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On the incidence map of incidence structures*

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Abstract

By using elementary linear algebra methods we exploit properties of the incidence map of certain incidence structures with finite block sizes. We give new and simple proofs of theorems of Kantor and Lehrer, and their infinitary version. Similar results are obtained also for diagrams geometries.

By mean of an extension of Block's Lemma on the number of orbits of an automorphism group of an incidence structure, we give informations on the number of orbits of: a permutation group (of possible infinite degree) on subsets of finite size; a collineation group of a projective and affine space (of possible infinite dimension) over a finite field on subspaces of finite dimension; a group of isometries of a classical polar space (of possible infinite rank) over a finite field on totally isotropic subspaces (or totally singular in case of a orthogonal space) of finite dimension.

Furthermore, when the structure is finite and the associated incidence matrix has full rank, we give an alternative proof of a result of Camina and Siemons. We then deduce that certain families of incidence structures have no sharply transitive sets of automorphisms acting on blocks.

Keywords: Incidence structure, incidence map, diagram geometry.

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1 Introduction

An *incidence structure* is a triple $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and I is a subset of $\mathcal{P} \times \mathcal{B}$. The elements of $\mathcal{P}$ are called *points*, those of $\mathcal{B}$ *blocks* and I defines the following *incidence relation*: the point P and the block B are *incident* if and only if $(P, B) \in \mathrm{I}$, and we will write $P \,\mathrm{I}\, B$. The incidence structure $\mathcal{I}$ has *finite block sizes* if $\{P \in \mathcal{P} : P \,\mathrm{I}\, B\}$ has finite size for all $B \in \mathcal{B}$; $\mathcal{I}$ is *finite* if $\mathcal{P}$ and $\mathcal{B}$, and hence also I, are finite sets. An *automorphism* of an incidence structure is a pair of permutations (π, β), with π acting on $\mathcal{P}$ and β on $\mathcal{B}$, such that $P \,\mathrm{I}\, B$ if and only if $P^{\pi} \,\mathrm{I}\, B^{\beta}$, for all $P \in \mathcal{P}$ and $B \in \mathcal{B}$. The group of all automorphisms is denoted by $\mathrm{Aut}\, \mathcal{I}$.

A finite incidence structure can be represented by a $(0, 1)$-matrix A with rows indexed by points and columns indexed by blocks, and with the (P, B)-entry equal to 1 if and only if P is incident with B. The incidence matrix A have been studied by many authors at least since the 1960s, and most of their investigations were on the rank of A. Dembowski in [12, p. 20] showed that the rank of the incidence matrix defined by the natural incidence relation of points versus i-dimensional subspaces of a finite d-dimensional projective or affine space is the number of points of the geometry. This result was generalized by Kantor in [14]. He showed that the incidence matrix defined by the incidence between the i-dimensional subspaces and the j-dimensional subspaces of a finite d-dimensional projective or affine space, with $0 \leq i < j \leq d - i - 1$, has full rank. Analogous results for the incidence matrices of all k-subsets versus all l-subsets of a m-set and for the incidence matrices arising from finite polar spaces were proved by Lehrer [16].

A *decomposition* of an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a partition of $\mathcal{P}$ into *point classes* together with a partition of $\mathcal{B}$ into *block classes*. A decomposition is said to be *block-tactical* if the number of points in a point class which lie in a block depends only on the class in which the block lies. When the incidence structure is finite then the fundamental *Block's Lemma* [2, Theorem 2.1] states that in a block-tactical decomposition the number of point classes differs from the number of block classes by at most the nullity of the incidence matrix of the structure. A principle example of block tactical decomposition is obtained by taking as the point and the block classes the orbits of any automorphism group of the structure. So, Block's Lemma naturally leads to consideration of the rank of the incidence matrix in order to study the number of orbits of an automorphism group of an incidence structure.

When $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is finite, and both permutation representations of any automorphism of $\mathcal{I}$ are regarded as linear representations of the automorphism group, then the incidence matrix A of $\mathcal{I}$ is an intertwining operator between the linear representations of the automorphism group on $\mathcal{P}$ and $\mathcal{B}$. Using this fact, Camina and Siemons [11] showed that when A has maximum rank then the permutation representation on points is a subrepresentation of the permutation representation on blocks. This containment relation implies the non-existence of sharply 1-transitive sets of automorphisms on blocks unless the number of points divides the number of blocks [19].

The aim of this paper is to bring together all the previous questions by providing a unified treatment. Our approach is different from those adopted by the authors referred to above: the main idea is to exploit properties of the incidence map of incidence structures by using elementary linear algebra methods. We find a new and simpler proof of Kantor's and Lehrer's theorems, beside giving the infinitary version of these results. We also provide some geometric version of the main result in [9] on the number of orbits of a permutation group on unordered sets by mean of an extension of Block's Lemma [2] on the number of orbits of an automorphism group of an incidence structure. Furthermore, when the structure is finite and the associated incidence matrix has full rank, we give an alternative proof of the result of Camina and Siemons [11].

We now give a summary of the present paper. In Section 2 we prove that the incidence map of certain (possibly infinite) incidence structures is one-to-one. The keystone is a result (Lemma 2.6) about the kernel of the incidence map from i-dimensional subspaces to $(i+1)$-dimensional subspaces of a finite d-dimensional projective space, where incidence is the inclusion relation. By replacing the dimension with size of a set and the Gaussian coefficients with binomial coefficients, we get the analogous result for the incidence map from k-sets to $(k+1)$-sets of an m-set, where incidence is the inclusion relation. This leads to an alternative proof of both of Kantor's theorems, on the incidence structures arising from projective and affine spaces, and of Lehrer's theorem [16] on the incidence structures arising from subsets. These results are summarized in Theorem 2.7. In Section 3 we illustrate some applications of Theorem 2.7. Under the hypothesis that every block is incident with a finite number of points we prove the infinitary version of the above results. From Kantor's theorem for projective spaces, and because of its infinitary version, we prove that the Lehrer result about incidence structures in finite classical polar spaces [16] holds also in case of polar spaces of infinite rank. Similar results are obtained for diagram geometries associated to certain finite Chevalley groups. If Δ denotes the diagram of the geometry, then by using [7, Theorem 2] we show that the k-varieties give rise to full substructures of the incidence structure of i-varieties versus j-varieties of the geometry, provided i and k lie in distinct connected components of $\Delta - \{j\}$. This gives plenty of scope to apply the main result (Lemma 3.1) of this section. It is conceivable that the weak conclusion that there are as many j-varieties as i-varieties could be useful to diagram geometers. Section 4 is related with Block's Lemma. In the function space and incidence map setting we prove a slight extension of this fundamental result. We then apply it to obtain informations on the number of orbits of: a permutation group (of possible infinite degree) on subsets of finite size; a collineation group of a projective and affine space (of possible infinite dimension) over a finite field on subspaces of finite dimension; a group of isometries of a classical polar space (of possible infinite rank) over a finite field on totally isotropic subspaces (or totally singular in case of a orthogonal space) of finite dimension. We point out that the result on permutation groups was obtained by Cameron in [9], where the theorem of Livingstone and Wagner [17] is proved to hold also for permutation groups of infinite degree. Section 5 is all in the finite setting. We provide an alternative proof of the result of Camina and Siemons [11] which states that if the incidence map of a finite incidence structure is one-to-one, then the permutation representation on points of any given automorphism group is a subrepresentation of the representation on blocks with equal or greater multiplicity. We then deduce that certain families of incidence structures have no sharply transitive sets of automorphisms acting on blocks.

Although some of the results presented here have been obtained by other authors and appear scattered over a large number of papers, in our opinion it is difficult to find a convenient reference for this knowledge with a presentation that doesn't assume a lot of the reader. This work can be considered as an attempt to providing such a reference.

2 The rank of incidence maps

In order to treat our arguments by linear algebra methods, we introduce the incidence map of a finite incidence structure. Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be an incidence structure. The *point space* of $\mathcal{I}$ is the vector space $\mathbb{Q}^{\mathcal{P}}$ of all functions $\mathcal{P} \to \mathbb{Q}$; the *block space* of $\mathcal{I}$ is the vector space $\mathbb{Q}^{\mathcal{B}}$ of all functions $\mathcal{B} \to \mathbb{Q}$. When $\mathcal{I}$ has finite block sizes, we define the (linear) *incidence map* $\alpha\colon \mathbb{Q}^{\mathcal{P}} \to \mathbb{Q}^{\mathcal{B}}$ of $\mathcal{I}$ by the rule

$$(f\alpha)(B) = \sum_{P \, \mathrm{I} \, B} f(P),$$

for all $B \in \mathcal{B}$ and $f \in \mathbb{Q}^{\mathcal{P}}$.

For any subset Y of a given set X the *characteristic function* $\chi_Y \in \mathbb{Q}^X$ of Y is defined as follows:

$$\chi_Y(x) = \begin{cases} 1 & \text{for } x \in Y; \\ 0 & \text{for } x \in X \setminus Y. \end{cases}$$

With this notation, the set $\{\chi_{\{P\}} : P \in \mathcal{P}\}$ is a basis for $\mathbb{Q}^{\mathcal{P}}$ and $\{\chi_{\{B\}} : B \in \mathcal{B}\}$ is a basis for $\mathbb{Q}^{\mathcal{B}}$; we refer to each of these bases as the *natural basis* of the corresponding space. If $\mathcal{I}$ is finite the matrix of the map α with respect to these bases is precisely the incidence matrix of $\mathcal{I}$, with multiplication being on the right (i.e., vectors regarded as rows).

We now exhibit some properties of the incidence maps of the incidence structures arising from subspaces of a finite dimensional projective space over a finite field.

Let $\mathrm{PG}(d, q)$ be the projective space of dimension d over the finite field with q elements. For $0 \le i \le d-1$, let F_i denote the set of all i-dimensional subspaces (or i-subspaces, for short) of $\mathrm{PG}(d, q)$. For $i \neq j$ we consider the incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ where $\mathcal{P} = F_i$, $\mathcal{B} = F_j$ and the incidence relation I is given by set-theoretic inclusion.

The following notation will be adopted in the rest of the paper:

- V_i denotes the vector space $\mathbb{Q}^{F_i}$ of functions from F_i to $\mathbb{Q}$;
- $\alpha_{i,j}$ denotes the incidence map from V_i to V_j, with $i \neq j$;
- $W_{-1} = V_{-1} = \{\emptyset\}$;
- W_i denotes the kernel of $\alpha_{i,i-1}$, for $i \ge 0$.

With the above notation, $\alpha_{i,i}$ is the identity map on V_i. For any $S_i \in F_i$, the coordinate array of $\chi_{\{S_i\}}\alpha_{i,j}$, whose entries are indexed by elements of V_j, is precisely the i-th row of the incidence matrix A of $\alpha_{i,j}$. In other words, if $i > j$ then the image under $\alpha_{i,j}$ of $\chi_{\{S_i\}}$ is the characteristic function of the set of j-subspaces contained in S_i. Similarly, if $i < j$ then the image under $\alpha_{i,j}$ of $\chi_{\{S_i\}}$ is the characteristic function of the pencil of j-subspaces passing through S_i.

In the following we need the q-analogs of binomial coefficients, which are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} (q^{n-i} - 1)/(q^{k-i} - 1)$$

for non-negative integers n, k with $n \ge k$. Note that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of $(k-1)$-subspaces of $\mathrm{PG}(n-1, q)$.

Lemma 2.1. *Let* $-1 \leq i \leq j \leq k \leq d-1$. *Then*

$$\alpha_{i,j}\alpha_{j,k} = \begin{bmatrix} k-i \\ j-i \end{bmatrix}_q \alpha_{i,k}.$$

Proof. By applying directly the definition of $\alpha_{i,j}$ we see that

$$(f\alpha_{i,j}\alpha_{j,k})(S_k) = \sum_{S_i \subseteq S_j \subseteq S_k} f(S_i)$$

holds for all $f \in V_i$. The result now follows by recalling that the number of j-subspaces in $\mathrm{PG}(d,q)$ through any given i-subspace which is in turn contained in a k-subspace is $\left[\begin{smallmatrix} k-i \\ j-i \end{smallmatrix}\right]_q$. □

Lemma 2.2. *For* $i = -1, \ldots, d$,

$$V_i = \bigoplus_{j=-1}^{i} W_j \alpha_{j,i}. \tag{2.1}$$

(Note that some of the summands may be 0).

Proof. For $i = -1$ the result is trivial. For $i = 0, \ldots, d-1$, we note that V_i is a vector space over a field of characteristic zero. Then the inner product defined by

$$\langle g, h \rangle_i = \sum_{S_i \in F_i} g(S_i)h(S_i), \tag{2.2}$$

for all $g, h \in V_i$, is a non-degenerate bilinear form. Since, in the natural bases of V_i and V_j, the matrix of $\alpha_{i-1,i}$ is the transpose of the matrix of $\alpha_{i,i-1}$, then $\langle f\alpha_{i-1,i}, g\rangle_i = \langle f, g\alpha_{i,i-1}\rangle_{i-1}$, for all $f \in V_{i-1}$ and $g \in V_i$, i.e. the incidence map $\alpha_{i-1,i}$ and the dual map $\alpha_{i,i-1}$ are adjoint.

We now show that $V_i = W_i \oplus \operatorname{Im}\alpha_{i-1,i}$. Let $\perp_i$ denote the polarity defined by the inner product $\langle -, - \rangle_i$. Since V_i is finite dimensional, then $V_i = \operatorname{Im}\alpha_{i-1,i} \oplus (\operatorname{Im}\alpha_{i-1,i})^{\perp_i}$. Furthermore, for all $g \in W_i$ and $f \in V_{i-1}$, $\langle f\alpha_{i-1,i}, g\rangle_i = \langle f, g\alpha_{i,i-1}\rangle_{i-1} = 0$ holds, giving $\operatorname{Im}\alpha_{i-1,i} \subseteq W_i^{\perp_i}$, or equivalently, $W_i \subseteq (\operatorname{Im}\alpha_{i-1,i})^{\perp_i}$. Conversely, if $g \in (\operatorname{Im}\alpha_{i-1,i})^{\perp_i}$, then $0 = \langle f\alpha_{i-1,i}, g\rangle_i = \langle f, g\alpha_{i,i-1}\rangle_{i-1}$, for all $f \in V_{i-1}$. By the non-degeneracy of $\langle -, - \rangle_{i-1}$, we get $g\alpha_{i,i-1} = 0$, and hence $g \in W_i$.

We now use induction on i. For $i = -1$ we have $V_{-1} = W_{-1}$. Assume the statement holds for V_{i-1}, that is $V_{i-1} = \bigoplus_{j=-1}^{i-1} W_j\alpha_{j,i-1}$. As $V_i = \operatorname{Im}\alpha_{i-1,i} \oplus W_i\alpha_{i,i}$, to conclude the proof we only need to prove that $\operatorname{Im}\alpha_{i-1,i} = \bigoplus_{j=-1}^{i-1} W_j\alpha_{j,i}$. But this easily follows from Lemma 2.1 since

$$\operatorname{Im}\alpha_{i-1,i} = V_{i-1}\alpha_{i-1,i} = \bigoplus_{j=-1}^{i-1} W_j\alpha_{j,i-1}\alpha_{i-1,i} = \bigoplus_{j=-1}^{i-1} W_j\alpha_{j,i}.$$ □

Remark 2.3. We point out that the bilinear form defined by (2.2) is an appropriate one for the permutation module V_i, in that permutations of the characteristic functions of singletons are isometries of the form. In the basis consisting of the characteristic functions of singletons, this is just a way of saying that permutation matrices are orthogonal in the usual sense of the term, that is $PP^T = I$.

Lemma 2.4. *For* $i = 0, \dots, d-1$,

$$\alpha_{i,i+1}\alpha_{i+1,i} = \alpha_{i,i-1}\alpha_{i-1,i} + \left(\begin{bmatrix} d-i \\ 1 \end{bmatrix}_q - \begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q \right) \alpha_{i,i}.$$

Proof. Let $S_i, S_i' \in F_i$. For any given $S_{i+1} \in F_{i+1}$ we have

$$(\chi_{\{S_i\}}\alpha_{i,i+1})(S_{i+1}) = \begin{cases} 1 & \text{if } S_i \subset S_{i+1}; \\ 0 & \text{otherwise.} \end{cases}$$

It easily follows that

$$(\chi_{\{S_i\}}\alpha_{i,i+1}\alpha_{i+1,i})(S_i') = \sum_{S_{i+1} \supset S_i'} (\chi_{\{S_i\}}\alpha_{i,i+1})(S_{i+1})$$

is the number of $(i+1)$-subspaces containing both S_i and S_i'. This number equals

$$\begin{array}{ll} 0 & \text{if } \dim(S_i \cap S_i') < i-1; \\ 1 & \text{if } \dim(S_i \cap S_i') = i-1; \\ \left[\begin{smallmatrix} d-i \\ 1 \end{smallmatrix}\right]_q & \text{if } S_i' = S_i. \end{array}$$

Applying similar arguments we see that $(\chi_{\{S_i\}}\alpha_{i,i-1}\alpha_{i-1,i})(S_i')$ is the number of $(i-1)$-subspaces contained in both S_i and S_i'. This number is

$$\begin{array}{ll} 0 & \text{if } \dim(S_i \cap S_i') < i-1; \\ 1 & \text{if } \dim(S_i \cap S_i') = i-1; \\ \left[\begin{smallmatrix} i+1 \\ 1 \end{smallmatrix}\right]_q & \text{if } S_i' = S_i. \end{array}$$

The result then follows. □

Lemma 2.5. *For* $j = -1, \dots, i$,

$$(\alpha_{i,i+1}\alpha_{i+1,i})|_{W_j\alpha_{j,i}} = \sum_{k=j}^{i} \left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q \right) \alpha_{i,i}.$$

Proof. We use induction on i. For $i = -1$ we have $W_{-1} = V_{-1} = \{\emptyset\}$ by definition. We also note that $\left[\begin{smallmatrix} d+1 \\ 1 \end{smallmatrix}\right]_q = (q^{d+1}-1)/(q-1)$ is the number of points in $\mathrm{PG}(d,q)$, that is the size of F_0. Then,

$$(\alpha_{-1,0}\alpha_{0,-1})|_{V_{-1}} = (q^{d+1}-1)/(q-1)\alpha_{-1,-1} = \begin{bmatrix} d+1 \\ 1 \end{bmatrix}_q \alpha_{-1,-1}.$$

Now let $i \geq 0$. For $j = i$, the result follows immediately from Lemma 2.4.

Let $j < i$. By Lemma 2.4 we have

$$(\alpha_{i,i+1}\alpha_{i+1,i})|_{W_j\alpha_{j,i}} = (\alpha_{i,i-1}\alpha_{i-1,i})|_{W_j\alpha_{j,i}} + \left(\begin{bmatrix} d-i \\ 1 \end{bmatrix}_q - \begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q \right) \alpha_{i,i}|_{W_j\alpha_{j,i}}.$$

To conclude the proof it is enough to show that

$$(\alpha_{i,i-1}\alpha_{i-1,i})|_{W_j\alpha_{j,i}} = \sum_{k=j}^{i-1}\left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q\right)\alpha_{i,i}.$$

By the inductive hypothesis

$$(\alpha_{i-1,i}\alpha_{i,i-1})|_{W_j\alpha_{j,i-1}} = \sum_{k=j}^{i-1}\left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q\right)\alpha_{i-1,i-1},$$

and Lemma 2.1 gives $\alpha_{j,i-1}\alpha_{i-1,i} = \begin{bmatrix} i-j \\ i-j-1 \end{bmatrix}_q \alpha_{j,i} = \begin{bmatrix} i-j \\ 1 \end{bmatrix}_q \alpha_{j,i}$. Hence, we may write

$$\begin{aligned} w_j\alpha_{j,i}\alpha_{i,i-1}\alpha_{i-1,i} &= \begin{bmatrix} i-j \\ 1 \end{bmatrix}_q^{-1} w_j\alpha_{j,i-1}(\alpha_{i-1,i}\alpha_{i,i-1})\alpha_{i-1,i} \\ &= \sum_{k=j}^{i-1}\left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q\right)\begin{bmatrix} i-j \\ 1 \end{bmatrix}_q^{-1} w_j\alpha_{j,i-1}\alpha_{i-1,i} \\ &= \sum_{k=j}^{i-1}\left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q\right) w_j\alpha_{j,i}, \end{aligned}$$

for $w_j \in W_j$. This implies

$$(\alpha_{i,i-1}\alpha_{i-1,i})|_{W_j\alpha_{j,i}} = \sum_{k=j}^{i-1}\left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q\right)\alpha_{i,i},$$

which is the desired result. □

Lemma 2.6. *Let* $i = 0, \dots, d-1$. *Then*

$$\ker\alpha_{i,i+1} = \begin{cases} 0 & \text{for } i < \frac{d-1}{2}; \\ W_{d-i-1}\alpha_{d-i-1,i} & \text{for } i \geq \frac{d-1}{2}. \end{cases}$$

Proof. It is clear that $\ker\alpha_{i,i+1} \leq \ker(\alpha_{i,i+1}\alpha_{i+1,i})$. In addition,

$$\dim\ker(\alpha_{i,i+1}\alpha_{i+1,i}) = \dim\ker\alpha_{i,i+1} + \dim(\ker\alpha_{i+1,i} \cap \operatorname{Im}\alpha_{i,i+1}).$$

From the proof of Lemma 2.2, we get $\ker\alpha_{i+1,i} \cap \operatorname{Im}\alpha_{i,i+1} = 0$. Therefore $\ker\alpha_{i,i+1} = \ker(\alpha_{i,i+1}\alpha_{i+1,i})$.

From Lemmas 2.2 and 2.5, the eigenvalues of $\alpha_{i,i+1}\alpha_{i+1,i}$ are the integers

$$\sum_{k=j}^{i}\left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q\right), \tag{2.3}$$

for $j = -1, \dots, i$, with the j-th eigenvalue corresponding to the summand $W_j\alpha_{j,i}$ in the decomposition (2.1) of V_i. For $i < (d-1)/2$ all these integers are non-zero, and therefore $\ker\alpha_{i,i+1} = 0$.

Let $i \geq (d-1)/2$. Two cases are treated separately according as d is odd or even. Let d be odd and assume $i = (d-1)/2$. It is easily seen that the only zero eigenvalue of $\alpha_{i,i+1}\alpha_{i+1,i}$ is for $j = i = d-i-1$, as $d-(d-1)/2 = (d-1)/2+1$. Therefore,

$$\ker \alpha_{\frac{d-1}{2},\frac{d+1}{2}} = W_{\frac{d-1}{2}}\alpha_{\frac{d-1}{2},\frac{d-1}{2}}.$$

Now let $i = (d-1)/2 + \delta$, for some integer $\delta > 0$. We note that the summand with $k = (d-1)/2$ in the expression (2.3) is zero. A straightforward calculation shows that for sufficiently small j, the summand with $k = (d-1)/2 - l$ in (2.3) erases with the summand with $k = (d-1)/2 + l$, for $1 \leq l \leq \delta$. This implies that the only zero eigenvalue of $\alpha_{i,i+1}\alpha_{i+1,i}$ is for $j = (d-1)/2 - \delta = d-i-1$. Hence, the kernel of $\alpha_{i,i+1}\alpha_{i+1,i}$ is $W_{d-i-1}\alpha_{d-i-1,i}$.

For d even, the above approach still works up to some differences. For completeness, we give all details.

If d is even, we write $i = \lceil \frac{d-1}{2} \rceil + \delta$, for some integer $\delta \geq 0$. For sufficiently small j, the summand with $k = \lceil \frac{d-1}{2} \rceil - l - 1$ in the expression (2.3) erases with the summand with $k = \lceil \frac{d-1}{2} \rceil + l$, for $0 \leq l \leq \delta$. This implies that the only zero eigenvalue of $\alpha_{i,i+1}\alpha_{i+1,i}$ is for $j = \lceil \frac{d-1}{2} \rceil - \delta - 1 = d-i-1$. Hence the kernel of $\alpha_{i,i+1}\alpha_{i+1,i}$ is $W_{d-i-1}\alpha_{d-i-1,i}$. ☐

The above Lemmata lead to the following fundamental theorem whose proof is new and, in our opinion, more elementary than those provided in [14] and [16].

Theorem 2.7. *The incidence map of the following incidence structures is one-to-one:*

(i) *i-sets versus j-sets of a d-set, with $i < j$ and $i + j \leq d < \infty$.*

(ii) *i-spaces versus j-spaces of* $\mathrm{PG}(d,q)$*, with* $0 \leq i < j \leq d-1$ *and* $i + j < d < \infty$.

(iii) *i-flats versus j-flats of the affine space* $\mathrm{AG}(d,q)$ *of dimension d over the finite field with q elements, with* $0 \leq i < j \leq d-1$ *and* $i + j < d < \infty$.

Proof. We first give the proof of (ii). We need to prove that $\ker \alpha_{i,j} = 0$, for $0 \leq i < j \leq d-1$ and $i + j < d$. We use induction on $j - i$.

If $j - i = 1$ then $\ker \alpha_{i,i+1} = 0$, by Lemma 2.6 as $i < (d-1)/2$. Now let $j - i > 1$ and assume $\ker \alpha_{i',j'} = 0$ for any pair (i',j') with $0 \leq i' < j' \leq d-1$, $i' + j' < d$ and $j' - i' < j - i$. By Lemma 2.1, we have $\ker \alpha_{i,j} = \ker \alpha_{i,i+1}\alpha_{i+1,j}$. In addition $\dim \ker \alpha_{i,i+1}\alpha_{i+1,j} = \dim \ker \alpha_{i,i+1} + \dim(\operatorname{Im} \alpha_{i,i+1} \cap \ker \alpha_{i+1,j})$.

Assume $i + j < d - 1$ so $i < (d-1)/2$ and $i + 1 + j < d$. Then $\ker \alpha_{i,i+1} = 0$ by Lemma 2.6, and $\ker \alpha_{i+1,j} = 0$ by inductive hypothesis. Hence $\ker \alpha_{i,j} = 0$ in this case.

Now assume $i + j = d - 1$. We will prove the result by calculating the dimension of $\operatorname{Im} \alpha_{i,d-i-1}$. By Lemma 2.1 and 2.2 we have

$$\operatorname{Im} \alpha_{i,d-i-1} = V_i \alpha_{i,d-i-1} = \bigoplus_{k=-1}^{i} W_k \alpha_{k,d-i-1}.$$

By the previous part, the map $\alpha_{k,d-i-1}$ is one-to-one for $k = -1, \ldots, i-1$ as $k+d-i-1 < d-1$. Then $\dim W_k\alpha_{k,d-i-1} = \dim W_k$, with $W_k = \ker \alpha_{k,k-1}$. By the arguments in the proof of Lemma 2.2 we get $\dim W_k = \dim V_k - \dim \operatorname{Im} \alpha_{k-1,k}$. By Lemma 2.6

the map $\alpha_{k-1,k}$ is one-to-one for $k = -1, \ldots, i-1$ as $k-1 < (d-1)/2$. Therefore $\dim \operatorname{Im} \alpha_{k-1,k} = \dim V_{k-1}$. This implies

$$\begin{aligned} \dim\, W_k \alpha_{k,d-i-1} &= \dim\, W_k \\ &= \dim\, V_k - \dim V_{k-1} \\ &= \begin{bmatrix} d+1 \\ k+1 \end{bmatrix}_q - \begin{bmatrix} d+1 \\ k \end{bmatrix}_q, \end{aligned}$$

for $k = -1, \ldots, i-1$. Therefore

$$\begin{aligned} \dim \operatorname{Im} \alpha_{i,d-i-1} &= \dim V_i \alpha_{i,d-i-1} \\ &= 1 + \sum_{k=0}^{i-1} \left(\begin{bmatrix} d+1 \\ k+1 \end{bmatrix}_q - \begin{bmatrix} d+1 \\ k \end{bmatrix}_q \right) + \dim W_i \alpha_{i,d-i-1} \\ &= \begin{bmatrix} d+1 \\ i \end{bmatrix}_q + \dim W_i \alpha_{i,d-i-1}. \end{aligned}$$

Still by the proof of Lemma 2.2, we may write $V_i = \operatorname{Im} \alpha_{i-1,i} \oplus W_i$, where $\alpha_{i-1,i}$ is one-to-one as $i < (d-1)/2$. Hence,

$$\dim W_i = \dim V_i - \dim V_{i-1} = \begin{bmatrix} d+1 \\ i+1 \end{bmatrix}_q - \begin{bmatrix} d+1 \\ i \end{bmatrix}_q.$$

This implies

$$\dim W_i \alpha_{i,d-i-1} = \begin{bmatrix} d+1 \\ i+1 \end{bmatrix}_q - \begin{bmatrix} d+1 \\ i \end{bmatrix}_q - \varepsilon,$$

for some $\varepsilon \geq 0$. Thus

$$\begin{aligned} \dim \operatorname{Im} \alpha_{i,d-i-1} &= \dim V_i \alpha_{i,d-i-1} \\ &= \begin{bmatrix} d+1 \\ i \end{bmatrix}_q + \dim W_i \alpha_{i,d-i-1} \\ &= \begin{bmatrix} d+1 \\ i \end{bmatrix}_q + \left(\begin{bmatrix} d+1 \\ i+1 \end{bmatrix}_q - \begin{bmatrix} d+1 \\ i \end{bmatrix}_q - \varepsilon \right) \\ &= \begin{bmatrix} d+1 \\ i+1 \end{bmatrix}_q - \varepsilon. \end{aligned}$$

As $\dim V_i = \left[\begin{smallmatrix} d+1 \\ i+1 \end{smallmatrix}\right]_q$, then $\dim \ker \alpha_{i,d-i-1} = \varepsilon$. At this point to finish the proof we need to evaluate $\dim W_i \alpha_{i,d-i-1}$. We have $\operatorname{Im} \alpha_{i,d-i-1} \leq V_{d-1-1}$, and $\dim V_{d-i-1} = \dim V_i$ by duality. Note that $W_i \alpha_{i,d-i-1}$ is a component of V_{d-i-1} by Lemma 2.1. Then

$$\dim V_{d-i-1} - \dim W_i \alpha_{i,d-i-1} = \dim V_i - \dim W_i \alpha_{i,i} = \begin{bmatrix} d+1 \\ i \end{bmatrix}_q.$$

Hence $\varepsilon = 0$ and this concludes the proof of (ii).

Similar arguments can be used to prove (i). We just need to replace the projective dimension with size of set minus one and the q-binomial coefficients with binomial coefficients.

We now prove (iii). Let $\alpha^A_{i,j}$ denote the incidence map of the i-flats versus the j-flats of $\mathrm{AG}(d,q)$. Embed $\mathrm{AG}(d,q)$ in $\mathrm{PG}(d,q)$ identifying every k-flat of $\mathrm{AG}(d,q)$ with the k-dimensional spaces of $\mathrm{PG}(d,q)$ it spans. Let H denote the hyperplane at infinity of $\mathrm{AG}(d,q)$. Let $f \in \ker \alpha^A_{i,j}$ and g be the extension of f on V_i defined as follows:

$$g(S_i) = \begin{cases} f(S_i) & \text{if } S_i \not\subseteq H; \\ 0 & \text{if } S_i \subseteq H. \end{cases}$$

Then

$$(g\alpha_{i,j})(S_j) = \sum_{S_i \subseteq S_j} g(S_i) = \begin{cases} (f\alpha^A_{i,j})(S_j) & \text{if } S_j \not\subseteq H; \\ 0 & \text{if } S_j \subseteq H. \end{cases}$$

Since $f \in \ker \alpha^A_{i,j}$, we get $g \in \ker \alpha_{i,j}$. By (ii) $g = 0$ and hence $f = 0$. □

Remark 2.8. For $2i + 1 \leq d$, the summands $W_j\alpha_{j,i}$ in the decomposition of V_i given in Lemma 2.2, are all the irreducible constituents of the permutation representation of $\mathrm{PGL}(d,q)$ on F_i. To see this, set $G = \mathrm{PGL}(d,q)$. From the proof of Lemma 2.1 we have $V_i = \operatorname{Im}\alpha_{i-1,i} \oplus W_i$. The map $\alpha_{i-1,i}$ is one-to-one, so the number of irreducible components in its image is precisely the number of the irreducible components of the permutation $\mathbb{Q}G$-module V_{i-1}. This number is $i+1$, being the dimension of the intersection of two $(i-1)$-subspaces a complete invariant. This shows that the modules in question are pairwise non-isomorphic, and irreducible. This was proved by Steinberg [22] using deeper representation theory.

An analogous result holds for the permutation $\mathbb{Q}G$-module defined by the symmetric group $\mathrm{Sym}(n)$ acting on the m-sets, with $2m \leq n$. Here the size of set minus one replaces the projective dimension, and binomial coefficients replace q-binomial coefficients.

Remark 2.9. For $2i+1 \leq d$, the summand $W_j\alpha_{j,i}$, for $j = 0, \dots, i$, in the decomposition of V_i given in Lemma 2.2, is the restriction over the rationals of the $(j+1)$-th eigenspace of the Bose-Mesner algebra of the association scheme on F_i [13, Theorem 2.7]. For a thorough treatment on association schemes we refer the reader to [1, 4].

3 Some applications of Theorem 2.7

The incidence structure $\mathcal{J} = (\mathcal{Q}, \mathcal{C}, \mathrm{J})$ is said to be a *substructure* of $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ if $\mathcal{Q} \subseteq \mathcal{P}$, $\mathcal{C} \subseteq \mathcal{B}$ and $\mathrm{J} = \mathrm{I} \cap (\mathcal{Q} \times \mathcal{C})$. The substructure $\mathcal{J}$ of $\mathcal{I}$ is said to be *full* if $\{P \in \mathcal{P} : P\mathrm{I}C\} \subseteq \mathcal{Q}$, for all $C \in \mathcal{C}$.

Lemma 3.1. *Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be an incidence structure with finite block sizes. Suppose that there is a set $\mathcal{F}$ of full substructures of $\mathcal{I}$, all of whose incidence maps are one-to-one, and such that, for any $P \in \mathcal{P}$ there exists $\mathcal{J} \in \mathcal{F}$ such that P is a point of $\mathcal{J}$. Then the incidence map of $\mathcal{I}$ is one-to-one.*

Proof. Let $\alpha_{\mathcal{I}}$ be the incidence map of $\mathcal{I}$ and $f \in \ker \alpha_{\mathcal{I}}$. For any given $P \in \mathcal{P}$, let $\mathcal{J} = (\mathcal{Q}, \mathcal{C}, \mathrm{J}) \in \mathcal{F}$ such that $P \in \mathcal{Q}$. Let $\alpha_{\mathcal{J}}$ be the incidence map of $\mathcal{J}$. Set $g = f|_{\mathcal{Q}}$. Since $\mathcal{J}$ is full we have

$$(g\alpha_{\mathcal{J}})(C) = \sum_{\substack{Q \in \mathcal{Q} \\ Q\,\mathrm{J}\,C}} g(Q) = \sum_{\substack{R \in \mathcal{P} \\ R\,\mathrm{I}\,C}} f(R) = (f\alpha_{\mathcal{I}})(C),$$

for all $C \in \mathcal{C}$. Since $f \in \ker(\alpha_{\mathcal{I}})$ we have $(g\alpha_{\mathcal{J}})(C) = 0$, for all $C \in \mathcal{C}$, that is $g \in \ker \alpha_{\mathcal{J}}$. Thus $g = 0$, and therefore $f(P) = g(P) = 0$. Since P is arbitrary, it follows that $f = 0$. □

The above Lemma allows to get the infinitary version of Theorem 2.7; this means that the incidence structures involved are over a set with infinite size (in case (i)), or a space with infinite dimension (in case (ii) and (iii)).

Theorem 3.2. *The incidence map of the following structures is one-to-one:*

(i) *i-sets versus j-sets of an infinite set, with $i < j < \infty$.*

(ii) *i-spaces versus j-spaces of a projective space of infinite dimension over a finite field, with $i < j < \infty$.*

(iii) *i-flats versus j-flats of an affine space of infinite dimension over a finite field, with $i < j < \infty$.*

Proof. We apply Lemma 3.1 and Theorem 2.7 to the above structures by taking the set $\mathcal{F}$ of full substructures as follows: all subsets of size $i + j$ for statement (i), all subspaces of dimension $i+j+1$ for statement (ii), all flats of dimension $i+j+1$ for statement (iii). □

Theorem 3.3. *Let $\mathcal{A}$ be a classical polar space of (possible infinite) rank m over a finite field. Then the incidence map of totally isotropic subspaces (or totally singular in case of a orthogonal space) of $\mathcal{A}$ of algebraic dimension k versus singular subspaces of algebraic dimension l is one-to-one, if $k < l < \infty$ and $k + l \leq m$.*

Proof. Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be the incidence structure defined by the subspaces of algebraic dimension k versus subspaces of algebraic dimension l of $\mathcal{A}$. Let $\mathcal{F}$ be the family of all the subspaces of $\mathcal{A}$ of algebraic dimension $k + l$. Since every element $\mathcal{J}$ of $\mathcal{F}$ is a full substructure of $\mathcal{I}$, we may apply Theorem 2.7 (ii), or Theorem 3.2 (ii) for the infinitary version, with $i = k-1$, $j = l-1$ and $d = k+l-1$. Thus we get that the incidence map $\alpha_{\mathcal{J}}$ of $\mathcal{J}$ is one-to-one. The result then follows by applying Lemma 3.1 to the family $\mathcal{F}$. □

Remark 3.4. For the case of finite rank the above theorem is due to Lehrer [16, Theorem 5.3]. Note that Lehrer mistakenly asserts that the incidence map of the incidence structure of singular 1-spaces versus singular $(n-1)$-spaces of the $O^{+}(2n, q)$ polar space is not one-to-one. This error is caused by confusing the $O^{+}(2n, q)$ polar space with the $D_n(q)$ building.

In the following we apply Lemma 3.1 to the incidence structures known as *diagram geometries*. For a thorough treatment on diagram geometries we refer the reader to [7, 8]; our notation is taken from [7].

Let $\Gamma = (S, \bar{\mathrm{I}}, \overline{\Delta}, \tau)$ be a diagram geometry of finite rank with diagram Δ, and $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be the incidence structure where $\mathcal{P}$ is the set of all i-varieties and $\mathcal{B}$ the set of all j-varieties of S; I is the restriction of $\bar{\mathrm{I}}$ on $\mathcal{P} \times \mathcal{B}$. Assume that blocks in $\mathcal{I}$ have finite size and let $k \in \overline{\Delta} \setminus \{j\}$ such that i and k lie in distinct components of the diagram $\Delta - \{j\}$. We now show that the set of k-varieties of S gives rise to a family $\mathcal{F}$ of full substructures of $\mathcal{I}$ with the property that for any point (i-variety) P of $\mathcal{I}$ there exists $\mathcal{J} \in \mathcal{F}$ such that P is a point of $\mathcal{J}$.

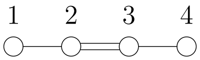
For any given k-variety Λ of S, set $\mathcal{J}_\Lambda = (\mathcal{P}_\Lambda, \mathcal{B}_\Lambda, \mathrm{I}_\Lambda)$ where $\mathcal{P}_\Lambda$ and $\mathcal{B}_\Lambda$ are the set of all i-varieties and j-varieties of S incident to Λ in Γ, respectively; I_Λ is the restriction of I on $\mathcal{P}_\Lambda \times \mathcal{B}_\Lambda$.

Let B be a j-variety in $\mathcal{B}_\Lambda$ and let Γ_B be the residue of B in Γ, that is the diagram geometry $(S', \mathrm{I}', \Delta', \tau')$ where S' is the set of all varieties of S of type $m \in \overline{\Delta} \setminus \{j\}$ which are incident with B, the incidence relation I' is the restriction of $\bar{\mathrm{I}}$ to S', $\Delta' = \tau(S')$ and τ' is the restriction of τ to S'. It is known that the diagram of Γ_B is $\Delta - \{j\}$ [7, Theorem 1]. Therefore the i-varieties of S' are precisely all elements (i-varieties) of $\mathcal{P}_\Lambda$ that are incident with B in $\mathcal{J}_\Lambda$. In addition, as Λ is incident with B, it is a k-variety of S'. Since i and k lie in distinct components of $\Delta - \{j\}$, by [7, Theorem 2] every i-variety of S' is incident with every k-variety, in particular every i-variety of S' is incident with Λ. This implies that $\{P \in \mathcal{P} : P \,\mathrm{I}\, B\}$ is a subset of $\mathcal{P}_\Lambda$. From the arbitrariness of B in $\mathcal{B}_\Lambda$ it follows that $\mathcal{J}_\Lambda$ is a full substructures of $\mathcal{I}$.

Let $\mathcal{F}$ be the family of the substructures $\mathcal{J}_\Lambda$, for all k-varieties Λ of S. Since the type map τ take all values of Δ on every maximal flag of Γ then for every i-variety P of S there exists a k-variety Λ such that P is a point of $\mathcal{J}_\Lambda$. These considerations together with Lemma 3.1 led to the following result.

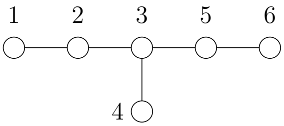
Theorem 3.5. *Let $\Gamma = (S, \bar{\mathrm{I}}, \overline{\Delta}, \tau)$ be the diagram geometry underlying the buildings of types F_4, E_6, E_7 and E_8. Then the incidence map of i-varieties versus j-varieties of Γ is one-to-one in the following cases:*

(i) F_4:



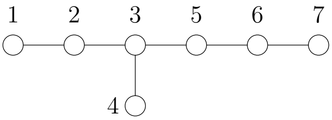
$(i, j) = (1, 2), (4, 3).$

(ii) E_6:



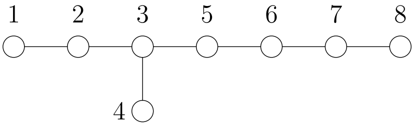
$(i, j) = (1, 2), (1, 3), (2, 3), (6, 5), (6, 3), (5, 3).$

(iii) E_7:



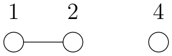
$(i, j) = (1, 2), (1, 3), (2, 3), (7, 6), (7, 5), (7, 3), (6, 5), (6, 3), (5, 3).$

(iv) E_8:



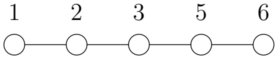
$(i, j) = (1, 2), (1, 3), (2, 3), (8, 7), (8, 6), (8, 5), (8, 3), (7, 6), (7, 5), (7, 3), (6, 5), (6, 3).$

Proof. Consider the diagram $\Gamma = (S, \bar{\mathrm{I}}, \overline{\Delta}, \tau)$ for F_4, and take $(i, j) = (1, 2)$, $k = 3$. Let $\mathcal{F}$ be the family of full substructures arising from the 3-varieties of S constructed as above. The points and blocks of any $\mathcal{J}_\Lambda \in \mathcal{F}$ are precisely the 1- and 2-varieties of S incident with Λ. By [7, Theorem 1], these are precisely the 1- and 2-varieties of the residue $R(\Lambda)$ of Λ in Γ, whose diagram is



Note that every 1- and 2-variety is incident with every 4-variety. This implies that the set of the 1- and 2-varieties of S incident with Λ form a finite projective plane, whose incidence map is injective by a result of Bruck and Ryser [5] and Bose [3]. Lemma 3.1 yields that the incidence map of 1-varieties versus 2-varieties of S is one-to-one in this case. Very similar argument can used with $(i, j) = (4, 3)$ and $k = 2$.

Now consider the diagram $\Gamma = (S, \bar{\mathrm{I}}, \overline{\Delta}, \tau)$ for E_6, and take $(i, j) = (1, 2)$, $k = 4$. As above the points and blocks of any $\mathcal{J}_\Lambda \in \mathcal{F}$ are precisely the 1- and 2-varieties of S incident with Λ, and these are precisely the 1- and 2-varieties of the residue $R(\Lambda)$ of Λ in Γ, whose diagram is



This implies that $R(\Lambda)$ has the geometry of a $\mathrm{PG}(5, q)$. We now apply Theorem 2.7 to conclude that the incidence map of $\mathcal{J}_\Lambda$ is incidence, and Lemma 3.1 yields that the incidence map of 1-varieties versus 2-varieties of S is one-to-one in this case. Very similar arguments apply for the remaining cases, and for the buildings E_7, E_8. $\square$

4 An extension of Block's Lemma

An *automorphism* of the incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a mapping g of $\mathcal{P} \cup \mathcal{B}$ such that g defines permutations on $\mathcal{P}$ and $\mathcal{B}$ such that $P\mathrm{I}B$ if and only if $P^g\mathrm{I}B^g$. The group of all automorphisms of $\mathcal{I}$ is denoted by $\mathrm{Aut}\,\mathcal{I}$.

A *decomposition* of an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a pair $(\mathcal{X}, \mathcal{Y})$, with $\mathcal{X}$ a partition of $\mathcal{P}$ and $\mathcal{Y}$ a partition of $\mathcal{B}$. A decomposition $(\mathcal{X}, \mathcal{Y})$ of an incidence structure with finite block sizes is *block-tactical* if

$$|\{P \in X : P\,\mathrm{I}\,B_1\}| = |\{P \in X : P\,\mathrm{I}\,B_2\}|,$$

for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, $B_1, B_2 \in Y$. An example of block tactical decomposition of an incidence structure $\mathcal{I}$ is obtained by taking the orbits on points and blocks of a subgroup of $\mathrm{Aut}\,\mathcal{I}$.

With a decomposition $(\mathcal{X}, \mathcal{Y})$ of $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ we associate the following subspaces of the point space $\mathbb{Q}^{\mathcal{P}}$ and the block space $\mathbb{Q}^{\mathcal{B}}$ of $\mathcal{I}$: the *point class space* $V_{\mathcal{X}}$ of all functions on $\mathcal{P}$ constant on each $X \in \mathcal{X}$, and the *block class space* $V_{\mathcal{Y}}$ of all functions on $\mathcal{B}$ constant on each $Y \in \mathcal{Y}$.

Lemma 4.1. *A decomposition $(\mathcal{X}, \mathcal{Y})$ of an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ with finite block sizes and incidence map α is block-tactical if and only if $V_{\mathcal{X}}\alpha \subseteq V_{\mathcal{Y}}$.*

Proof. Suppose $(\mathcal{X}, \mathcal{Y})$ is block-tactical and $f \in V_{\mathcal{X}}$. For each $X \in \mathcal{X}$, let P_X be a fixed chosen point in X. As f is constant on X, then $f(P) = f(P_X)$ for all $P \in X$. Let $Y \in \mathcal{Y}$

and $B_1, B_2 \in Y$. Then $|\{Q \in X : Q\mathrm{I}B_1\}| = |\{Q \in X : Q\mathrm{I}B_2\}|$ and therefore

$$\begin{aligned}
(f\alpha)(B_1) &= \sum_{P\,\mathrm{I}\,B_1} f(P) = \sum_{X\in\mathcal{X}} \sum_{\substack{P\in X\\ P\mathrm{I}B_1}} f(P) \\
&= \sum_{X\in\mathcal{X}} |\{Q \in X : Q\mathrm{I}B_1\}| f(P_X) \\
&= \sum_{X\in\mathcal{X}} |\{Q \in X : Q\mathrm{I}B_2\}| f(P_X) \\
&= \sum_{X\in\mathcal{X}} \sum_{\substack{P\in X\\ P\mathrm{I}B_2}} f(P) = \sum_{P\mathrm{I}B_2} f(P) = (f\alpha)(B_2).
\end{aligned}$$

Hence $f\alpha$ is constant on Y. So $f\alpha \in V_{\mathcal{Y}}$, giving $V_{\mathcal{X}}\alpha \subseteq V_{\mathcal{Y}}$.

Conversely, suppose that $V_{\mathcal{X}}\alpha \subseteq V_{\mathcal{Y}}$. Let $X \in \mathcal{X}$ and $\chi_X \in \mathbb{Q}^{\mathcal{P}}$ denote the characteristic function of X. Then, χ_X can be naturally considered as an element of $V_{\mathcal{X}}$, thus $\chi_X\alpha \in V_{\mathcal{Y}}$ by hypothesis. Therefore, we have

$$|\{P \in X : P\mathrm{I}B_1\}| = (\chi_X\alpha)(B_1) = (\chi_X\alpha)(B_2) = |\{P \in X : P\mathrm{I}B_2\}|,$$

for each $Y \in \mathcal{Y}$ and $B_1, B_2 \in \mathcal{Y}$. Hence $(\mathcal{X}, \mathcal{Y})$ is block-tactical. □

The following result is a slight extension of a fundamental result of R. E. Block [2, Theorem 2.1] often known as “Block’s Lemma”.

Lemma 4.2. *Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be an incidence structure with finite block sizes and $(\mathcal{X}, \mathcal{Y})$ a block-tactical decomposition of $\mathcal{I}$. Let α denote the incidence map of $\mathcal{I}$. Then*

$$\dim V_{\mathcal{X}} \le \dim V_{\mathcal{Y}} + \dim(\ker\alpha).$$

Proof. By Lemma 4.1, we have $V_{\mathcal{X}}\alpha \subseteq V_{\mathcal{Y}}$, so $\dim(V_{\mathcal{X}}\alpha) \le \dim V_{\mathcal{Y}}$. Now $\dim V_{\mathcal{X}} = \dim(V_{\mathcal{X}}\alpha) + \dim(V_{\mathcal{X}} \cap \ker\alpha) \le \dim V_{\mathcal{Y}} + \dim(\ker\alpha)$. □

Theorem 4.3. *Let G be one of the following groups:*

(i) *a permutation group of finite degree d;*

(ii) *a group of collineations of $\mathrm{PG}(d, q)$, $d < \infty$;*

(iii) *a group of affine collineations of $\mathrm{AG}(d, q)$, $d < \infty$;*

(iv) *a group of semi-linear isometries of a classical polar space of finite rank d over a finite field.*

For any given non-negative integer $i < d$, let n_i denote the number of orbits on i-sets for (i), on subspaces of dimension i for (ii), on flats of dimension i for (iii), on totally isotropic subspaces (or totally singular in case of a orthogonal space) of dimension i for (iv). Then $n_i \le n_j$, for $i < j$ and $i + j < d$.

Proof. Let $\mathcal{X}_i$ be the set of the orbits of G on the corresponding family of objects indexed by i. For any $i < j < d$, put $(\mathcal{X}, \mathcal{Y}) = (\mathcal{X}_i, \mathcal{X}_j)$. The set of all characteristic functions χ_X, $X \in \mathcal{X}$, is a basis for $V_{\mathcal{X}}$. Hence, $\dim V_{\mathcal{X}} = |\mathcal{X}| = n_i$. Similarly, $\dim V_{\mathcal{Y}} = |\mathcal{Y}| = n_j$, and Lemma 4.2 gives $|\mathcal{X}| \leq |\mathcal{Y}| + \dim(\ker \alpha)$ since the point- and block-orbits of any subgroup of the full automorphism group of an incidence structure form a block-tactical decomposition. The result is obtained by applying Theorems 2.7 and 3.3. ☐

The following is the infinite version of the previous result.

Theorem 4.4. *Let G be one of the following groups:*

(i) *a permutation group of infinite degree;*

(ii) *a group of collineations of a projective space of infinite dimension over a finite field;*

(iii) *a group of affine collineations of an affine space of infinite dimension over a finite field;*

(iv) *a group of semi-linear isometries of a classical polar space of infinite rank over a finite field.*

For any given non-negative integer i, let n_i denote the number of orbits on i-sets for (i), on subspaces of dimension i for (ii), on flats of dimension i for (iii), on totally isotropic subspaces (or totally singular in case of a orthogonal space) of dimension i for (iv). Let l be the least index such that n_l is infinite. Then $n_0 \leq n_1 \leq \cdots \leq n_{l-1}$ and n_k is infinite for all $k \geq l$.

Proof. Let $\mathcal{X}_i$ be the set of the orbits of G on the corresponding family of objects indexed by i.

Let $i < j \leq l - 1$. We apply very similar arguments as in the proof of Theorem 4.3 to the block-tactical decomposition $(\mathcal{X}, \mathcal{Y}) = (\mathcal{X}_i, \mathcal{X}_j)$. Then Theorems 3.2 and 3.3 give $n_i \leq n_j$.

Let $l \leq i < j < \infty$. Since the incidence map of the incidence structure associated with $(\mathcal{X}_i, \mathcal{X}_j)$ has trivial kernel by Theorems 3.2 and 3.3, we may apply Proposition 2.1 in [9] (where ρ is the incidence relation). ☐

Remark 4.5. Theorem 4.4 (i) is due to Cameron [9, Theorem 2.2].

Remark 4.6. By using the Generalized Continuum Hypothesis it is possible to give a slight improvement of the previous result when n_i and n_j, $i < j$, are infinite.

From Lemma 4.2 we get $\dim V_{\mathcal{X}_i} \leq \dim V_{\mathcal{X}_j} + \dim(\ker \alpha)$, and it is known that $\dim V = |V|$ when V is an infinite dimensional vector space over an infinite field F such that $|V| > |F|$.

Set $n_i = \aleph_{\beta_i}$, $\beta_i \geq 0$. Thus, $|V_{\mathcal{X}_i}| = |\mathbb{Q}^{\mathcal{X}_i}| = \aleph_0^{\aleph_{\beta_i}} = \aleph_{\beta_i+1} = 2^{\aleph_{\beta_i}} > \aleph_0 = |\mathbb{Q}|$ by the Generalized Continuum Hypothesis. Therefore, $\dim V_{\mathcal{X}_i} = 2^{\aleph_{\beta_i}}$, and similarly, $\dim V_{\mathcal{X}_j} = 2^{\aleph_{\beta_j}}$. Hence Lemma 4.2 yields

$$2^{\aleph_{\beta_i}} \leq 2^{\aleph_{\beta_j}} + \dim(\ker \alpha).$$

Theorems 3.2 and 3.3 yield $2^{\aleph_{\beta_i}} \leq 2^{\aleph_{\beta_j}}$, and the Generalized Continuum Hypothesis implies $\aleph_{\beta_i} \leq \aleph_{\beta_j}$, that is $n_i \leq n_j$.

Remark 4.7. In the paper [18], examples of infinite Desarguesian projective planes with collineation groups having three orbits on points and two on lines are provided, solving a problem posed by Cameron [10] and attributed to Kantor.

5 Incidence structures and permutation representations

Block's Lemma leads to consideration of $\ker\alpha$. It is particularly nice when $\ker\alpha$ is trivial, and the following lemma also emphasizes this case.

Lemma 5.1. *Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a finite incidence structure whose incidence map is one-to-one. For any given automorphism group G of $\mathcal{I}$ the permutation representation of G on $\mathcal{P}$ is a subrepresentation of the permutation representation of G on $\mathcal{B}$ (considered as linear representation over a field of characteristic zero).*

Proof. The point space $\mathbb{Q}^{\mathcal{P}}$ is the permutation $\mathbb{Q}$-module for G on $\mathcal{P}$, and the block space $\mathbb{Q}^{\mathcal{B}}$ is the permutation $\mathbb{Q}$-module for G on $\mathcal{B}$. Since G preserves the incidence, we have

$$(f^g\alpha)(B) = \sum_{P\,\mathrm{I}\,B} f^g(P) = \sum_{P\,\mathrm{I}\,B} f(P^{g^{-1}}) = \sum_{P\,\mathrm{I}\,B^{g^{-1}}} f(P) = (f\alpha)(B^{g^{-1}}) = (f\alpha)^g(B),$$

for all $f \in \mathbb{Q}^{\mathcal{P}}$ and $g \in G$. Therefore, α is a $\mathbb{Q}G$-homomorphism from $\mathbb{Q}^{\mathcal{P}}$ to $\mathbb{Q}^{\mathcal{B}}$. As α is one-to-one, the permutation representation of G on $\mathcal{P}$ is a subrepresentation of the permutation representation of G on $\mathcal{B}$ (over $\mathbb{Q}$). For other fields of characteristic zero, we need only tensor up. □

Lemma 5.2. *Let G be a group acting as a transitive permutation group on a finite set X of size n. Let S be a subset of G such that $\sum_{s\in S} s$ is mapped to the 0-matrix under every irreducible non-principal representation. Then $|X|$ divides $|S|$.*

Proof. Let $\{x_1, \dots, x_n\}$ be the natural basis of the permutation $\mathbb{Q}G$-module on X. The matrix representation with respect this basis of any element $s \in G$ on the trivial module is $1/|X|J$, where J is the all-one $n \times n$ matrix. This implies that the matrix representation of the endomorphism $\sum_{s\in S} s$ on the trivial module is $|S|/|X|J$.

On the other hand, the matrix representation of $\sum_{s\in S} s$ in the basis $\{x_1, \dots, x_n\}$ is $P_S = \sum_{s\in S} P(s)$, where $P(s)$ is the permutation matrix representing $s \in G$. Note that the entries in P_S are positive integers. Since $\sum_{s\in S} s$ is mapped to the 0-matrix under every irreducible non-principal representation, we have $P_S = |S|/|X|J$. The result then follows. □

Theorem 5.3 ([19]). *Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a finite incidence structure with incidence map one-to-one. If the automorphism group of $\mathcal{I}$ contains a subset which is sharply transitive on blocks, then $|\mathcal{P}|$ divides $|\mathcal{B}|$.*

Proof. Set $G = \operatorname{Aut}\mathcal{I}$ and $S \subset G$ be sharply transitive on blocks. Hence, $|S| = |\mathcal{B}|$. By [19, Lemma 1], the endomorphism $\sum_{s\in S} s$ of the permutation $\mathbb{Q}G$-module $\mathbb{Q}^{\mathcal{B}}$ on blocks is mapped to the 0-matrix under every irreducible non-principal representation. By Lemma 5.1, every irreducible submodule of $\mathbb{Q}^{\mathcal{P}}$ is a submodule of $\mathbb{Q}^{\mathcal{B}}$ with less or equal multiplicity. Hence, $\sum_{s\in S} s$ acting on $\mathbb{Q}^{\mathcal{P}}$ is mapped to the 0-matrix under every irreducible non-principal representation in $\mathbb{Q}^{\mathcal{P}}$. By Lemma 5.2, we have $|\mathcal{P}|$ divides $|\mathcal{B}|$. □

Corollary 5.4. *Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a finite incidence structure with incidence map one-to-one and automorphism group G acting transitively on blocks. If $|\mathcal{P}|$ does not divide $|\mathcal{B}|$, then G does not contain a subset acting sharply transitive on blocks.*

The above result can be restated as follows.

Corollary 5.5. *Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a finite incidence structure with incidence incidence map one-to-one and automorphism group $\operatorname{Aut} \mathcal{I}$ acting transitively on blocks. Let H denote the one-block stabilizer in $\operatorname{Aut} \mathcal{I}$. If $|\mathcal{P}|$ does not divide $|\mathcal{B}|$, then the permutation representation of $\operatorname{Aut} \mathcal{I}$ on the cosets of H contains no sharply transitive subset.*

Remark 5.6. Corollary 5.4 applies to the following incidence structures as their incidence map is one-to-one: combinatorial designs, linear spaces and circular spaces (see [6]); incidence structures in projective and affine spaces (see [14] and Theorem 2.7); incidence structures in classical polar spaces (see [16] and Theorem 3.3); incidence structures on subsets ([9, 14, 15, 20] and Theorem 2.7); nonbipartite graphs (see [21]).

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