Also available at http://amc.imfm.si
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 5 (2012) 325–331

On the core of a unicyclic graph

Vadim E. Levit
Ariel University Center of Samaria, Israel

Eugen Mandrescu
Holon Institute of Technology, Israel

Received 23 February 2011, accepted 22 December 2011, published online 10 April 2012

Abstract

A set $S \subseteq V$ is *independent* in a graph $G = (V, E)$ if no two vertices from S are adjacent. By $\text{core}(G)$ we mean the intersection of all maximum independent sets. The *independence number* $\alpha(G)$ is the cardinality of a maximum independent set, while $\mu(G)$ is the size of a maximum matching in G.

A connected graph having only one cycle, say C, is a *unicyclic graph*. In this paper we prove that if G is a unicyclic graph of order n and $n - 1 = \alpha(G) + \mu(G)$, then $\text{core}(G)$ coincides with the union of cores of all trees in $G - C$.

Keywords: Maximum independent set, core, matching, unicyclic graph, König-Egerváry graph.

Math. Subj. Class.: 05C69, 05C70

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X. By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. For $F \subset E(G)$, by $G - F$ we denote the partial subgraph of G obtained by deleting the edges of F, and we use $G - e$, if $W = \{e\}$. If $A, B \subset V$ and $A \cap B = \emptyset$, then (A, B) stands for the set $\{e = ab : a \in A, b \in B, e \in E\}$. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, and $N(A) = \cup\{N(v) : v \in A\}$, $N[A] = A \cup N(A)$ for $A \subset V$. By C_n, K_n we mean the chordless cycle on $n \geq 4$ vertices, and respectively the complete graph on $n \geq 1$ vertices.

A set S of vertices is *independent* if no two vertices from S are adjacent, and an independent set of maximum size will be referred to as a *maximum independent set*. The

E-mail addresses: levitv@ariel.ac.il (Vadim E. Levit), eugen_m@hit.ac.il (Eugen Mandrescu)

independence number of G, denoted by $\alpha(G)$, is the size of a maximum independent set of G. Let $\Omega(G)$ denote the family $\{S : S \textit{ is a maximum independent set of } G\}$, while

$$\text{core}(G) = \cap\{S : S \in \Omega(G)\} \text{ [11]}.$$

An edge $e \in E(G)$ is *α-critical* whenever $\alpha(G - e) > \alpha(G)$. Notice that the inequalities $\alpha(G) \leq \alpha(G - e) \leq \alpha(G) + 1$ hold for each edge e.

A matching (i.e., a set of non-incident edges of G) of maximum cardinality $\mu(G)$ is a *maximum matching*, and a *perfect matching* is one covering all vertices of G. An edge $e \in E(G)$ is *μ-critical* provided $\mu(G - e) < \mu(G)$.

Theorem 1.1. [13] For every graph G no α-critical edge has an endpoint in $N[\text{core}(G)]$.

It is well-known that

$$\lfloor n/2 \rfloor + 1 \leq \alpha(G) + \mu(G) \leq n$$

hold for every graph G with n vertices. If $\alpha(G) + \mu(G) = n$, then G is called a *König-Egerváry graph* [3, 19]. Several properties of König-Egerváry graphs are presented in [6, 9, 10, 12, 15, 16].

It is known that every bipartite graph is a König-Egerváry graph as well [5, 8]. This class includes also non-bipartite graphs (see, for instance, the graph G in Figure 1).

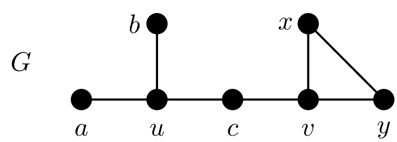

Figure 1: A König-Egerváry graph with $\alpha(G) = |\{a, b, c, x\}|$ and $\mu(G) = |\{au, cv, xy\}|$.

Theorem 1.2. If G is a König-Egerváry graph, then

(i) [12] every maximum matching matches $N(\text{core}(G))$ into $\text{core}(G)$;

(ii) [13] $H = G - N[\text{core}(G)]$ is a König-Egerváry graph with a perfect matching and each maximum matching of H can be enlarged to a maximum matching of G.

The graph G is called *unicyclic* if it is connected and has a unique cycle, which we denote by $C = (V(C), E(C))$. Let

$$N_1(C) = \{v : v \in V(G) - V(C), N(v) \cap V(C) \neq \emptyset\},$$

and $T_x = (V_x, E_x)$ be the tree of $G - xy$ containing x, where $x \in N_1(C), y \in V(C)$.

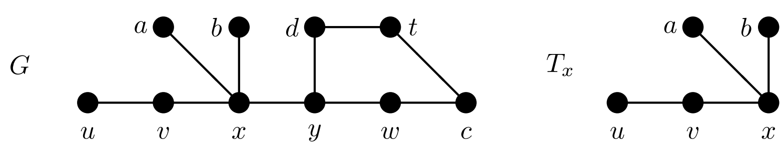

Figure 2: G is a unicyclic non-König-Egerváry graph with $V(C) = \{y, d, t, c, w\}$.

Unicyclic graphs keep enjoying plenty of interest, as one can see, for instance, in [1, 4, 7, 14, 18, 20, 21].

In this paper we analyze the structure of $\text{core}(G)$ for a unicyclic graph G.

2 Results

If G is a unicyclic graph, then there is an edge $e \in E(C)$, such that $\mu(G - e) = \mu(G)$, because for each pair of edges, consecutive on C, at most one could be μ-critical. Let us mention that $\alpha(G) \leq \alpha(G - e) \leq \alpha(G) + 1$ holds for each edge $e \in E(G)$. Every edge of the unique cycle could be α-critical; e.g., the graph G from Figure 2, which has also additional α-critical edges (e.g., the edge uv).

Notice that the bipartite graph T_x from Figure 2 has only two maximum matchings, namely, $M_1 = \{ax, uv\}$ and $M_2 = \{bx, uv\}$, while for each maximum matching there is a vertex in $\mathrm{core}(T_x) = \{a, b\}$ not saturated by that matching.

Lemma 2.1. For every bipartite graph G, a vertex $v \in \mathrm{core}(G)$ if and only if there exists a maximum matching that does not saturate v.

Proof. Since $v \in \mathrm{core}(G)$, it follows that $\alpha(G - v) = \alpha(G) - 1$. Consequently, we have

$$\alpha(G) + \mu(G) - 1 = |V(G)| - 1 = |V(G - v)| = \alpha(G - v) + \mu(G - v)$$

which implies that $\mu(G) = \mu(G - v)$. In other words, there is a maximum matching in G not saturating v.

Conversely, suppose that there exists a maximum matching in G that does not saturate v. Since, by Theorem 1.2*(i)*, $N(\mathrm{core}(G))$ is matched into $\mathrm{core}(G)$ by every maximum matching, it follows that $v \notin N(\mathrm{core}(G))$.

Assume that $v \notin \mathrm{core}(G)$. By Theorem 1.2*(ii)*, every maximum matching M of G is of the form $M = M_1 \cup M_2$, where M_1 matches $N(\mathrm{core}(G))$ into $\mathrm{core}(G)$, while M_2 is a perfect matching of $G - N[\mathrm{core}(G)]$. Thus v is saturated by every maximum matching of G, in contradiction with the hypothesis on v. □

Remark 2.2. Lemma 2.1 fails for non-bipartite König-Egerváry graphs; e.g., every maximum matching of the graph G from Figure 1 saturates $c \in \mathrm{core}(G) = \{a, b, c\}$.

Lemma 2.3. If G is a unicyclic graph of order n, then $n - 1 \leq \alpha(G) + \mu(G) \leq n$.

Proof. If $e = xy \in E(C)$, then $G - e$ is a tree, because G is connected. Hence, $\alpha(G - e) + \mu(G - e) = n$. Clearly, $\alpha(G - e) \leq \alpha(G) + 1$, while $\mu(G - e) \leq \mu(G)$. Consequently, we get that

$$n = \alpha(G - e) + \mu(G - e) \leq \alpha(G) + \mu(G) + 1,$$

which leads to $n - 1 \leq \alpha(G) + \mu(G)$. The inequality $\alpha(G) + \mu(G) \leq n$ is true for every graph G. □

Remark 2.4. If G has n vertices, p connected components, say $H_i, 1 \leq i \leq p$, and each component contains only one cycle, then one can easily see that $n - p \leq \alpha(G) + \mu(G) \leq n$, because $\alpha(G) = \sum_{i=1}^{p} \alpha(H_i)$ and $\mu(G) = \sum_{i=1}^{p} \mu(H_i)$.

While C_{2k}, $k \geq 2$, has no α-critical edge at all, each edge of every odd cycle C_{2k-1}, $k \geq 2$, is α-critical. This property is partially inherited by unicyclic graphs.

Lemma 2.5. Let G be a unicyclic graph of order n. Then $n - 1 = \alpha(G) + \mu(G)$ if and only if each edge of its unique cycle is α-critical.

Proof. Assume that $n-1=\alpha(G)+\mu(G)$. Since G is connected, for each $e\in E(C)$ the graph $G-e$ is a tree. Hence, we have

$$\alpha(G-e)-\alpha(G)+\mu(G-e)-\mu(G)=1,$$

which implies $\mu(G-e)=\mu(G)$ and $\alpha(G-e)=\alpha(G)+1$, since

$$-1\leq\mu(G-e)-\mu(G)\leq 0\leq\alpha(G-e)-\alpha(G)\leq 1.$$

In other words, every $e\in E(C)$ is α-critical.

Conversely, let $e\in E(C)$ be such that $\mu(G-e)=\mu(G)$; such an edge exists, because no two consecutive edges on C could be μ-critical. Since e is α-critical, and $G-e$ is a tree, we infer that

$$n-1=\alpha(G-e)+\mu(G-e)-1=\alpha(G)+\mu(G),$$

and this completes the proof. □

Combining Lemma 2.5 and Theorem 1.1, we infer the following.

Corollary 2.6. If G is a unicyclic non-König-Egerváry graph, then no vertex of its unique cycle belongs to $N[\text{core}(G)]$.

Remark 2.7. Corollary 2.6 is true also for some unicyclic König-Egerváry graphs; e.g., the graph H_1 from Figure 3. However, the König-Egerváry graph H_2 from the same figure satisfies $N[\text{core}(H_2)]\cap V(C)=\{u\}\neq\emptyset$.

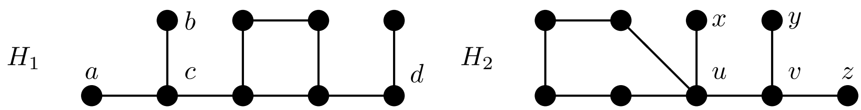

Figure 3: H_1 and H_2 have $N[\text{core}(H_1)]=\{a,b,c\}$, $N[\text{core}(H_2)]=\{x,y,z,u,v\}$.

Lemma 2.8. Let G be a unicyclic graph of order n. If there exists some $x\in N_1(C)$, such that $x\in\text{core}(T_x)$, then G is a König-Egerváry graph.

Proof. Let $x\in\text{core}(T_x)$, $y\in N(x)\cap V(C)$, and $z\in N(y)\cap V(C)$. Suppose, to the contrary, that G is not a König-Egerváry graph. By Lemmas 2.3 and 2.5, the edge yz is α-critical. Hence $y\notin\text{core}(G)$, which implies that $\alpha(G)=\alpha(G-y)$. In accordance with Lemma 2.1, there exists a maximum matching M_x of T_x not saturating x. Combining M_x with a maximum matching of $G-y-T_x$ we get a maximum matching M_y of $G-y$. Hence $M_y\cup\{xy\}$ is a matching of G, which results in $\mu(G)\geq\mu(G-y)+1$. Therefore, using Lemma 2.3 and having in mind that $G-y$ is a forest of order $n-1$, we get the following contradiction

$$n-1=\alpha(G)+\mu(G)\geq\alpha(G-y)+\mu(G-y)+1=n-1+1=n,$$

that completes the proof. □

Remark 2.9. The converse of Lemma 2.8 is not generally true; e.g., the graph H_1 from Figure 3 is a unicyclic König-Egerváry graph, while both $c \notin \text{core}(T_c) = \{a, b\}$, and $d \notin \text{core}(T_d) = \emptyset$.

Theorem 2.10. If G is a unicyclic non-König-Egerváry graph, then

$$\text{core}\,(G) = \cup \{\text{core}\,(T_x) : x \in N_1(C)\}\,.$$

Proof. *Claim 1*. Every maximum independent set of T_x may be enlarged to some maximum independent set of G, for each $x \in N_1(C)$.

Let $A \in \Omega(T_x)$, $y \in N(x) \cap V(C)$, and $z \in N(y) \cap V(C)$. According to Lemma 2.5, the edge yz is α-critical. Hence there exist $S_y \in \Omega(G)$, $S_{yz} \in \Omega(G - yz)$, such that $y \in S_y$ and $y, z \in S_{yz}$.

Case 1. Assume that $x \notin A$.

If $|S_y - V(T_x)| < \alpha(G - T_x) = |S_0|$, where $S_0 \in \Omega(G - T_x)$, then the set $S_1 = S_0 \cup (S_y \cap V(T_x))$ is independent in G, and we get the contradiction

$$\alpha\,(G) = |S_y - V(T_x)| + |S_y \cap V(T_x)| < |S_0| + |S_y \cap V(T_x)| = |S_1|\,.$$

Therefore, we have $|S_y - V(T_x)| = \alpha(G - T_x)$. Then $A \cup (S_y - V(T_x)) \in \Omega(G)$, otherwise we obtain the following contradiction

$$|S_y - V(T_x)| + |A| < \alpha(G) \le \alpha(G - T_x) + \alpha(T_x) = |S_y - V(T_x)| + |A|\,.$$

Case 2. Assume now that $x \in A$.

Then we have $|A| \ge |S_{yz} \cap V(T_x)|$, because $S_{yz} \cap V(T_x)$ is independent in T_x. Hence we infer

$$\begin{aligned}\alpha\,(G) = |S_{yz} - \{y\}| \le |(S_{yz} - \{y\} - (S_{yz} \cap V(T_x))) \cup A| = \\ = |(S_{yz} - \{y\} - V(T_x)) \cup A|\,.\end{aligned}$$

Since $W = (S_{yz} - \{y\} - V(T_x)) \cup A$ is independent and its size is $\alpha(G)$ at least, it follows that W is also a maximum independent set, i.e., we have $A \subseteq W \in \Omega(G)$, as needed.

Claim 2. $S \cap V(T_x) \in \Omega(T_x)$ for every $S \in \Omega(G)$ and each $x \in N_1(C)$.

Let $S \in \Omega(G)$, and suppose, to the contrary, that $A = S \cap V(T_x) \notin \Omega(T_x)$. By Lemma 2.8, $x \notin \text{core}(T_x)$. Thus we can change A for some $B \in \Omega(T_x)$ not containing x. The set $(S - A) \cup B$ is clearly independent in G, and this leads to the contradiction $|(S - A) \cup B| = |S - A| + |B| > |S| = \alpha(G)$.

Combining *Claims 1* and *2*, we infer that:

$$\begin{aligned}\text{core}\,(T_x) &= \cap\{A : A \in \Omega(T_x)\} = \cap\{S \cap V(T_x) : S \in \Omega(G)\} \\ &= (\cap\{S : S \in \Omega(G)\}) \cap V(T_x) = \text{core}\,(G) \cap V(T_x)\,,\end{aligned}$$

which clearly implies

$$\text{core}\,(G) = \cup\{\text{core}\,(T_x) : x \in N(V(C)) - V(C)\}$$

as required. □

Remark 2.11. The assertion in Theorem 2.10 may fail for:

(i) bipartite unicyclic graphs; for example, the graphs H_1, H_2 from Figure 4 satisfy

$$\text{core}\,(H_1) = \cup\,\{\text{core}\,(T_x) : x \in N_1(C)\}\,, \text{ and}$$
$$\text{core}\,(H_2) \neq \{x, z\} = \cup\,\{\text{core}\,(T_x) : x \in N_1(C)\}\,;$$

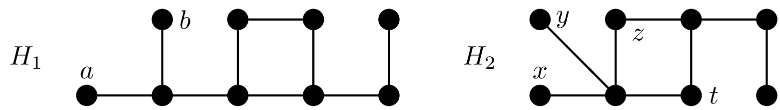

Figure 4: H_1, H_2 are bipartite unicyclic graphs, core$(H_1) = \{a, b\}$, core$(H_2) = \{t, x, y, z\}$.

(ii) non-bipartite König-Egerváry unicyclic graphs; for instance,

$$\text{core}\,(G_2) \neq \{t, z\} = \cup\,\{\text{core}\,(T_x) : x \in N_1(C)\}\,, \text{ while}$$
$$\text{core}\,(G_1) = \cup\,\{\text{core}\,(T_x) : x \in N_1(C)\}\,,$$

where G_1 and G_2 are from Figure 5.

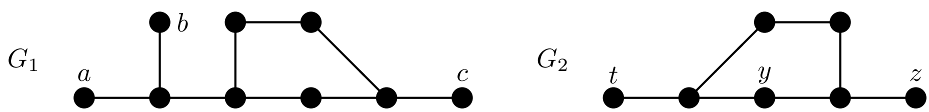

Figure 5: G_1, G_2 are König-Egerváry graphs, core$(G_1) = \{a, b, c\}$, core$(G_2) = \{t, y, z\}$.

It is worth mentioning that the problem of whether there are vertices in a given graph G belonging to core (G) is **NP**-hard [2]. In [17] we have presented both sequential and parallel algorithms finding core (G) in polynomial time for König-Egerváry graphs. By Theorem 2.10, a unicyclic graph is either a König-Egerváry graph or its core (G) equals a union of cores of a finite number of some special subtrees. Therefore, we get the following.

Corollary 2.12. If G is a unicyclic graph, then core (G) is computable in polynomial time.

3 Conclusions

The main purpose of this paper is to investigate the structure of core (G) for unicyclic graphs. One the one hand, we have succeeded to represent core (G) as the union of cores of some specific subtrees of a non König-Egerváry unicyclic graph G. On the other hand, it is still not clear if there exists a characterization of this kind for bipartite unicyclic graphs and/or non-bipartite König-Egerváry graphs.

References

[1] F. Belardo, M. Li, M. Enzo, S. K. Simić and J. Wang, On the spectral radius of unicyclic graphs with prescribed degree sequence, *Linear Algebra Appl.* **432** (2010), 2323–2334.

[2] E. Boros, M. C. Golumbic and V. E. Levit, On the number of vertices belonging to all maximum stable sets of a graph, *Discrete Appl. Math.* **124** (2002), 17–25.

[3] R. W. Deming, Independence numbers of graphs - an extension of the König-Egerváry theorem, *Discrete Math.* **27** (1979), 23–33.

[4] Z. Du, B. Zhou and N. Trinajstić, Minimum sum-connectivity indices of trees and unicyclic graphs of a given matching number, *J. Math. Chem.* **47** (2010), 842–855.

[5] E. Egerváry, On combinatorial properties of matrices, *Matematikai Lapok* **38** (1931), 16–28.

[6] F. Gavril, Testing for equality between maximum matching and minimum node covering, *Inform. Process. Lett.* **6** (1977), 199–202.

[7] B. Huo, S. Ji and X. Li, Note on unicyclic graphs with given number of pendent vertices and minimal energy, *Linear Algebra Appl.* **433** (2010), 1381–1387.

[8] D. König, Graphen und Matrizen, *Matematikai Lapok* **38** (1931), 116–119.

[9] E. Korach, T. Nguyen and B. Peis, Subgraph characterization of red/blue-split graphs and König-Egerváry graphs, *Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms*, ACM Press, 2006, pp. 842–850.

[10] C. E. Larson, The critical independence number and an independence decomposition, *Eur. J. Combin.* **32** (2011), 294–300.

[11] V. E. Levit and E. Mandrescu, Combinatorial properties of the family of maximum stable sets of a graph, *Discrete Appl. Math.* **117** (2002), 149–161.

[12] V. E. Levit and E. Mandrescu, On α^+-stable König-Egerváry graphs, *Discrete Math.* **263** (2003), 179–190.

[13] V. E. Levit and E. Mandrescu, On α-critical edges in König-Egervary graphs, *Discrete Math.* **306** (2006), 1684–1693.

[14] V. E. Levit and E. Mandrescu, Greedoids on vertex sets of unicycle graphs, *Congressus Numerantium* **197** (2009), 183–191.

[15] V. E. Levit and E. Mandrescu, A characterization of König-Egerváry graphs using a common property of all maximum matchings, *Electronic Notes in Discrete Math.* **38** (2011), 565–570.

[16] V. E. Levit and E. Mandrescu, Critical independent sets and König-Egerváry graphs, *Graph. Combinator.* **28** (2012), 243–250.

[17] V. E. Levit and E. Mandrescu, An algorithm computing the core of a König-Egerváry graph, 2011, arXiv:1102.1141 [cs.DM], 8 pp.

[18] J. Li, J. Guo and W. C. Shiu, *The smallest values of algebraic connectivity for unicyclic graphs*, *Discrete Appl. Math.* **158** (2010), 1633–1643.

[19] F. Sterboul, A characterization of the graphs in which the transversal number equals the matching number, *J. Comb. Theory B* **27** (1979), 228–229.

[20] Y. Wu and J. Shu, The spread of the unicyclic graphs, *Eur. J. Combin.* **31** (2010), 411–418.

[21] M. Zhai, R. Liu and J. Shu, Minimizing the least eigenvalue of unicyclic graphs with fixed diameter, *Discrete Math.* **310** (2010), 947–955.