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Graphs with maximum degree 5 are acyclically 7-colorable*

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Abstract

An acyclic coloring is a proper coloring with the additional property that the union of any two color classes induces a forest. We show that every graph with maximum degree at most 5 has an acyclic 7-coloring. We also show that every graph with maximum degree at most r has an acyclic $(1 + \lfloor \frac{(r+1)^2}{4} \rfloor)$-coloring.

Keywords: Acyclic coloring, maximum degree.

Math. Subj. Class.: 05C15, 05C35

1 Introduction

A *proper coloring* of the vertices of a graph $G = (V, E)$ is an assignment of colors to the vertices of the graph such that no two adjacent vertices receive the same color. A proper coloring of a graph G is *acyclic* if the union of any two color classes induces a forest. The *acyclic chromatic number*, $a(G)$, is the smallest integer k such that G is acyclically k-colorable. The notion of acyclic coloring was introduced in 1973 by Grünbaum [8] and turned out to be interesting and closely connected to a number of other notions in graph coloring. Several researchers felt the beauty of the subject and started working on problems and conjectures posed by Grünbaum. Michael Albertson was among the enthusiasts and wrote in total four papers on the topic [1, 2, 3, 4].

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In particular, Grünbaum studied $a(r)$ – the maximum value of the acyclic chromatic number over all graphs G with maximum degree at most r. He conjectured that for every r, $a(r) = r + 1$ and proved that his conjecture holds for $r \leq 3$. In 1979, Burstein [6] proved the conjecture for $r = 4$. This result was proved independently by Kostochka [10]. It was also proved in [10] that for every $k \geq 3$, the problem of deciding whether a graph is acyclically k-colorable is NP-complete. It turned out that for large r, Grünbaum's conjecture is incorrect in a strong sense. Albertson and Berman mentioned in [1] that Erdős proved that $a(r) = \Omega(r^{4/3-\epsilon})$ and conjectured that $a(r) = o(r^2)$. Alon, McDiarmid and Reed [5] sharpened Erdős' lower bound to $a(r) \geq c\,r^{4/3}/(\log r)^{1/3}$ and proved that

$$a(r) \leq 50\,r^{4/3}. \tag{1.1}$$

This established almost the order of the magnitude of $a(r)$ for large r. Recently, the problem of estimating $a(r)$ for small r was considered again.

Fertin and Raspaud [7] showed among other results that $a(5) \leq 9$ and gave a linear-time algorithm for acyclic 9-coloring of any graph with maximum degree 5. Furthermore, for every fixed $r \geq 3$, they gave a fast algorithm that uses at most $r(r-1)/2$ colors for acyclic coloring of any graph with maximum degree r. Of course, for large r this is much worse than the upper bound (1.1), but for $r < 1000$, it is better. Hocquard and Montassier [9] showed that every 5-connected graph G with $\Delta(G) = 5$ has an acyclic 8-coloring. Kothapalli, Varagani, Venkaiah, and Yadav [12] showed that $a(5) \leq 8$. Kothapalli, Satish, and Venkaiah [11] proved that every graph with maximum degree r is acyclically colorable with at most $1 + r(3r + 4)/8$ colors. This is better than the bound $r(r-1)/2$ in [7] for $r \geq 8$. The main result of this paper is

Theorem 1.1. *Every graph with maximum degree* 5 *has an acyclic* 7*-coloring, i.e.,* $a(5) \leq 7$.

We do not know whether $a(5)$ is 7 or 6, and do not have a strong opinion about it.

Our proof is different from that in [7, 9, 12] and heavily uses the ideas of Burstein [6]: he started from an uncolored graph G with maximum degree 4 and colored step by step more and more vertices (with some recolorings) so that each of partial acyclic 5-colorings of G had additional good properties that enabled him to extend the coloring further. The proof yields a linear-time algorithm for acyclic coloring with at most 7 colors of any graph with maximum degree 5. Using this approach we also show that for every fixed $r \geq 6$, there exists a linear-time algorithm giving an acyclic coloring of any graph with maximum degree r with at most $1 + \lfloor \frac{(r+1)^2}{4} \rfloor$ colors. This is better than the bounds in [7] and [11] cited above for every $r \geq 6$.

In the next section we introduce notation, prove two small lemmas and state the main lemma. In Section 3 we prove Theorem 1.1 modulo the main lemma. In Section 4 we derive linear-time algorithms for acyclic coloring of graphs with bounded maximum degree. In the last section we give the proof of the main lemma.

2 Preliminaries

Let G be a graph. A *partial coloring* of G is a coloring of some subset of the vertices of G. A *partial acyclic coloring* is then a proper partial coloring of G containing no bicolored cycles.

Given a partial coloring f of G, a vertex v is
(a) *rainbow* if all colored neighbors of v have distinct colors;
(b) *almost rainbow* if there is a color c such that exactly two neighbors of v are colored with c and all other colored neighbors of v have distinct colors;
(c) *admissible* if it is either rainbow or almost rainbow;
(d) *defective* if v is an uncolored almost rainbow vertex such that at least one of the two of its neighbors receiving the same color is admissible.

A partial acyclic coloring f of a graph G is *rainbow* if f is a partial acyclic coloring of G such that every uncolored vertex is rainbow.

A partial acyclic coloring f of a graph G is *admissible* if either f is rainbow or one vertex is defective and all other uncolored vertices are rainbow. In these terms, a coloring is rainbow if it is admissible and has no defective vertices. Note that both, rainbow and admissible colorings are partial acyclic colorings where additional restrictions are put only on uncolored vertices. The advantage of using admissible colorings is that they provide a stronger induction condition that places additional restrictions only on coloring of neighbors of uncolored vertices. So, the fewer uncolored vertices remain, the weaker these additional restrictions are.

All colorings in this section will be from the set $\{1, 2, \ldots, 7\}$.

Lemma 2.1. *Let v be a vertex of degree 4 in a graph G with $\Delta(G) \leq 5$. Let f be an admissible (respectively, rainbow) coloring in which v is colored with color c_1, each of the neighbors of v is colored, and exactly 3 colors appear on the neighbors of v. If at least one of the two neighbors of v receiving the same color and one of the other two neighbors of v each have a second (i.e., distinct from v) neighbor with color c_1, then we can recolor v and at most one of its neighbors so that the coloring remains admissible (respectively, rainbow). In particular, the new partial acyclic coloring has no new defective vertices. Moreover, if we need to recolor a vertex other than v, then we may choose a vertex with 5 colored neighbors and recolor it with a color incident to v in f.*

Proof. Let $N(v) = \{z_1, z_2, z_3, z_4\}$, $f(z_1) = f(z_2) = c_2$, $f(z_3) = c_3$, $f(z_4) = c_4$. Let z_2 and z_3 be the neighbors of v with colors c_2, and c_3 that are also adjacent to another vertex of color c_1. We may assume that z_2 is adjacent to a vertex of color c_5, since otherwise when we recolor v with c_5, no bicolored cycles appear and the coloring remains admissible (respectively, rainbow). Similarly, we may assume that z_2 is adjacent to vertices of colors c_6 and c_7. Then we may recolor z_2 with c_3 and repeat the above argument to get that z_3 also is adjacent to vertices with colors c_5, c_6, and c_7. In this case, we may change the original coloring by recoloring z_3 with c_2 and v with c_3. So, in this case only v and z_3 change colors. Note that either only v changes its color, or z_2 receives color c_3, or z_3 receives color c_2. □

For partial colorings f and f' of a graph G, we say that f' is *larger than* f if it colors more vertices.

Lemma 2.2. *Let v be a vertex of degree 4 in a graph G with $\Delta(G) \leq 5$. Let f be a rainbow coloring in which v is colored with color c_1, the neighbors z_1, z_2, and z_3 of v receive the distinct colors c_2, c_3, and c_4, the neighbor z_4 of v is an uncolored rainbow vertex. Then either G has a rainbow coloring f_1 that colors the same vertices and differs from f only at v, or G has a rainbow coloring f' larger than f. Moreover, if the former does not hold,* *then z_4 has degree 5 and exactly one uncolored neighbor, say $z_{4,4}$, and we can choose the larger coloring f' so that all the following are true:*

1. *Every vertex colored in f is still colored.*
2. *Vertex z_4 is colored.*
3. *The only uncolored vertex apart from z_4 that may get colored is $z_{4,4}$, and it does only if it has neighbors of colors c_1, c_2, c_3, and c_4.*
4. *Apart from v, only one vertex w may change its color, and if it does, then (a) w is a neighbor of z_4, (b) w has four colored neighbors, (c) it changes a color in $\{c_5, c_6, c_7\}$ to another color in $\{c_5, c_6, c_7\}$, and (d) z_4 gets the former color of w. In particular, v is admissible in f'.*

Proof. Let v, z_1, z_2, z_3, and v_4 be as in the hypothesis. We may assume that z_4 is adjacent to a vertex $z_{4,1}$ of color c_5: otherwise, since v_4 is rainbow, when we recolor v with c_5, the new coloring will be rainbow. Similarly, we may assume that z_4 is adjacent to vertices $z_{4,2}$, and $z_{4,3}$ of colors c_6 and c_7. If z_4 has no other neighbors, then we can recolor v with c_5 and color z_4 with c_1. So, assume that z_4 has the fifth neighbor, $z_{4,4}$. If $z_{4,4}$ is colored, then $f(z_{4,4}) \in \{c_2, c_3, c_4\}$, since z_4 is rainbow. In this case, we let $f'(z_4) = c_1$ and $f'(v) = c_5$. So, we may assume that $z_{4,4}$ is not colored. If $z_{4,4}$ has no neighbor of color c_2, then coloring z_4 with c_2 leaves the coloring rainbow and makes it larger than f. Thus, we may assume that $z_{4,4}$ has a neighbor of color c_2 and similarly neighbors of colors c_3 and c_4. If $z_{4,4}$ has no neighbor of color c_1, then we let $f'(z_4) = c_1$ and $f'(v) = c_5$. So, let $z_{4,4}$ have such a neighbor.

If $z_{4,1}$ has no neighbor of color c_2, then by coloring z_4 with c_2 and $z_{4,4}$ with c_5, we get a rainbow coloring larger than f. So, we may assume (by symmetry) that $z_{4,1}$ has neighbors of colors c_2, c_3, c_4. If $z_{4,1}$ has no neighbor of color c_1, then we let $f'(z_4) = c_1$, $f'(z_{4,4}) = c_5$, and $f'(v) = c_6$. Finally, if $z_{4,1}$ also has a neighbor of color c_1, then we let $f'(z_{4,1}) = c_6$ and $f'(z_4) = c_5$. □

The next lemma is our main lemma. We will use it in the next section and prove in Section 5.

Lemma 2.3. *Let f be an admissible partial coloring of a 5-regular graph G. Then G has a rainbow coloring f' that colors at least as many vertices as f.*

3 Proof the the Theorem

For convenience, we restate Theorem 1.1.

Theorem. Every graph with maximum degree 5 has an acyclic 7-coloring.

Proof. Let G be such a graph. If G is not 5-regular, form G' from two disjoint copies of G by adding for each $v \in V(G)$ of degree less than 5 an edge between the copies of v. Repeating this process at most five times gives a 5-regular graph G^* containing G as a subgraph. Since an acyclic 7-coloring of G^* yields an acyclic 7-coloring of its subgraph G, we may assume that G is 5-regular.

Let f be an admissible coloring of G from the set $\{1, 2, \ldots, 7\}$ with the most colored vertices. By Lemma 2.3, we may assume that f is rainbow.

Let H be the subgraph of G induced by the vertices left uncolored by f. Let x be a vertex of minimum degree in H. We consider several cases according to the degree $d_H(x)$.

Case 1: $d_H(x) = 0$. Since f is rainbow, any color in $\{1, 2, \dots, 7\} - f(N_G(x))$ can be used to color x contradicting the maximality of f.

Case 2: $d_H(x) = 1$. Since f is rainbow, we may assume that x is adjacent to vertices of colors $1, 2, 3,$ and 4. Let y be the uncolored neighbor of x. Since y is rainbow, coloring x with 5 gives either a rainbow coloring or an admissible coloring with the defective vertex y having the admissible neighbor x, a contradiction to the maximality of f.

Case 3: $d_H(x) = 2$. We may assume that x is adjacent to vertices with colors $1, 2, 3,$ and two uncolored vertices y_1 and y_2. Since in our case y_1 is adjacent to at most 3 colored vertices, some color $c \in \{4, 5, 6, 7\}$ does not appear on the neighbors of y_1. Coloring x with c then yields either a rainbow coloring, or an admissible coloring with defective vertex y_2 and its admissible neighbor x, a contradiction to the maximality of f.

Case 4: $d_H(x) = 3$. We may assume that x is adjacent to vertices of colors 1 and 2. By the choice of x, each uncolored vertex of G has at most 2 colored neighbors. Since the three uncolored neighbors of x have at most 6 colored neighbors in total, some color $c \in \{3, 4, 5, 6, 7\}$ is present at most once among these 6 neighbors. Then coloring x with c again yields an admissible coloring, a contradiction to the maximality of f.

Case 5: $d_H(x) \geq 4$. Since each vertex of G has at most one colored neighbor, at most 5 colors are used in the second neighborhood of x. Hence x may be colored to give a rainbow coloring with more colored vertices.

We conclude that H is empty and that f is an acyclic 5-coloring of G. □

4 Algorithms

Theorem 4.1. *There exists a linear time algorithm for finding an acyclic 7-coloring of a graph with maximum degree 5.*

Proof. The proof of the Theorem 1.1, along with Lemmas 2.1–2.3 gives an algorithm. In order to control the efficiency of the algorithm we make the following modification: whenever the proof checks whether a vertex v is in a two-colored cycle, we check only for such a cycle of length at most 12, and if we do not find such a short cycle, then check whether two bicolored paths of length 6 leave v. This is enough, since the existence of such paths already makes the proofs of Theorem 1.1 and all the lemmas work. So, we need only to consider a bounded (at most 5^6) number of vertices around our vertex. It then suffices to compute the running time of this algorithm. Let n be the number of vertices in G. The process of creating a 5-regular graph takes $O(n)$ time since we apply this process at most 5 times, each time on at most $2^5 n$ vertices, each of degree at most 5. We may now assume that G is a 5-regular graph. We then create and maintain 6 databases D_j, $j = 0, 1, \dots, 5$ (say doubly linked lists), each for the set of vertices with degree j in the current H. At the beginning, all vertices are in D_5, and it is possible to update the databases in a constant amount of time each time a vertex gains or loses a colored neighbor. Since there are at most $2^5 n$ possible searches for a vertex with the minimum number of uncolored neighbors, all the searches and updates will take $O(n)$ time. Note that the processes of Lemma 2.1 and Lemma 2.2 also take a constant amount of time to complete. Observe that each of the cases in Lemma 2.3 either finds a rainbow coloring, or finds an admissible coloring with more colored vertices, or reduces to a previous case in an amount of time bounded by a constant. Also when Lemma 2.3 processes a defective vertex, it yields either a rainbow coloring, or a larger admissible coloring and the next defective vertex in a constant time. Finally, since we start from an uncolored graph and color each additional vertex in a constant time, the implied algorithm colors all vertices in $O(n)$ time. □

For a partial coloring f of a graph G and a vertex $v \in V(G)$, we say that $u \in V(G)$ is *f-visible from v*, if either $vu \in E(G)$ or v and u have a common uncolored neighbor.

Theorem 4.2. *For every fixed r, there exists a linear (in n) algorithm finding an acyclic coloring for any n-vertex graph G with maximum degree r using at most $1 + \lfloor \frac{(1+r)^2}{4} \rfloor$ colors.*

Proof. We start from the partial coloring f_0 that has no colored vertices, and for $i = 1, \dots, n$ at Step i obtain a rainbow partial acyclic coloring f_i from f_{i-1} by coloring one more vertex (without recoloring). The algorithm proceeds as follows: at Step i, choose an uncolored vertex v_i with the most colored neighbors. Greedily color v_i with a color α_i in $C := \{1, \dots, 1 + \lfloor \frac{(1+r)^2}{4} \rfloor\}$ that is distinct from the colors of all vertices f_{i-1}-visible from v_i. We claim that we always can find such α_i in C.

Suppose that at Step i, v_i has exactly k colored neighbors. Then it has at most $r-k$ uncolored neighbors, and each of these uncolored neighbors has at most k colored neighbors. So, the total number of vertices f_{i-1}-visible from v_i is at most

$$k + (r-k)k = k(r+1-k) \leq \left\lfloor \frac{(r+1)^2}{4} \right\rfloor = |C| - 1,$$

and we can find a suitable color α_i for v_i.

It now suffices to show that for each i, the coloring f_i is rainbow and acyclic. For f_0, this is obvious. Assume now that f_{i-1} is rainbow and acyclic. Since v_i is rainbow in f_{i-1}, coloring it with α_i does not create bicolored cycles. Thus, f_i is acyclic. Also since α_i is distinct from the colors of all vertices f_{i-1}-visible from v_i, f_i is rainbow.

For the runtime, note that at Step i the algorithm considers only v_i and vertices at distance at most 2 from v_i. As in the proof of Theorem 4.1, it is sufficient to maintain $r+1$ databases each containing all vertices with a given number of colored neighbors. This allows a constant time search for a vertex with the greatest number of colored neighbors. Moving a vertex as its number of colored neighbors changes takes a constant amount of time. Choosing and coloring v_i together with updating the databases then takes $O(r^2)$ time. Hence the running time of the algorithm is at most $c_r n$, where c_r depends on r. □

5 Proof of Lemma 2.3

We will prove that under the conditions of the lemma, either its conclusion holds or there is an admissible coloring f'' larger than f. Since G is finite, repeating the argument eventually yields either an acyclic coloring of the whole G or a rainbow coloring. In both cases we do not have defective vertices.

Let H be the subgraph of G induced by the uncolored vertices. Let x be the sole defective vertex under f and let $y_1, y_2, \dots, y_5$ be its neighbors. By the definition of a defective vertex, x has two neighbors of the same color. We will assume that $f(y_1) = f(y_2) = 1$ and that y_1 is admissible. When more then two neighbors of x are colored, we assume for $i = 3, 4, 5$ that if y_i is colored, then $f(y_i) = i - 1$. Also for $i = 1, \dots, 5$, the four neighbors of y_i distinct from x will be denoted by $y_{i,1}, \dots, y_{i,4}$ (some vertices will have more than one name, since they may be adjacent to more than one y_i). We consider several cases depending on $d_H(x)$.

Case 1: $d_H(x) = 0$. First we try to color x with colors $5, 6$, and 7. If this is not allowed, then for $j = 5, 6, 7$, G has a $1, j$-colored y_1, y_2-path. This forces that both of y_1 and y_2 have neighbors with colors $5, 6$, and 7, each of which is adjacent to another vertex of color 1. In particular, both y_1 and y_2 are admissible. For $i = 1, 2$ and $j = 1, 2, 3$, we suppose that $f(y_{i,j}) = j + 4$ and $y_{i,j}$ is adjacent to another vertex of color 1.

Case 1.1: For some $i \in \{1, 2\}$, $y_{i,4}$ is colored and $f(y_{i,4}) \notin \{5, 6, 7\}$. By symmetry, we may assume that $i = 1$ and $f(y_{1,4}) = 2$. Recolor y_1 with 3 and call the new admissible coloring f'. If we can now recolor y_2 so that the resulting coloring f'' is rainbow on $G - xy_2 - xy_1$ or the only defective vertex in f'' on $G - xy_2 - xy_1$ is $y_{2,4}$, then we do this recoloring and color x with 1. Since y_1 and y_2 have no neighbors of color 1 apart from x, we obtained an admissible coloring of G larger than f. If we cannot recolor y_2 to get such a coloring, then $y_{2,4}$ is colored with a color $c \in \{5, 6, 7\}$. Moreover, in this case by Lemma 2.1 applied to y_2 in coloring f' of $G - xy_2 - xy_1$, we can change the colors of only y_2 and some $y \in \{y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}\}$ to get an admissible coloring f_1 of $G - xy_2 - xy_1$. Moreover, by Lemma 2.1, $f_1(y) \in \{5, 6, 7\}$. Then by coloring x with 1 we obtain a rainbow coloring of G, as above.

Case 1.2: $y_{1,4}$ is not colored. By Lemma 2.2 for vertex y_1 in $G - xy_1$, either $G - xy_1$ has a rainbow coloring f' that differs from f only at y_1 (in which case by symmetry, we may assume that $f'(y_1) = 3$ and proceed further exactly as in Case 1.1), or $G - xy_1$ has a larger rainbow coloring f' satisfying statements 1)–4) of Lemma 2.2. In particular, by 4), none of y_2, y_3, y_4, y_5 changes its color and y_1 remains admissible. This finishes Case 1.2.

By the symmetry between y_1 and y_2, the remaining subcase is the following.

Case 1.3: $f(y_{1,4}) = 5$ and $f(y_{2,4}) = c \in \{5, 6, 7\}$. By Lemma 2.1 applied to y_1 in $G - xy_1$, we can recolor y_1 and at most one other vertex (a neighbor of y_1) to obtain another admissible coloring f'. If $f'(y_1) \in \{5, 6, 7\}$, then f' is a rainbow coloring, as claimed. So, we may assume that $f'(y_1) = c_1 \in \{2, 3, 4\}$. If all the colors $5, 6, 7$ are present on neighbors of y_2, then again by Lemma 2.1 (applied now to y_2 in coloring f' of $G - xy_2$), G has an admissible coloring f'' that differs from f' only at y_2 and maybe at one neighbor of y_2. Then coloring x with 1 we get a rainbow coloring. So, some color in $\{5, 6, 7\}$ is not present in $f'(N(y_2))$. By Lemma 2.1, this may happen only if $y_{1,1}$ is a common neighbor of y_1 and y_2, and $c = f(y_{2,4}) \neq 5$. In particular, in this case, $y_{1,1}$ has neighbors of colors 1 (they are y_1 and y_2), $2, 3$, and 4. Since $c \neq 5$, we may assume that $c = 6$. By the symmetry between y_1 and y_2, we conclude that, in f, vertex $y_{2,2}$ also is a common neighbor of y_1 and y_2 and has neighbors of colors 1 (they are y_1 and y_2), $2, 3$, and 4. Returning to coloring f', we see that y_2 has no neighbors of color 5, and its neighbors $y_{1,1}$ (formerly of color 5) and $y_{2,2}$ (by the previous sentence) also have no neighbors of color 5. So, recoloring y_2 with 5 yields an admissible coloring of G. Now coloring x with 1 creates a larger rainbow coloring.

Case 2: $d_H(x) = 1$. We first try to color x with 4. If no bicolored cycle is formed, then either we have a rainbow coloring or an admissible coloring with defective vertex y_5 and an admissible neighbor x. Hence we may assume that coloring x with 4 creates a bicolored cycle. This then gives each of y_1 and y_2 a neighbor of color 4. A similar argument gives each of y_1 and y_2 a neighbor of color $5, 6$, and 7, i.e., both y_1 and y_2 are rainbow. Recoloring y_1 with color 2 allows us to repeat the argument at y_3. Then y_3 also has neighbors of each of the colors $4, 5, 6$, and 7. If y_5 has no neighbor of color

2, then recoloring (in the original coloring f) y_3 with 1, and coloring x with 2 yields a rainbow coloring. So, by the symmetry between colors $1, 2$, and 3, we may assume that for $i \in \{1, 2, 3\}$, $f(y_{5,i}) = i$. Since y_5 is rainbow, by the symmetry between colors $4, 5, 6$, and 7, we may assume that either $f(y_{5,4}) = 4$, or $y_{5,4}$ is not colored. In both cases, recolor (in the original coloring f) y_3 with 1, color x with 2 and y_5 with 5. We get an admissible coloring larger than f, where only $y_{5,4}$ may be defective.

Case 3: $d_H(x) = 3$. If one of the uncolored neighbors y_3, y_4, y_5 (say, y_3) of x has 4 colored neighbors, then we may color y_3 with some $c \notin f(N(y_3)) \cup \{1\}$ and thus create an admissible coloring larger than f. Hence we may assume that each of y_3, y_4, and y_5 has at most 3 colored neighbors.

Case 3.1: One of y_1 and y_2 has three neighbors of different colors such that each of these neighbors has another neighbor of color 1. Suppose for example that for $j = 1, 2, 3$, $f(y_{1,j}) = 1 + j$ and $y_{1,j}$ has another neighbor of color 1. If y_1 has a fourth color, say c, in its neighborhood, then we recolor y_1 with a color $c' \notin \{1, c, 5, 6, 7\}$ and get a rainbow coloring of G. Suppose now that color $c \in \{5, 6, 7\}$ appears twice on $N(y_1)$. Then by Lemma 2.1 applied to y_1 in $G - xy_1$, we can change the color of y_1 and at most one other vertex that is a neighbor of y_1 not adjacent to uncolored vertices to get another rainbow coloring of $G - xy_1$. Then this coloring will also be a rainbow coloring of G. Finally, suppose that y_1 has an uncolored neighbor $y_{1,4}$. Applying Lemma 2.2 to y_1 in $G - xy_1$ we either recolor only y_1 and get a rainbow coloring of G (finishing the case), or obtain a rainbow coloring f' of $G - xy_1$ larger than f satisfying the conclusions of the lemma. Since each of y_3, y_4 and y_5 has at least two neighbors left uncolored by f, none of them may play role of z_4 or $z_{4,4}$ in Lemma 2.2 when they get colored. Then f' is an admissible coloring of G where only x could be a defective vertex with admissible neighbor v. This proves Case 3.1.

Let T be the set of colors c such that more than one of the vertices y_3, y_4 and y_5 has a neighbor of color c. Since y_3, y_4 and y_5 have in total at most 9 colored neighbors, $|T| \leq 4$.

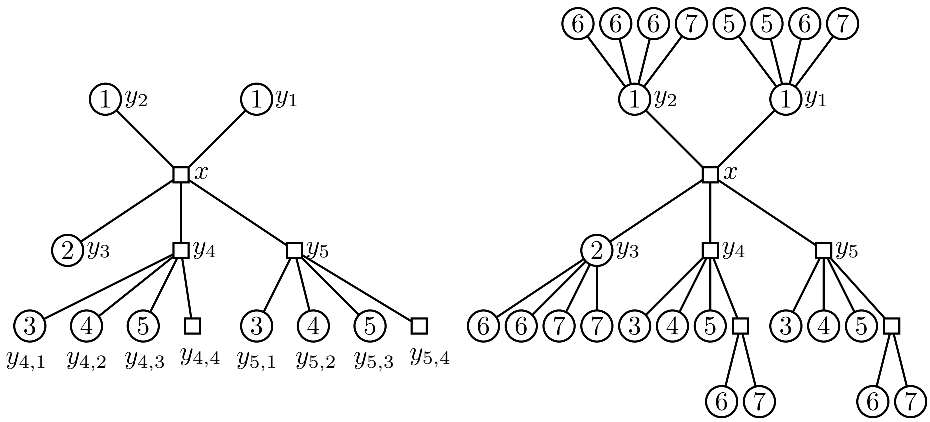
Case 3.2: $|T| \leq 3$. By symmetry, we may assume that $T \subseteq \{2, 3, 4\}$. If coloring x with $c \in \{5, 6, 7\}$ does not create a bicolored cycle, then it will yield an admissible coloring larger than f. So, we may assume that each of y_1 and y_2 has in its neighborhood vertices of colors $5, 6$, and 7, each of which is adjacent to another vertex of color 1. So, we have Case 3.1.

Case 3.3: $|T| = 4$. Let $T = \{2, 3, 4, 5\}$. As in Case 3.1, we may assume that each of y_1 and y_2 is adjacent to vertices of colors 6 and 7, each of which have another neighbor of color 1.

Let y_3 have exactly 3 colored neighbors labeled $y_{3,1}, y_{3,2}, y_{3,3}$ with colors $2, 3, 4$. Let $y_{3,4}$ be the uncolored neighbor of y_3. Then if $y_{3,4}$ has no neighbor of color 5, we may color y_3 with 5 to get a new admissible coloring. Hence $y_{3,4}$ is adjacent to a vertex of color 5. Similarly, $y_{3,4}$ has neighbors of color 6 and 7. By symmetry, we may assume that a vertex of color 2 is adjacent to at most one of y_4 and y_5.

Case 3.3.1: $y_{3,4}$ has no neighbor of color 1. We try to color y_3 with 1 and x with 2. If this does not produce a new admissible coloring, then one of y_1 or y_2, say y_1, has a neighbor of color 2 that is adjacent to another vertex of color 1. So, we again get Case 3.1.

Case 3.3.2: $y_{3,4}$ has a neighbor of color 1. If $y_{3,1}$ has no neighbor of color 1, then we again try to color y_3 with 1 and x with 2, but also color $y_{3,4}$ with 2. Then we simply repeat the argument of Case 3.3.1. So, suppose that $y_{3,1}$ has a neighbor of color 1. If $y_{3,1}$ has no neighbor of some color $\alpha \in \{5, 6, 7\}$, then we color $y_{3,4}$ with 2 and y_3 with α. Thus $y_{3,1}$



Case 4.3. Case 4.3.4.3.

Figure 1: Cases 4.3 and 4.3.4.3 from the proof of Lemma 2.3.

has neighbors of colors $1, 5, 6, 7$. Then we recolor $y_{3,1}$ with 3 and color y_3 with 2.

Case 4: $d_H(x) = 2$. As at the beginning of Case 3, we conclude that each of the uncolored vertices y_4 and y_5 has at least one uncolored neighbor besides x.

Let B be the set of colors appearing in the neighborhoods of both, y_4 and y_5. By the previous paragraph, $|B| \leq 3$.

Case 4.1: $|B| \leq 1$. We may assume that $\{4, 5, 6, 7\} \cap B = \emptyset$. Try to color x with 4. By the definition of B, either a two-colored cycle appears, or we get a new admissible coloring larger than f. Hence we may assume that coloring x with 4 creates a bicolored cycle. Since this cycle necessarily goes through y_1, y_1 is adjacent to a vertex with color 4. Similarly, y_1 is adjacent to vertices with colors $5, 6$, and 7. Then recoloring y_1 with 3 yields a rainbow coloring of G.

Case 4.2: $|B| = 2$. If $1 \in B$ or $2 \in B$, then the argument of Case 4.1 holds. Assume that $B = \{3, 4\}$. Similarly to Case 4.1, we may assume that for $i = 1, 2$ and $j = 1, 2, 3$, y_i is adjacent to a vertex $y_{i,j}$ of color $j + 4$ that is adjacent to another vertex of color 1 (in particular, y_1 and y_2 may have a common neighbor of color $j + 4$).

If y_1 is rainbow, then uncoloring y_1 and coloring x with 7 gives Case 1 or Case 2. Thus we may assume that y_1 and (by symmetry) y_2 are not rainbow. So, we may assume that for $i = 1, 2$, the fourth neighbor $y_{i,4}$ of y_i distinct from x has color $c_i \in \{5, 6, 7\}$. By symmetry, we may assume that $c_1 = 5$. Similarly to Case 1.3, by Lemma 2.1 applied to y_1 in $G - xy_1$, we can recolor y_1 and at most one other vertex (a neighbor of y_1) to obtain another rainbow coloring f' of $G - xy_1$. If $f'(y_1) \in \{3, 4, 5, 6, 7\}$, then f' is a rainbow coloring of G, as claimed. So, we may assume that $f'(y_1) = 2$. Now practically repeating the argument of Case 1.3, we find a promised coloring.

Case 4.3: $|B| = 3$ (see Figure 1). If $2 \in B$, then we can repeat the argument of Case 4.2 for $B' = B - \{2\}$. Hence we may assume that $B \subseteq \{1, 3, 4, 5, 6, 7\}$.

Case 4.3.0: $1 \in B$. Let $B = \{1, 3, 4\}$. Then some color in $\{5, 6, 7\}$, say 7, is not present on $N(y_4) \cup N(y_5)$. Again, we may assume that for $i = 1, 2$ and $j = 1, 2, 3$, y_i is adjacent to a vertex $y_{i,j}$ of color $j+4$ that is adjacent to another vertex of color 1. If y_1 is rainbow, then we may uncolor y_1 and color x with 7 to get Case 1 or Case 2. Suppose now that y_1 and y_2 are not rainbow. By Lemma 2.1 applied to y_1 in $G - xy_1$, we can recolor y_1 and at most one other vertex (a neighbor of y_1) to obtain another admissible coloring f'. If $f'(y_1) \in \{3, 4, 5, 6, 7\}$, then f' is a rainbow coloring, as claimed. So, we may assume that $f'(y_1) = 2$. But then we can use the argument of Case 4.2 with the roles of y_3 and y_2 switched. This proves Case 4.3.0.

So, from now on, $B = \{3, 4, 5\}$. For $i = 4, 5$ and $j = 1, 2, 3$, let $y_{i,j}$ be the neighbor of y_i of color $j+2$. We write *the* neighbor, since y_4 and y_5 are rainbow. As observed at the beginning of Case 4, y_4 and y_5 each have another uncolored neighbor, call them $y_{4,4}$ and $y_{5,4}$. In particular, y_4 and y_5 have no neighbors colored with 6 or 7. If x can be colored with either of 6 or 7 without creating a two-colored cycle, then we obtain a rainbow coloring. Hence we assume that for $i = 1, 2$ and $j = 1, 2$, $f(y_{i,j}) = j+5$ and $y_{i,j}$ has a neighbor of color 1 distinct from y_i.

Case 4.3.1: One of y_1 or y_2, say y_1, is rainbow. If $y_{4,4}$ has no neighbor of color $c \in \{6, 7\}$, then we can color y_4 with c, a contradiction to the maximality of f. If $y_{4,4}$ has no neighbor of color $c' \in \{1, 2\}$, then by uncoloring y_1 and coloring y_4 with c' and x with 6, we obtain an admissible coloring larger than f. So, $f(N(y_{4,4})) = \{1, 2, 6, 7\}$. Then we may color $y_{4,4}$ with 3 and uncolor y_1 to get a new admissible coloring as large as f with one defective vertex y_4, for which Case 2 holds. This finishes Case 4.3.1.

So, below y_1 and y_2 are not rainbow and hence each of them is adjacent to at least three colored vertices.

Case 4.3.2: One of y_1 or y_2, say y_1, is adjacent to an uncolored vertex $y_{1,4} \neq x$. We may assume that $f(y_{1,1}) = f(y_{1,2}) = 6$ and $f(y_{1,3}) = 7$. First, we try to color x with 7 and y_1 with 3. Since the new coloring has at most one defective vertex, we may assume that a two-colored cycle is created. Hence each of $y_{1,1}$ and $y_{1,2}$ is adjacent to a vertex of color 3. The same argument gives these vertices neighbors of colors 4 and 5. Recall that one of $y_{1,1}$ and $y_{1,2}$, say $y_{1,1}$, has another neighbor of color 1. Then recoloring $y_{1,1}$ with 2 gives an admissible coloring in which y_1 is rainbow. Hence Case 4.3.1 applies to this new coloring.

So, from now on each of y_1 and y_2 has 4 colored neighbors. Since y_1 is admissible we may assume w.l.o.g. that y_1 is adjacent either to the colors $5, 6, 6, 7$ or the colors $5, 5, 6, 7$.

Case 4.3.3: y_1 has one neighbor of color 5 and three neighbors with colors 6 or 7. We may assume that $f(y_{1,1}) = 5$, $f(y_{1,2}) = f(y_{1,3}) = 6$, and $f(y_{1,4}) = 7$. If coloring y_1 with 3 or 4 yields an admissible coloring, then we are done; so we may assume that a two-colored cycle is formed in each case. It follows that each of $y_{1,2}$ and $y_{1,3}$ has neighbors colored with 3 and 4. By the symmetry between $y_{1,2}$ and $y_{1,3}$, we may assume that $y_{1,3}$ has a neighbor of color 1 other than y_1. If $y_{1,3}$ is almost rainbow, then we can uncolor it, recolor y_1 with 3, and color x with 7: this will give an admissible coloring with the same number of colored vertices as in f, and the only defective vertex $y_{1,3}$. Then either Case 1 or Case 2 applies to this new coloring. Hence we may assume that $y_{1,3}$ has two neighbors other than y_1 that receive the same color. Then since $y_{1,3}$ has no neighbor of color 2, y_1 may now be recolored with color 2 without creating a bicolored cycle. Repeating the above argument we derive that $y_{1,2}$ has neighbors of colors $2, 3$, and 4, and one of these colors appears twice on $N(y_{1,2}) - y_1$. By Lemma 2.1 applied to $y_{1,3}$ in the graph $G - y_{1,3}y_1$ for the original coloring, we can change its color and the color of at most one other vertex (that is a neighbor of $y_{1,3}$, all of whose neighbors are colored) to get an admissible coloring of

$G - y_{1,3}y_1$. Since y_2 and y_3 are adjacent to the uncolored vertex x, their colors are not changed. If $y_{1,3}$ receives color 1, then we recolor y_1 with 3 and get a rainbow coloring of G. If $y_{1,3}$ receives a color other than 1, then we color x with 6 and again get a rainbow coloring of G.

Case 4.3.4: y_1 has two neighbors of color 5 (see Figure 1). We may assume that $f(y_{1,1}) = f(y_{1,2}) = 5$, $f(y_{1,3}) = 6$, and $f(y_{1,4}) = 7$. If y_1 can be recolored with either 3 or 4, this would give a rainbow coloring f'. Hence we assume that both of $y_{1,1}$ and $y_{1,2}$ are adjacent to vertices with colors 3 and 4.

Case 4.3.4.1: One of $y_{1,1}$ or $y_{1,2}$, say $y_{1,1}$, is rainbow. Then uncoloring $y_{1,1}$ and coloring y_1 with 3 and x with 7 yields either a rainbow coloring f' or a new admissible coloring (with the same number of colored vertices) with the defective vertex $y_{1,1}$ and admissible colored neighbor y_1. In the former case, we are done. In the latter, if one of the previous cases occurs, then we are done again. So, we may assume that Case 4.3.4 occurs. By the symmetry between colors 3 and 4, we may assume that apart from y_1, vertex $y_{1,1}$ has a neighbor of color 3, a neighbor of color 4, and two uncolored neighbors, say z_1 and z_2, each of whose has another uncolored neighbor and 3 colored neighbors. Moreover, the same 3 colors appear on the neighborhoods of z_1 and z_2, and since Case 4.3.4 holds, by the symmetry between colors 6 and 7, both of them are among these 3 colors. Then either coloring $y_{1,1}$ with 1 yields a rainbow coloring or coloring $y_{1,1}$ with 2 does.

Case 4.3.4.2: Each of $y_{1,1}$ and $y_{1,2}$ has a neighbor of color 2 that has another neighbor of color 5. Since $y_{1,1}$ is not rainbow, the fourth neighbor of $y_{1,1}$ has color $c \in \{2, 3, 4\}$. Since y_1 cannot be recolored with 3 or 4, some neighbor, say r, of $y_{1,1}$ of color c has another neighbor of color 5. If in the graph $G - y_1y_{1,1}$, $y_{1,1}$ can be recolored with 1, then we may recolor y_1 with 3 and get a rainbow coloring of G. If $y_{1,1}$ can be recolored with either of 6 or 7, then we have Case 4.3.3. To disallow coloring $y_{1,1}$ with $1, 6$, and 7, r must be adjacent to vertices with each of these colors. By the symmetry between colors 3 and 4, we assume that $f(r) \neq 4$. If the neighbor r' of $y_{1,1}$ with $f(r') = 4$ has no neighbor of color $c' \in \{6, 7\}$, then we recolor r with 4 and $y_{1,1}$ with c' thus getting Case 4.3.3. If r' has no neighbor of color 1, then we recolor r with 4, $y_{1,1}$ with 1, and y_1 with 3 obtaining a rainbow coloring. Finally if $f(N(r') - y_{1,1}) = \{1, 5, 6, 7\}$, then we recolor r' with 3, $y_{1,1}$ with 4, and y_1 with 3.

The last subcase is:

Case 4.3.4.3: $y_{1,1}$ has no neighbor of color 2 that has another neighbor of color 5. Then recoloring y_1 with 2 creates another admissible coloring f'. We may then repeat our previous argument with y_3 playing the role of y_2 to conclude that y_3 has neighbors of color 6 and 7. If y_3 is admissible, then repeating the above argument we conclude that y_3 may be recolored with color 1 in the original coloring f. Then after this recoloring, by coloring x with 2 we get a rainbow coloring. Also, if y_2 is admissible in f, then we may recolor both of y_1 and y_2 with 2 and color x with 1 to get a rainbow coloring. Hence we may assume that all the neighbors of y_2 and y_3 apart from x are colored with 6 or 7. Recall that for $i = 4, 5$ and $j = 1, 2, 3$, $f(y_{i,j}) = j + 2$ and $y_{i,4}$ is uncolored. If for some $i \in \{4, 5\}$, $y_{i,4}$ has no neighbor of color $c \in \{6, 7\}$, then we can color y_i with c and get a better admissible coloring. Since none of y_1, y_2, or y_3 has a neighbor with color 3, if $y_{4,4}$ has no neighbor of color 1 or $y_{5,4}$ has no neighbor of color 2, then by coloring y_4 with 1, y_5 with 2 and x with 3 creates an admissible coloring with more colored vertices. By the symmetry between colors 1 and 2, each of $y_{4,4}$ and $y_{5,4}$ has neighbors of colors $1, 2, 6$, and 7.

If $y_{4,1}$ does not have a neighbor of color $c' \in \{1, 2, 6, 7\}$, then coloring $y_{4,4}$ with 3, y_4

with c' and x with 4 yields an admissible coloring. Otherwise, we recolor $y_{4,1}$ with 4 and color y_4 with 3. This proves the lemma.

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