-1)m} + \frac{n}{2} \left(\frac{1}{3} - \frac{1}{n+1} \right) \\
&\quad - \frac{1}{2} \sum_{m=4}^{n+1} \frac{(-1)^m (4m^2 - 12m + 6)}{(m-3)(m-1)m} - \frac{1}{2} \sum_{m=1}^{n+1} \frac{1}{m} \\
&\quad + \frac{3}{4} + \frac{1}{2(n+1)} + \left(\frac{1}{3} - \frac{1}{n+1} \right)
\end{aligned}$$

which yields the desired result (3.2).

Now we are in a position to prove (3.3). Using the software *Mathematica* or series theory, one has

$$\sum_{m=4}^{n+1} \frac{(-1)^m (4m^2 - 12m + 6)}{(m-3)(m-2)(m-1)m} = \frac{2}{3} + o\left(\frac{1}{n}\right) \quad (3.6)$$

and

$$\sum_{m=4}^{n+1} \frac{(-1)^m (2m^2 - 6m + 3)}{(m-3)(m-1)m} = \frac{7}{12} + o(1). \quad (3.7)$$

Combining (3.6)–(3.7), (2.10) and (3.2), we complete the proof of (3.3). $\square$

4 Some remarks

Bouquets and dipoles are two important classes of graphs in topological graph theory. Their average genera are of independent interest. In this paper, we obtain explicit formulas for $\gamma_{\text{avg}}(B_n)$ and $\gamma_{\text{avg}}(D_n)$. By Theorems 2.2 and 3.2, we have the following relation between $\gamma_{\text{avg}}(B_n)$ and $\gamma_{\text{avg}}(D_n)$,

$$\gamma_{\text{avg}}(B_n) = \gamma_{\text{avg}}(D_n) + \frac{1 - \ln 2}{2} + o(1).$$

It follows that the difference of $\gamma_{\text{avg}}(B_n)$ and $\gamma_{\text{avg}}(D_n)$ tends to the constant $\frac{1 - \ln 2}{2}$ when n tends to infinity.

Since both B_n and D_n are upper-embeddable, the maximum genera of B_n and D_n are $\lfloor \frac{n}{2} \rfloor$ and $\lfloor \frac{n-1}{2} \rfloor$, respectively. Recall that the minimum genera of B_n and D_n equal 0. Therefore, also by Theorems 2.2 and 3.2, we have

$$\lim_{n \rightarrow \infty} \frac{\gamma_{\text{avg}}(B_n)}{\lfloor \frac{n}{2} \rfloor} = 1 \text{ and } \lim_{n \rightarrow \infty} \frac{\gamma_{\text{avg}}(D_n)}{\lfloor \frac{n-1}{2} \rfloor} = 1.$$

This implies that the average genus of B_n (D_n) is closer to the maximum genus than to the minimum genus.

ORCID iDs

Xuhui Peng <https://orcid.org/0000-0002-2443-4896>

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