ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA
https://doi.org/10.26493/1855-3974.1988.220
(Also available at http://amc-journal.eu)

The cubical matching complex revisited*

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Received 15 April 2019, accepted 29 April 2021

Abstract

Ehrenborg noted that all tilings of a bipartite planar graph are encoded by its cubical matching complex and claimed that this complex is collapsible. We point out to an oversight in his proof and explain why these complexes can be the disjoint union of two or more collapsible complexes. We also prove that all links in these complexes are suspensions up to homotopy. Furthermore, we extend the definition of a cubical matching complex to planar graphs that are not necessarily bipartite, and show that these complexes are either contractible or a disjoint union of contractible complexes. For a simple connected region that can be tiled with dominoes (2×1 and 1×2) and 2×2 squares, let f_i denote the number of tilings with exactly i squares. We prove that $f_0 - f_1 + f_2 - f_3 + \cdots = 1$ (established by Ehrenborg) is the only linear relation for the numbers f_i.

Keywords: Domino tilings, independence complexes, matching, cubical complexes.

Math. Subj. Class. (2020): 52C20, 05C70

1 Introduction

Let $G = (V, E)$ be a bipartite planar graph that admits a perfect matching. Assume that G is embedded in the plane. This embedding splits the plane into the *regions*, the connected components of $\mathbb{R}^2 \setminus |G|$ (here $|G|$ denotes the embedding of G into $\mathbb{R}^2$). An *elementary cycle* of G is a cycle that encircles a single region R different from the outer region R^*. Throughout this paper, we identify an elementary cycle with the region it encircles as well as with its set of vertices or edges.

A *tiling* of G is a partition of the vertex set V into disjoint blocks of the following two types:

(1) an edge $\{x, y\}$ of G; or

*We are very grateful to the referee for numerous remarks and suggestions which considerably improved the presentation of results in the paper.

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(2) an elementary cycle R (the set of vertices of R).

The set of all tilings of G form a cubical complex $\mathcal{C}(G)$ (called the *cubical matching complex*) defined by Ehrenborg in [6]. Note that $\mathcal{C}(G)$ depends not only on G, but also on the choice of the embedding of that graph in the plane.

A face F of $\mathcal{C}(G)$ has the form $F = M_F \cup C_F = (M_F, C_F)$, where C_F is a collection $C_F = \{R_1, R_2, \dots, R_t\}$ of vertex-disjoint elementary cycles of G, and M_F is a perfect matching on $G \setminus \big(R_1 \cup R_2 \cup \dots \cup R_t\big)$. The dimension of F is $|C_F|$, and the vertices of $\mathcal{C}(G)$ are the perfect matchings of G.

All tilings of G covered by $F = (M_F, C_F)$ can be obtained by deleting an elementary cycle R from C_F, and adding every other edge of R into M_F (there are two possibilities to do this). Therefore, for two faces $F_1 = (M_{F_1}, C_{F_1})$ and $F_2 = (M_{F_2}, C_{F_2})$, we have that

$$\big(F_1 \subset F_2\big) \iff \Big(C_{F_1} \subset C_{F_2} \text{ and } M_{F_1} \supset M_{F_2}\Big). \tag{1.1}$$

Let G° denote the weak dual graph of a planar graph G. The vertices of G° are all bounded regions of G, and two regions that share a common edge are adjacent in G°.

The *independence complex* of a graph H is a simplicial complex $I(H)$ whose faces are the independent subsets of vertices of H. Note that for any face $F = (M_F, C_F)$ of $\mathcal{C}(G)$, the set C_F contains independent vertices of G°, i.e., C_F is a face of $I(G^\circ)$.

Since two elementary cycles of G sharing a common edge cannot be in a common face of $\mathcal{C}(G)$, it may seem at the first glance that $\mathcal{C}(G)$ can be computed from the independence complex $I(G^\circ)$.

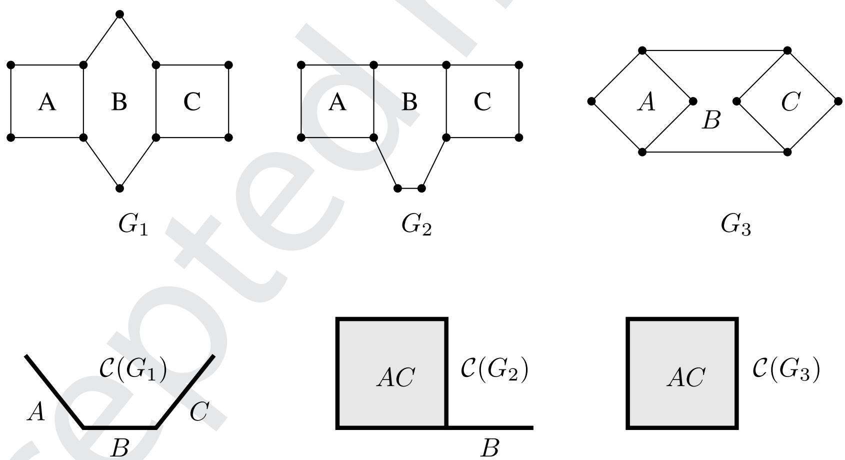


Figure 1: The three graphs with the same weak dual, but different cubical matching complexes.

However, Figure 1 shows three graphs with the same weak dual but different cubical matching complexes. The facets of the complexes on Figure 1 are labeled by corresponding subsets of pairwise disjoint elementary regions. This example points out that the requirement that $G \setminus (R_1 \cup \dots \cup R_t)$ admits a perfect matching is a substantial one.

Example 1.1. Let $\mathcal{L}_n$ and $\mathcal{C}_n$ denote the independence complexes of P_n and C_n (the path and cycle with n vertices) respectively. The homotopy types of these complexes are determined by Kozlov in [10]:

$$\mathcal{L}_n \simeq \begin{cases} \text{a point}, & \text{if } n = 3k+1; \\ S^{\lfloor \frac{n-1}{3} \rfloor}, & \text{otherwise.} \end{cases} \quad \mathcal{C}_n \simeq \begin{cases} S^{k-1}, & \text{if } n = 3k \pm 1; \\ S^{k-1} \vee S^{k-1}, & \text{if } n = 3k. \end{cases}$$

We will use these complexes later, see Corollary 2.2 and Remark 2.5. An interested reader can find more details about combinatorial and topological properties of $\mathcal{L}_n$ and $\mathcal{C}_n$ (and about independence complexes in general) in [7], [8] and [9].

There are some cubical complexes that cannot be realized as subcomplexes of the d-cube $C^d = [0,1]^d$, see Chapter 4 of [5].

Proposition 1.2. *Let G be a bipartite planar graph that has a perfect matching. If G has d elementary regions, then its cubical matching complex $\mathcal{C}(G)$ can be embedded into C^d.*

Proof. We use an idea from [11] to describe the coordinates of vertices of $\mathcal{C}(G)$ explicitly. Let $R_1, R_2, \ldots, R_d$ be a fixed linear order of elementary regions of G. We choose an arbitrary perfect matching M_0 of G (a vertex of $\mathcal{C}(G)$) to be the origin $\mathbf{0} = (0, 0, \ldots, 0)$ in $\mathbb{R}^d$. For another vertex M of $\mathcal{C}(G)$ the symmetric difference $M \bigtriangleup M_0$ is a disjoint union of cycles. Now, to a given perfect matching M of G, we assign the vertex $V_M = (x_1, \ldots, x_d)$ of C^d, where

$$x_i = \begin{cases} 1, & \text{if } R_i \text{ is contained in an odd number of cycles of } M \bigtriangleup M_0; \\ 0, & \text{otherwise.} \end{cases}$$

If M' and M'' are two perfect matchings of G such that $M' \bigtriangleup M'' = R_j$ (meaning that these two matchings differ just on an elementary region R_j), then their corresponding vertices $V_{M'}$ and $V_{M''}$ of C^d differ only at the j-th coordinate.

Therefore, all 1-dimensional faces of $\mathcal{C}(G)$ that correspond to the same region R_i are mutually parallel edges of C^d. The face $F = (M_F, C_F)$ of $\mathcal{C}(G)$ is embedded in C^d as the convex hull of its $2^{|C(F)|}$ vertices. □

2 The local structure of $\mathcal{C}(G)$

The *star* of a face F in a cubical complex $\mathcal{C}$ is the set of all faces of $\mathcal{C}$ that contain F

$$\operatorname{star}(F) = \{F' \in \mathcal{C} : F \subset F'\}.$$

The *link* of a vertex v in a cubical complex $\mathcal{C}$ is the simplicial complex $\operatorname{link}_{\mathcal{C}}(v)$ that can be realized in $\mathcal{C}$ as a "small sphere" around the vertex v. More formally, the vertices of $\operatorname{link}_{\mathcal{C}}(v)$ are the edges of $\mathcal{C}$ containing v. A subset of vertices of $\operatorname{link}_{\mathcal{C}}(v)$ is a face of $\operatorname{link}_{\mathcal{C}}(v)$ if and only if the corresponding edges belong to a common face of $\mathcal{C}$.

The *link* of a face F in a cubical complex $\mathcal{C}$ is defined in a similar way. The set of vertices of $\operatorname{link}_{\mathcal{C}}(F)$ is

$$\{F' \in \mathcal{C} : F \subset F' \text{ and } \dim F' = 1 + \dim F\},$$

and a subset A of the set of vertices is a face of $link_{\mathcal{C}}(F)$ if and only if all elements of A are contained in a same face of $\mathcal{C}$.

Ehrenborg investigated the links of the cubical complexes associated to tilings of a region by dominos or lozenges.

Here we describe the links in the cubical matching complex $\mathcal{C}(G)$ for any bipartite planar graph G. For a face $F = (M_F, C_F)$ of $\mathcal{C}(G)$, let $\mathcal{R}_F$ denote the set of all elementary regions of G for which every second edge is contained in M_F. Further, let G_F denote the subgraph of the weak dual graph G° induced with the regions from $\mathcal{R}_F$.

From the definition of the link in a cubical complex and (1.1), we obtain the next statement.

Proposition 2.1. *For any face $F = (M_F, C_F)$ of $\mathcal{C}(G)$ we have that*

$$\mathrm{link}_{\mathcal{C}}(F) \cong I(G_F).$$

The above proposition explains the appearance of complexes $\mathcal{L}_n$ and $\mathcal{C}_n$ as links in cubical the matching complexes, see Theorem 3.3 and Section 4 in [6].

Assume that all elementary regions of G are quadrilaterals. In that case, for any face F of $\mathcal{C}(G)$, a region in $\mathcal{R}_F$ has exactly two opposite edges in M_F, and the degree of a vertex in G_F is at most two. Therefore, G_F is a disjoint union of paths and cycles. If the regions (quadrilaterals) $R_1, R_2, \dots, R_t$ are vertices of a cycle in G_F, then the edges of these regions that are not in M_F form two cycles of length t in G. As G is bipartite, the length of any cycle in G_F is even.

Corollary 2.2. *If all elementary regions of G are quadrilaterals, then $\mathrm{link}_{\mathcal{C}}(F)$ is a join of complexes $\mathcal{L}_p$ and $\mathcal{C}_{2q}$.*

Theorem 2.3. *Let G be a bipartite planar graph that has a perfect matching. For any face $F = (M_F, C_F)$ of $\mathcal{C}(G)$ the graph G_F is bipartite.*

Proof. Assume that G_F contains an odd cycle $R_1, R_2, \dots, R_{2m+1}$. Recall that R_i is an elementary region of G and the that every second edge of R_i is contained in M_F. Two neighborly regions R_{i-1} and R_i have to share the odd number of edges, the first and the last of their common edges belong to M_F. Therefore, for each region R_i, there is an odd number of common edges of R_i and R_{i-1} that belong to M_F. Obviously, the same holds for R_i and R_{i+1}.

So, we can conclude that there is an odd number of edges of R_i that are between $R_i \cap R_{i-1}$ and $R_i \cap R_{i+1}$ (the first and the last one of these edges are not in M_F). Therefore, the union of these edges (for all regions R_i) forms an odd cycle in G. This is a contradiction with the assumption that G is a bipartite graph. □

Barmak proved in [1] (see also in [12]) that the independence complexes of bipartite graphs are suspensions up to homotopy type. This implies the next result.

Corollary 2.4. *All links in $\mathcal{C}(G)$ are homotopy equivalent to suspensions. Therefore, the link of any face in $\mathcal{C}(G)$ has at most two connected components.*

For any simplicial complex K there exists a bipartite graph G such that the independence complex of G is homotopy equivalent to the suspension over K, see [1]. Skwarski proved in [13] (see also [1]) that there exists a planar graph G whose independence complex is homotopy equivalent to an iterated suspension of K.

We prove that the links of faces in cubical matching complexes are independence complexes of bipartite planar graphs. What can be said about homotopy types of these complexes?

Remark 2.5. There is a natural question, posed by Ehrenborg in [6]: *For what graphs G would the cubical matching complex $\mathcal{C}(G)$ be pure and/or shellable?* The complexes $\mathcal{L}_n$ are non-pure for $n > 4$, and the complexes $\mathcal{C}_n$ are non-shellable for $n > 5$. Therefore, these complexes can be used to show that the cubical matching complex of a concrete graph is non-pure or non-shellable.

3 Collapsibility and contractibility of cubical matching complexes

The next statement that we discuss was the main result of [6]. We identify a problem with the proof, describe counterexamples (infinitely many), recover a weaker result, and give a generalization.

Theorem 3.1 (Theorem 1.2 in [6]). *For a planar bipartite graph G that has a perfect matching, the cubical matching complex $\mathcal{C}(G)$ is collapsible.*

The proof of the above statement is based on the following two results:

(i) (Propp, Theorem 2 in [11]) *The set of all perfect matchings of a bipartite planar graph is a distributive lattice (under a certain ordering, the details of which may be found in [11]).*

(ii) (Kalai, see in [14], Solution to Exercise 3.47 c) *The cubical complex associated (see [14]) to a meet-distributive lattice is collapsible.*

Note however that Propp in his proof of (i) assumed the following two additional conditions for bipartite planar graph G:

$(*)$ Graph G is connected, and

$(**)$ Any edge of G is contained in some matching of G but not in others.

The next statement is the correct version of Theorem 3.1.

Theorem 3.2. *For a connected planar bipartite graph G that has a perfect matching and whose any edge is contained in some matching of G but not in others, the cubical matching complex $\mathcal{C}(G)$ is collapsible.*

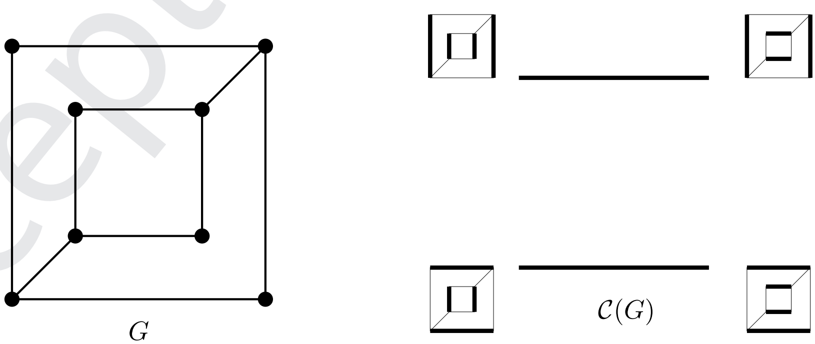


Figure 2: A bipartite planar graph G for which $\mathcal{C}(G)$ is not collapsible.

Example 3.3. The above figure shows a bipartite planar graph whose cubical matching complex is not collapsible. Note that the subdividing of the edges between the inner and outer quadrilaterals in Figure 2 gives us an infinite family of counterexamples for Theorem 3.1. Also, we can use this example to obtain a graph whose cubical complex has arbitrarily many connected components. Simply, we continue by inserting a new square into the smallest quadrilateral of the already constructed graph, and connect two non-adjacent vertices of the new square with the corresponding vertices of the old graph. This counterexample is motivated by the Jockusch example (page 27 in [11]). In his example we find a bipartite planar graph with 20 edges, but just 12 of them can be used in a perfect matching, and its cubical matching complex is a disjoint union of four segments.

Now we prove a weaker version of Theorem 3.1.

Theorem 3.4. *For a planar bipartite graph G that has a perfect matching, the cubical matching complex $\mathcal{C}(G)$ is either collapsible or a disjoint union of collapsible complexes.*

The proof will be established in a series of lemmas. Through these lemmas we assume that G is a planar bipartite graph that has a perfect matching.

The edges that do not appear in any perfect matching of G (the forbidden edges) can be deleted. Also, if the edge xy is a forced edge (xy appears in all perfect matching of G), then we may consider the graph $G - \{x, y\}$.

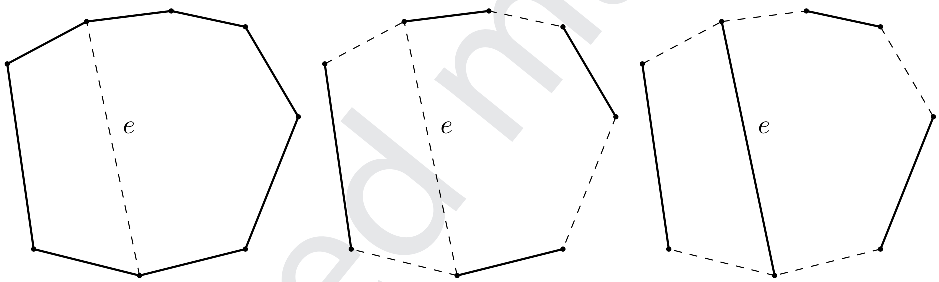


Figure 3: If a new region can be included in a tiling of $G - e$, then e is not forbidden.

Lemma 3.5. *Let e denote a forbidden edge in G and let $G' = G - e$. The possible new elementary region of G', that appears after we delete e, can not be included in a tiling of G'.*

Proof. Assume that a new region R that contains e can be included in a tiling of G'. Then e divides R into two regions of the old graph G, and we can find a perfect matching of G that contains e, see Figure 3. □

In a similar way we may prove that a new region appearing after deleting the vertices of a forced edge can not be included in a tiling of new graph.

Corollary 3.6. *Let $\overline{G}$ denote the graph obtained by deleting all forced and forbidden edges from G. Then the cubical matching complexes of $\overline{G}$ and G are isomorphic.*

Lemma 3.7. *Assume that G is a not connected, and let $G_1, G_2, \dots, G_k$ be the connected components of G. If these components are separated (there is no component of G that is contained in an elementary region of another component) then $\mathcal{C}(G) \cong \mathcal{C}(G_1) \times \mathcal{C}(G_2) \times \cdots \times \mathcal{C}(G_k)$.*

Proof. For a tiling $F = (M_F, C_F)$ of G let $F_i = (M_i, C_i)$ denote the corresponding face of $C(G_i)$ (i.e., $M_i = M_F \cap E(G_i)$ and C_i is the set of regions of G_i that are included into C_F). Then we have that $F \cong F_1 \times F_2 \times \cdots \times F_k$. □

Lemma 3.8. *Assume that G has two different connected components G_1 and G_2 such that G_1 is contained in an elementary region R of G_2. Then we have that*

$$\mathcal{C}(G) = \mathcal{C}(G_1) \times (\mathcal{C}(G_2) \setminus \{R\}) . \tag{3.1}$$

If there exists a tiling of G_2 that uses the region R, then $\mathcal{C}(G)$ is a disjoint union of collapsible complexes.

Here $\mathcal{C}(G_2) \setminus \{R\}$ denotes the cubical complex obtained from $\mathcal{C}(G_2)$ by deleting the faces (tilings) that contain R.

Proof. The proof of (3.1) is the same as in the previous lemma. Recall that $\mathcal{C}(G_2)$ can be embedded in a cube, and that the edges corresponding to R are mutually parallel, see Proposition 1.2. Therefore, $\mathcal{C}(G_2) \setminus \{R\}$ is a disjoint union of collapsible complexes. □

Now, we consider the cubical matching complex for all planar graphs that have a perfect matching (but that are not necessarily bipartite).

Definition 3.9. Let G be a planar graph that admits a perfect matching. A tiling of G is a partition of the vertex set V into disjoint blocks of the following two types:

- an edge $\{x, y\}$ of G; or
- the set of vertices $\{v_1, v_2, \dots, v_{2m}\}$ of an even elementary cycle R.

Let $\mathcal{C}(G)$ denote the set of all tilings of G. Note that $\mathcal{C}(G)$ is also a cubical complex.

Example 3.10. If G is a graph of a triangular prism (embedded in the plane so that the outer region is a triangle), then $\mathcal{C}(G)$ is a union of three 1-dimensional segments that share the same vertex, see the left side of Figure 4. Each segment of $\mathcal{C}(G)$ corresponds to a rectangle of prism. The link of the common vertex of these segments is a 0-dimensional complex with three points. This situation, where a link has 3 connected components, is not possible in a bipartite planar graph, as shown by Corollary 2.4. Further, the planar graph on the right-hand side on Figure 4 satisfies the conditions $(*)$ and $(**)$, but the corresponding cubical complex is not collapsible, it is a union of three disjoint edges. Therefore, the assumption that G is a bipartite graph is substantial in Theorem 3.2.

The next theorem describe the homotopy type of the cubical matching complex associated to a planar graph that admits a perfect matching.

Theorem 3.11. *Let G be a planar graph that has a perfect matching. The cubical complex $\mathcal{C}(G)$ is contractible or a disjoint union of contractible complexes.*

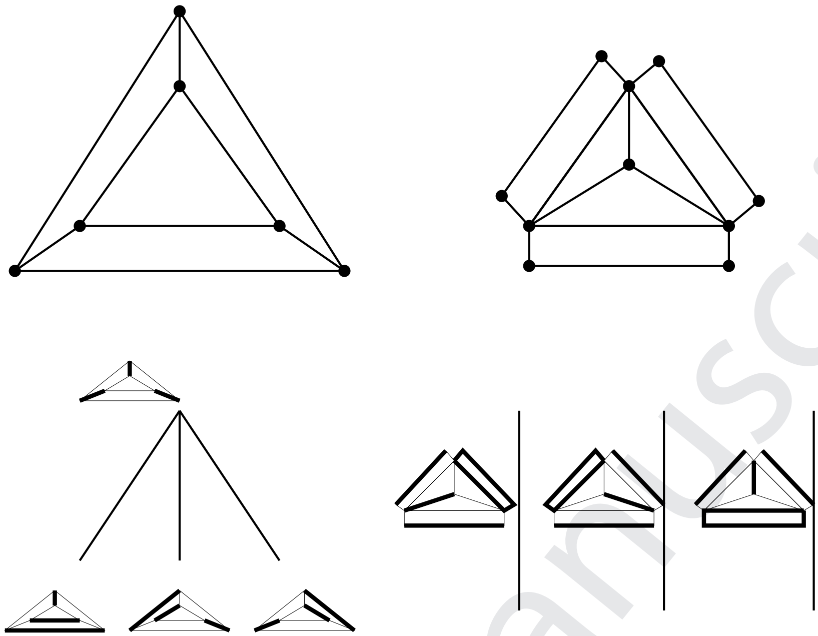


Figure 4: Non-bipartite graphs and their cubical matching complexes.

While contractibility is weaker than collapsibility, we partly relax the bipartite condition and obtain a weaker version of a corrected Theorem 3.1, with a different proof.

Proof. We use induction on the number of edges of G. Let $e = xy$ denote an edge that belongs to the outer region R^*. Let $R \neq R^*$ denote the elementary region that contains e. If R is an odd region, then all tilings of G can be divided into two disjoint classes:

(a) The tilings of G that do not use e. These tilings are just the tilings of $G \setminus e$.

(b) The tilings of G that contain e as an edge in a partial matching correspond to the tilings of $G \setminus \{x, y\}$, and the subcomplex of $\mathcal{C}(G)$ generated by these tilings is isomorphic to $\mathcal{C}(G \setminus \{x, y\})$.

In that case we obtain that $\mathcal{C}(G) = \mathcal{C}(G\setminus\{x, y\}) \cup \mathcal{C}(G\setminus e)$ is a disjoint union of contractible complexes by inductive assumption.

If R is an even elementary region, then some tilings of G may contain R, and these tilings are not considered in (a) and (b). Note that there is a bijection between tilings of G that contain R and all tilings of $G \setminus R$ (the graph obtained from G by deleting all vertices from R). The subcomplex of $\mathcal{C}(G)$ generated by tilings that contain R forms a prism over $\mathcal{C}(G \setminus R)$, i.e., this subcomplex is isomorphic to $Prism(\mathcal{C}(G \setminus R)) = \mathcal{C}(G \setminus R) \times [0, 1]$. Therefore, we obtain that

$$\mathcal{C}(G) = \mathcal{C}(G \setminus \{x, y\}) \cup \mathcal{C}(G \setminus e) \cup Prism(\mathcal{C}(G \setminus R)). \tag{3.2}$$

Let $\mathcal{C}_e$ denote the subcomplex of $\mathcal{C}(G\setminus e)$ formed by all tilings that contain every second edge of R (but do not contain e, obviously). Further, let $\mathcal{C}_{x,y}$ denote the subcomplex of

$\mathcal{C}(G \setminus \{x,y\})$, defined by tilings that contain every second edge of R (these tilings have to contain e). Note that both of complexes $\mathcal{C}_e$ and $\mathcal{C}_{x,y}$ are isomorphic to $\mathcal{C}(G \setminus R)$, and

$$\mathcal{C}(G \setminus e) \cap Prism(\mathcal{C}(G \setminus R)) = \mathcal{C}_e \text{ and } \mathcal{C}(G \setminus \{x,y\}) \cap Prism(\mathcal{C}(G \setminus R)) = \mathcal{C}_{x,y}.$$

The complexes on the right-hand side of (3.2) are disjoint unions of contractible complexes by the inductive hypothesis. Assume that

$$\mathcal{C}(G \setminus \{x,y\}) = A_1 \cup A_2 \cup \cdots \cup A_s \text{ and } \mathcal{C}_{x,y} = B_1 \cup B_2 \cup \cdots \cup B_t,$$

where A_i and B_j denote the contractible components of corresponding complexes. Obviously, each complex B_j is contained in some A_i. Now, we need the following lemma.

Lemma 3.12. *Each connected component of $\mathcal{C}(G \setminus \{x,y\})$ contains at most one component of $\mathcal{C}_{x,y}$.*

Proof of Lemma: Assume that a component of $\mathcal{C}(G \setminus \{x,y\})$ contains two components of $\mathcal{C}_{x,y}$. In that case, there are two vertices of $\mathcal{C}_{x,y}$ (perfect matchings of G that contain xy) that are in different components of $\mathcal{C}_{x,y}$, but in the same component of $\mathcal{C}(G \setminus \{x,y\})$. Assume that M' and M'' are two such vertices, chosen so that the distance between them in $\mathcal{C}(G \setminus \{x,y\})$ is minimal. Let

$$M' = M_0 \overset{R_0}{—} M_1 — \ldots — M_i \overset{R_i}{—} M_{i+1} — \ldots — M_n \overset{R_n}{—} M_{n+1} = M'' \qquad (3.3)$$

denote the shortest path from M' to M'' in $\mathcal{C}(G \setminus \{x,y\})$. The perfect matching M_{i+1} is obtained from M_i by removing the edges of M_i contained in an elementary region R_i, and by inserting the complementary edges. In other words, we have that $M_{i+1} = M_i \vartriangle R_i$, for an elementary region R_i contained in $\mathcal{R}_{F_i} \cap \mathcal{R}_{F_{i+1}}$.

Note that R_0 must be adjacent (share a common edge) with R. Otherwise, both of vertices M_0 and M_1 belong to the same component of $\mathcal{C}_{x,y}$, and we obtain a contradiction with the assumption that the distance between M' and M'' is minimal.

In a similar way, we obtain that for any $i = 1, 2, \ldots, n$, the region R_i must be adjacent with at least one of regions $R, R_0, R_1, \ldots, R_{i-1}$. If not, we have that the perfect matching $\overline{M} = M_0 \vartriangle R_i$ belongs to $\mathcal{C}_{x,y}$, and $\overline{M}$ and M' are contained in the same component of $\mathcal{C}_{x,y}$. In that case we obtain a contradiction, because the path

$$\overline{M} = \overline{M}_0 \overset{R_0}{—} \overline{M}_1 — \ldots — \overline{M}_{i-1} \overset{R_{i-1}}{—} \overline{M}_{i+1} \overset{R_{i+1}}{—} \ldots — \overline{M}_n \overset{R_n}{—} \overline{M}_{n+1} = M''$$

is shorter than (3.3). Here we let that $\overline{M}_{j+1} = \overline{M}_j \vartriangle R_j$.

Let e' denote a common edge of regions R_0 and R that is contained in M'. Note that e' is not contained in M_1. However, this edge is again contained in M'', and we conclude that the region R_0 has to reappear again in (3.3).

Let $R_{i_0} = R_0$ denote the first appearance of R_0 in (3.3) after the first step. There are the following three possible situations that enable the reappearance of R_0:

(a) The regions $R_1, R_2, \ldots, R_{i_0-1}$ are disjoint with R_0.
In that case, we can omit the steps in (3.3) labelled by R_0 and R_{i_0}, and obtain a shorter path between M' and M''.

(b) Each of regions that shares at least one edge with R_0 appears an odd number of times between R_0 and R_{i_0}.
This is impossible, because R (that share an edge with R_0) can not appear in (3.3).

(c) There is $t < i_0$ such that the region $R_t = \bar{R}$ shares an edge with R_0, but the fragment of the sequence (3.3) between R_0 and R_{i_0} does not contain all region that shares an edge with R_0.
Then the same region $\bar{R}$ has to appear again as R_s, for some s such that $t < s < i_0$. Again, if all regions R_j are disjoint with $\bar{R}$ (for $j = t+1, \dots, s-1$), we can omit R_t and R_s, and obtain a contradiction. If not, there exist indices t' and s' such that $t < t' < s' < s$ and $R'_t = R'_s$. We continue in the same way, and from the finiteness of the path, obtain a shorter path than (3.3). □

Proof of Theorem 3.6, continued:
We built $\mathcal{C}(G)$ by starting with $\mathcal{C}(G \setminus e)$, that is a disjoint union of contractible complexes by assumption. Then we glue the components of $Prism(\mathcal{C}(G \setminus R))$ one by one.

After that, we glue all components of $\mathcal{C}(G \setminus \{x, y\})$. At each step we are gluing two contractible complexes along a contractible subcomplex, or we just add a new contractible complex, disjoint with previously added components. From the Gluing Lemma (see Lemma 10.3 in [4]) we obtain that $\mathcal{C}(G)$ is contractible, or a disjoint union of contractible complexes. □

Corollary 3.13. *If G has two odd elementary regions that share the same edge $e = xy$, then its cubical complex $\mathcal{C}(G) = \mathcal{C}(G \setminus \{x, y\}) \cup \mathcal{C}(G \setminus e)$ has at least two connected components. The same holds if there is an odd elementary region of G that shares an edge with the outer region R^*.*

4 The f-vector of domino tilings

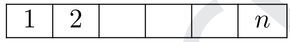
The concept of tilings of a bipartite planar graph generalizes the notion of domino tilings. Let $\mathcal{R}$ be a simple connected region, compound of unit squares in the plane, that can be tiled with domino tiles 1×2 and 2×1. The set of all tilings of $\mathcal{R}$ by domino tiles and 2×2 squares defines a cubical complex, denoted by $\mathcal{C}(\mathcal{R})$. If we consider $\mathcal{R}$ as a planar graph (all of its elementary regions are unit squares), and if G denotes the weak dual graph of $\mathcal{R}$ (the unit squares of $\mathcal{R}$ are vertices of G), then $\mathcal{C}(\mathcal{R})$ is isomorphic to the cubical matching complex $\mathcal{C}(G)$, see Section 3 in [6] for details. Note that the number of i-dimensional faces of $\mathcal{C}(G)$ counts the number of tilings of $\mathcal{R}$ with exactly i squares 2×2.

Ehrenborg used collapsibility of $\mathcal{C}(G)$ to conclude (see Corollary 3.1. in [6]) that the entries of f-vector of $f(\mathcal{C}(G)) = (f_0, f_1, \dots, f_d)$ satisfy

$$f_0 - f_1 + f_2 - \cdots + (-1)^d f_d = 1. \tag{4.1}$$

Let G denote the weak dual graph of a region $\mathcal{R}$ that admits a domino tiling. Choose a concrete perfect matching M of G, and let $e = xy$ denote the edge in M that contains the vertex (square) in the left corner of the top row of $\mathcal{R}$. The complex $\mathcal{C}(G \setminus \{x, y\})$ is nonempty and contractible by induction. The simple connectivity of $\mathcal{R}$ implies that the other two complexes that appear on the right-hand side of the relation (3.2) are either both empty or contractible (by induction). If both of these complexes are nonempty, when we glue them as in the proof of Theorem 3.11, we obtain that $\mathcal{C}(G)$ is contractible. So, we conclude that the relation (4.1) is true in any case, disregarding possible problems with Theorem 3.1.

In this section we will prove that (4.1) is the only linear relation for f-vectors of cubical complexes of domino tilings. We follow the idea from [2], where Bayer and Billera determine the affine span of the flag f-vectors of polytopes by constructing polytopes whose flag f-vectors are affinely independent. Here we describe $d+1$ simple connected regions whose cubical complexes are d-dimensional and their f-vectors are affinely independent.

For all $n \in \mathbb{N}$, we let G_n denote the following graph 

. This graph (also known as the ladder graph) has $2n+2$ vertices, $3n+1$ edges and n elementary regions (squares). For $i = 1, 2, \dots, n$, let $G_{n,i}$ denote the graph obtained by adding one unit square below the i-th square of G_n.

Now, we describe some recursive relations for f-vectors of $\mathcal{C}(G_n)$ and $\mathcal{C}(G_{n,i})$.

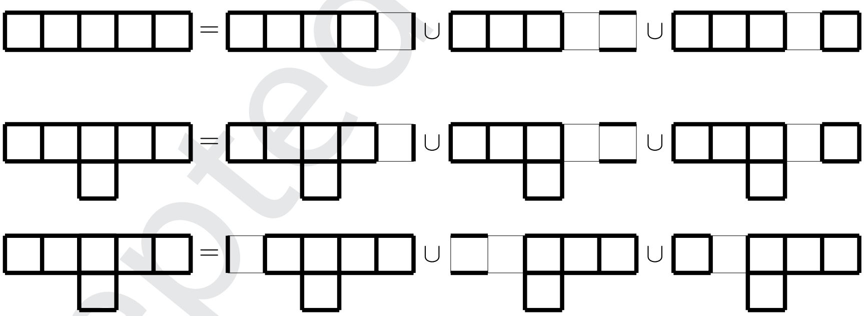
Proposition 4.1. *For all positive integers n the entries of f-vectors of $\mathcal{C}(G_n)$ and $\mathcal{C}(G_{n,i})$ satisfy the following recurrences:*

$$f_i(\mathcal{C}(G_{n+2})) = f_i(\mathcal{C}(G_{n+1})) + f_i(\mathcal{C}(G_n)) + f_{i-1}(\mathcal{C}(G_n)), \tag{4.2}$$

$$f_i(\mathcal{C}(G_{n+2,i})) = f_i(\mathcal{C}(G_{n+1,i})) + f_i(\mathcal{C}(G_{n,i})) + f_{i-1}(\mathcal{C}(G_{n,i})), \tag{4.3}$$

$$f_i(\mathcal{C}(G_{n+2,i})) = f_i(\mathcal{C}(G_{n+1,i-1})) + f_i(\mathcal{C}(G_{n,i-2})) + f_{i-1}(\mathcal{C}(G_{n,i-2})). \tag{4.4}$$

Proof. All formulas follow from relation (3.2), see the proof of Theorem 3.11. To obtain the formula (4.2), we apply (3.2) on G_{n+2}. The rightmost vertical edge and the rightmost unit square in G_{n+2} act as e and R in (3.2), see the first row on the next figure.



(4.2)

(4.3)

(4.4)

Figure 5: The "geometric proof" of recursive relations for $f(\mathcal{C}(G_n))$ and $f(\mathcal{C}(G_{n,i}))$.

In the same way we can prove the remaining two relations. For each relation, we choose an adequate elementary region R, a corresponding edge e of R, and use relation (3.2), see Figure 5. □

The f-vector $(f_0, f_1, f_2, \dots, f_{\lceil \frac{n}{2} \rceil})$ of $\mathcal{C}(G_n)$ can be encoded by the polynomial F_n:

$$F_n = F_{\mathcal{C}(G_n)}(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_{\lceil \frac{n}{2} \rceil} x^{\lceil \frac{n}{2} \rceil}.$$

Similarly, we define the polynomials $F_{n,i}$ to encode the f-vector of $\mathcal{C}(G_{n,i})$. Directly from (4.2) and (4.3) we obtain that

$$F_{n+2}(x) = F_{n+1}(x) + (x+1)F_n(x), \quad F_{n+2,i}(x) = F_{n+1,i}(x) + (x+1)F_{n,i}(x).$$

Now, we define new polynomials P_n and $P_{n,i}$ by

$$P_n = P_n(x) = F_n(x-1), \qquad P_{n,i} = P_{n,i}(x) = F_{n,i}(x-1).$$

This is a variant of h-polynomial associated to a cubical complex.
From Proposition 4.1 it follows that the polynomials P_n and $P_{n,i}$ satisfy the following recurrences

$$P_{n+2}(x) = P_{n+1}(x) + xP_n(x), \tag{4.5}$$

$$P_{n+2,i}(x) = P_{n+1,i}(x) + xP_{n,i}(x), \tag{4.6}$$

$$P_{n+2,i}(x) = P_{n+1,i-1}(x) + xP_{n,i-2}(x). \tag{4.7}$$

Remark 4.2. We can use (4.5) to obtain the polynomials P_n explicitly

$$P_{2d-1} = \binom{d}{d}x^d + \cdots + \binom{d+k}{d-k}x^k + \cdots + \binom{2d-1}{1}x + \binom{2d}{0}, \text{ and}$$

$$P_{2d} = \binom{d+1}{d}x^d + \cdots + \binom{d+k+1}{d-k}x^k + \cdots + \binom{2d}{1}x + \binom{2d+1}{0}.$$

Note that the polynomials P_n are related with Fibonacci polynomials, see Section 9.4 in [3] for the definition and a combinatorial interpretation of coefficients. The coefficients of these polynomials are positive integers and the sum of coefficients of P_n is a Fibonacci number. Note that this is just the number of vertices in $\mathcal{C}(G_n)$.

Assume that we embedded $\mathcal{C}(G_n)$ into n-cube as in Proposition 1.2, so that the perfect matching $M_0 =$  of G_n is the vertex in the origin. Now, the coefficient of x^k in P_n counts the number of vertices of $\mathcal{C}(G_n)$ for which the sum of coordinates is k, i.e., it is the number of vertices of $\mathcal{C}(G_n)$ whose distance from M_0 is k.

Also, following [3], we can recognize the coefficient of x^k in P_n as the number of k-element subsets of $[n]$ that do not contain two consecutive integers. Similarly, we can interpret the coefficient of x^k in $P_{n,i}$ as the number of k-element subsets of the multiset $M = \{1, 2, \dots, i-1, i, i, i+1, \dots, n\}$ that do not contain two consecutive integers. Note that the multiplicity of i in M is two, and all other elements have the multiplicity one.

Definition 4.3. Let $\mathcal{P}^d$ denote the vector space of all polynomials of degree at most d. We define the linear map $A_d : \mathcal{P}^d \to \mathcal{P}^{d+1}$ recursively by

$$A_d(x^k) = xA_{d-1}(x^{k-1}) \text{ for all } k > 0, \tag{4.8}$$

$$A_0(1) = 1 + 2x \text{ and } A_d(1) = P_{2d+1} - A_d(P_{2d-1} - 1). \tag{4.9}$$

Lemma 4.4. *For any non-negative integer d, we have that*

$$A_d(P_{2d-1}) = P_{2d+1}, A_d(P_{2d}) = P_{2d+2} \textit{ and } A_{d+1}(P_{2d}) = P_{2d+2}.$$

Proof. From (4.9) it follows that $A_d(P_{2d-1}) = P_{2d+1}$. For the proof of the second formula we use (4.5), (4.8) and induction

$$A_d(P_{2d}) = A_d(P_{2d-1} + xP_{2d-2}) = P_{2d+1} + xA_{d-1}(P_{2d-2}) = P_{2d+1} + xP_{2d} = P_{2d+2}.$$

The last formula in this lemma follows from (4.5) and earlier proved formulas

$$A_{d+1}(P_{2d}) = A_{d+1}(P_{2d+1} - xP_{2d-1}) =$$
$$P_{2d+3} - xA_d(P_{2d-1}) = P_{2d+3} - xP_{2d+1} = P_{2d+2}.$$

□

Lemma 4.5. *For all integers i and d such that $1 \leqslant i \leqslant \lfloor \frac{d}{2} \rfloor$, the following holds:*

$$A_d(P_{2d-1,i}) = P_{2d+1,i} \text{ and } A_d(P_{2d,i}) = P_{2d+2,i}.$$

Proof. For $i = 1$ and $i = 2$ we apply relation (3.2) in a similar way as in the proof of Proposition 4.1. We just delete the only square in the second row of $G_{n,1}$ and $G_{n,2}$, and obtain that

$$P_{2d-1,1} = P_{2d-1} + xP_{2d-3}, P_{2d-1,2} = P_{2d-1} + xP_{2d-4}.$$

By using Lemma 4.4, we obtain that

$$A_d(P_{2d-1,1}) = A_d(P_{2d-1} + xP_{2d-3}) = P_{2d+1} + xP_{2d-1} = P_{2d+1,1}, \text{ and}$$
$$A_d(P_{2d-1,2}) = A_d(P_{2d-1} + xP_{2d-4}) = P_{2d+1} + xA_{d-1}(P_{2d-4}) =$$
$$= P_{2d+1} + xP_{2d-2} = P_{2d+1,2}.$$

In a similar way, we can prove that

$$A_d(P_{2d,1}) = P_{2d+2,1} \text{ and } A_d(P_{2d,2}) = P_{2d+2,2}.$$

Assume that the statement of this lemma is true for $P_{2d-1,j}$ and $P_{2d,j}$ when $j < i + 1$. Now, we use (4.7) and induction to calculate

$$A_d(P_{2d,i+1}) = A_d(P_{2d-1,i} + xP_{2d-2,i-1}) = A_d(P_{2d-1,i}) + xA_{d-1}(P_{2d-2,i-1}) =$$
$$= P_{2d+1,i} + xP_{2d,i-1} = P_{2d+2,i+1}.$$

From (4.6) we obtain that

$$A_d(P_{2d-1,i+1}) = A_d(P_{2d,i+1} - xP_{2d-2,i+1}) = A_d(P_{2d,i+1}) - xA_{d-1}(P_{2d-2,i+1}) =$$
$$= P_{2d+2,i+1} - xP_{2d,i+1} = P_{2d+1,i+1}.$$

□

From Definition 4.3 and Remark 4.2 we can obtain the concrete formula for the linear map A_d.

Proposition 4.6. *For all $d, k \in \mathbb{N}$ such that $d \geqslant k \geqslant 1$, we have that:*

$$A_d(x^k) = x^k \left(1 + 2x - x^2 + 2x^3 - 5x^4 + 14x^5 - 42x^6 + \cdots + (-1)^{d-k} C_{d-k} x^{d-k+1}\right).$$

Here C_m denotes the m-th Catalan number.

Proof. From (4.8) it is enough to prove that

$$A_d(1) = 1 + 2x - x^2 + 2x^3 - 5x^4 + \cdots + (-1)^d C_d x^{d+1}. \tag{4.10}$$

For all integers n and k such that $n \geqslant k \geqslant 1$ (by using the induction and the Pascal's Identity), we can obtain the next relation

$$\binom{n}{k} = \sum_{i=0}^{k} (-1)^i \binom{n+1+i}{k-i} C_i. \tag{4.11}$$

Now, we assume that (4.10) is true for all positive integers less than d, and calculate $A_d(1)$ by definition:

$$A_d(1) = P_{2d+1} - A_d(P_{2d-1} - 1) =$$
$$= \sum_{i=0}^{d+1} \binom{2d+2-i}{i} x^i - \sum_{i=1}^{d} \binom{2d-i}{i} x^i A_{d-i}(1).$$

The coefficients of $1, x$ and x^2 in $A_d(1)$ are respectively:

$$\binom{2d+2}{0} = 1, \binom{2d+1}{1} - \binom{2d-1}{1} = 2, \binom{2d}{2} - \binom{2d-2}{2} - 2\binom{2d-1}{1} = -1.$$

For $k > 1$ the coefficient of x^{k+1} in the polynomial $A_d(1)$ is

$$\binom{2d+1-k}{k+1} - \binom{2d-k-1}{k+1} - 2\binom{2d-k}{k} - \sum_{i=1}^{k-1} (-1)^i \binom{2d-k+i}{k-i} C_i.$$

From (4.11) we obtain that the coefficient of x^{k+1} in $A_d(1)$ is $(-1)^k C_k$. □

Corollary 4.7. *For any positive integer d the linear map A_d is injective.*

Now, we consider all simple connected regions for which the degree of the associated polynomial $P_{\mathcal{R}}(x) = F_{\mathcal{R}}(x-1)$ is equal to d. Let $\mathcal{F}^d$ denote the affine subspace of $\mathcal{P}^d$ spanned by these polynomials.

Lemma 4.8. *The polynomial $P_{2d+1,d}$ is not contained in $A_d(\mathcal{F}^d)$.*

Proof. From (4.7) and (4.6) it follows that

$$P_{2d+1,d} - P_{2d+1,d-1} = (P_{2d,d-1} + xP_{2d-1,d-2}) - (P_{2d,d-1} + xP_{2d-1,d-1}) =$$
$$-x(P_{2d-1,d-1} - P_{2d-1,d-2}) = (-1)^{d+1}(x^{d+1} + x^d).$$

We know that $P_{2d+1,d-1} = A_d(P_{2d-1,d-1})$. If there exists a polynomial $p \in \mathcal{F}^d$ such that $A_d(p) = P_{2d+1,d}$ then we obtain

$$x^{d+1} + x^d = \pm A_d(p - P_{2d-1,d-1}),$$

which is impossible from Proposition 4.6. □

Theorem 4.9. *The polynomials $P_{2d-1}, P_{2d}, P_{2d-1,1}, \ldots, P_{2d-1,d-1}$ are affinely independent in $\mathcal{F}^d$.*

Proof. We use induction on the degree. Assume that d polynomials $P_{2d-3}, P_{2d-2}, P_{2d-3,1}, \ldots, P_{2d-3,d-2}$ are affinely independent in $\mathcal{F}^{d-1}$. From Lemmas 4.4 and 4.5 and Corollary 4.7, we conclude that $P_{2d-1}, P_{2d}, P_{2d-1,1}, \ldots, P_{2d-1,d-2}$ are affinely independent. These polynomials span a $(d-1)$-dimensional affine subspace of $\mathcal{F}^d$. From Lemma 4.8 follows that $P_{2d-1,d-1}$ is not contained in $A_{d-1}(\mathcal{F}^{d-1})$. □

Corollary 4.10. *The Euler-Poincare relation (4.1) is the only linear relation for the f-vectors of tilings.*

This answer the question of Ehrenborg question about numerical relations between the numbers of different types of tilings, see [6].

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