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The Sierpiński product of graphs*

Jurij Kovič
Institute of Mathematics, Physics, and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia, and
University of Primorska, FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia

Tomaž Pisanski
University of Primorska, FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia, and
Institute of Mathematics, Physics, and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia

Sara Sabrina Zemljič
Cambridge Quantum Computing Ltd, 32 St James's Street, London, SW1A 1HD, United Kingdom

Arjana Žitnik †
University of Ljubljana, Faculty of Mathematics and Physics, Jadranska 19, 1000 Ljubljana, Slovenia, and
Institute of Mathematics, Physics, and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia

Dedicated to Professor Wilfried Imrich on the occasion of his 80th birthday.

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Abstract

In this paper we introduce a product-like operation that generalizes the construction of the generalized Sierpiński graphs. Let G, H be graphs and let $f\colon V(G) \to V(H)$ be a function. Then the *Sierpiński product of graphs G and H with respect to f*, denoted by $G \otimes_f H$, is defined as the graph on the vertex set $V(G) \times V(H)$, consisting of $|V(G)|$

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†Corresponding author.

copies of H; for every edge $\{g, g'\}$ of G there is an edge between copies gH and $g'H$ of form $\{(g, f(g'), (g', f(g))\}$.

Some basic properties of the Sierpiński product are presented. In particular, we show that the graph $G \otimes_f H$ is connected if and only if both graphs G and H are connected and we present some conditions that G, H must fulfill for $G \otimes_f H$ to be planar. As for symmetry properties, we show which automorphisms of G and H extend to automorphisms of $G \otimes_f H$. In several cases we can also describe the whole automorphism group of the graph $G \otimes_f H$.

Finally, we show how to extend the Sierpiński product to multiple factors in a natural way. By applying this operation n times to the same graph we obtain an alternative approach to the well-known n-th generalized Sierpiński graph.

Keywords: Sierpiński graphs, graph products, connectivity, planarity, symmetry.

Math. Subj. Class. (2020): 05C76, 05C10, 05C40, 20B25

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1 Introduction

The motivation for this paper is the study of Sierpiński graphs and their generalizations. Although the body of this work is essentially self contained, a few remarks about the role of Sierpiński graphs seem to be appropriate. The family of *Sierpiński graphs* S_p^n was first introduced by Klavžar and Milutinović in [16] as a variant of the Tower of Hanoi problem. The Sierpiński graphs can be defined recursively as follows: S_p^1 is isomorphic to the complete graph K_p and S_p^{n+1} is constructed from p copies of S_p^n by adding exactly one edge between every pair of copies of S_p^{n+1} in a well-defined manner. The Sierpiński graphs S_3^1, S_3^2, and S_3^3 are depicted in Figure 1. In the "classical" case, when $p = 3$, the Sierpiński graphs are isomorphic to the Hanoi graphs. More about Sierpiński graphs and their connections to the Hanoi graphs can be found in the recent second edition of the book about the Tower of Hanoi puzzle by Hinz et al. [10].

Sierpiński graphs have been extensively studied in most graph-theoretical aspects as well as in other areas of mathematics and even psychology. Some notable papers are [11, 13, 15, 17, 18, 22, 26, 27]. An extensive summary of topics studied on and around Sierpiński graphs is available in the survey paper by Hinz, Klavžar and Zemljič [12]. In that paper the authors introduced *Sierpiński-type graphs* as graphs that are derived from or lead to the Sierpiński triangle fractal.

These families of graphs have been generalized by Gravier, Kovše and Parreau to a family called *generalized Sierpiński graphs* [8]. Instead of the complete graph, an arbitrary graph G is used as a base graph to form a self-similar graph in the same way as the Sierpiński graphs are derived from the complete graph: the generalized Sierpiński graph S_G^1 is isomorphic to the graph G. To construct the n-th iteration generalized Sierpiński graph S_G^n for $n > 1$, take $|V(G)|$ copies of S_G^{n-1} and add to the labels of vertices in copy x of S_G^{n-1} the letter x at the beginning. Then, for any edge $\{x, y\}$ of G, add an edge between the vertex $xy \dots y$ and the vertex $yx \dots x$. See Figure 2 for an example of the second iteration generalized Sierpiński graph, where the base graph is the house graph, i.e. the cycle on five vertices with a chord.

E-mail addresses: jurij.kovic@siol.net (Jurij Kovič), tomaz.pisanski@upr.si (Tomaž Pisanski), sara.zemljic@gmail.com (Sara Sabrina Zemljič), arjana.zitnik@fmf.uni-lj.si (Arjana Žitnik)

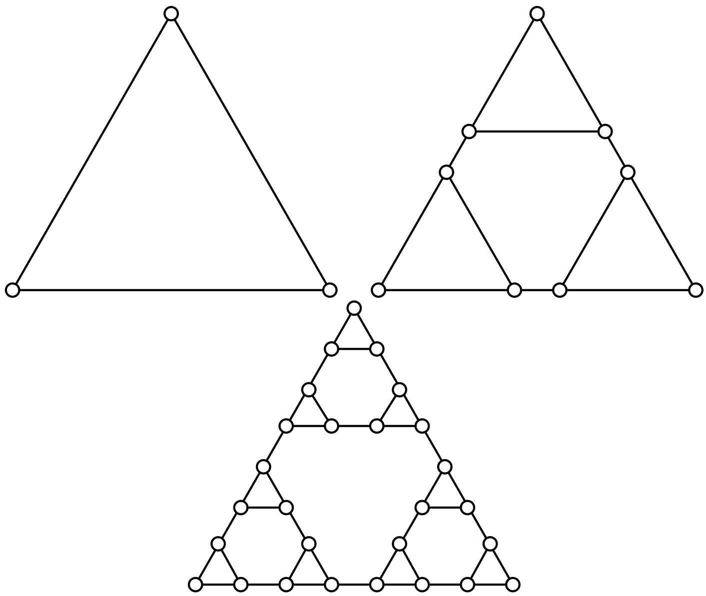

Figure 1: The Sierpiński graphs S_3^1, S_3^2, and S_3^3.

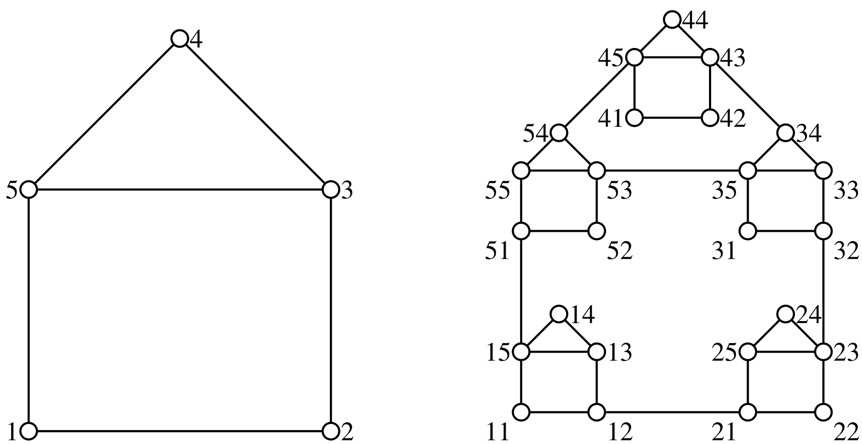

Figure 2: The generalized Sierpiński graphs S_G^1 and S_G^2 when G is the house graph.

The generalized Sierpiński graphs have been extensively studied in the past few years on topics including chromatic number, Randić index, vertex cover number, clique number, domination number and metric properties; see, for example, [5, 7, 24, 25].

At this point we would like to mention another approach towards the Sierpiński graphs. The graphs S_3^n appear naturally as subgraphs of repeated truncations of cubic graphs. More generally, by applying truncation to a p-valent vertex of a graph n times, i.e. by replacing each p-valent vertex of the graph by the complete graph K_p repeatedly, the corresponding part of the obtained graph looks like S_p^n; see Pisanski and Tucker [23] and Alspach and Dobson [1]. The truncation operation on graphs has been generalized in several ways; see Boben, Jajcay and Pisanski [2] and Exoo and Jajcay [6]. In [4], Eiben, Jajcay and Šparl study automorphisms of generalized truncations. These constructions show significant similarities to our construction, although they are distinct in general. A related construction, called the *clone cover*, is considered by Malnič, Pisanski and Žitnik in [20].

In this paper we generalize the generalized Sierpiński graphs even further. Notice that S_G^n can be viewed as an operation of S_G^{n-1} and G. This was a motivation for our introduction of the *Sierpiński product* of graphs. If we take any two graphs G and H, the resulting product locally has the structure of H, but globally it is similar to G. The formal definition of the Sierpiński product is given in Section 2.

The Sierpiński product shows some features of classical graph products, for instance the vertex set of the Sierpiński product of two graphs G and H is $V(G) \times V(H)$. However, to define the Sierpiński product of two graphs G and H, one needs some extra information besides G and H. This information can be encoded as a function $f\colon V(G) \to V(H)$. Furthermore, the product is defined so that we can extend it to multiple factors. We will see that by definition the Sierpiński product of two graphs is always a subgraph of their lexicographic product. Such layer-like structure also plays an important role in studying symmetries of the Sierpiński product. For extensive information about graph products, see the monograph by Hammack, Imrich and Klavžar [9].

The paper is organized as follows. In Section 2 we give a formal definition of the Sierpiński product of two graphs G and H with respect to $f\colon V(G) \to V(H)$; this product is denoted by $G \otimes_f H$. We explore some graph-theoretical properties such as connectedness and planarity of a Sierpiński product. In particular, we show that $G \otimes_f H$ is connected if and only if both G and H are connected and we present some necessary and sufficient conditions that G and H must fulfill in order for $G \otimes_f H$ to be planar. In Section 3 we study symmetries of the Sierpiński product of two graphs. We focus on the automorphisms of $G \otimes_f H$ that arise from the automorphisms of its factors and study the group generated by these automorphisms. In several cases we can also describe the whole automorphism group of $G \otimes_f H$. Finally, in Section 4 we consider the Sierpiński product of more than two graphs. A special case of n equal factors is considered.

2 Definition of the Sierpiński product and basic properties

Let us first review some necessary notions. All the graphs we consider are undirected, simple and finite. Let G be a graph and x be a vertex of G. By $N(x)$ we denote the set of vertices of G that are adjacent to x, i.e. the neighbourhood of x. Vertices in this paper will usually be tuples, but instead of writing them in vector form $(x_m, \ldots, x_1)$, we will sometimes write them as words $x_m \ldots x_1$ (especially in figures). More precisely, vertices $(0, 0, 0)$ or $(0, (0, 0))$ will simply be denoted by 000, except in the case when we will want to emphasize their origins. The number of vertices of a graph G, i.e. the order of G, will be denoted by $|G|$, and the number of edges of G, i.e. the size of G, will be denoted by $||G||$. For other graph theory concepts not defined here we refer the reader to [21].

Definition 2.1. Let G, H be graphs and let $f\colon V(G) \to V(H)$ be a function. Then the *Sierpiński product of the graphs G and H with respect to f*, denoted by $G \otimes_f H$, is defined as the graph on the vertex set $V(G) \times V(H)$ with two types of edges:

- $\{(g, h), (g, h')\}$ is an edge of $G \otimes_f H$ for every vertex $g \in V(G)$ and every edge $\{h, h'\} \in E(H)$,
- $\{(g, f(g')), (g', f(g))\}$ is an edge of $G \otimes_f H$ for every edge $\{g, g'\} \in E(G)$.

If $V(G) \subseteq V(H)$ and f is the identity function on its domain, we will skip the index f and denote the corresponding Sierpiński product of G and H simply by $G \otimes H$. Note that there are no restrictions on the function $f\colon V(G) \to V(H)$. However, sometimes it is convenient to assume that for every $g, g_1, g_2 \in V(G)$, $g_1 \neq g_2$, the following property holds: if $g_1, g_2 \in N(g)$, then $f(g_1) \neq f(g_2)$. In this case we say that f is *locally injective*.

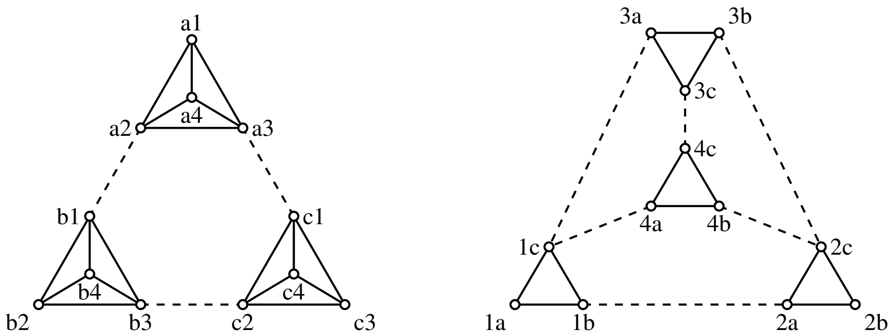

Figure 3: The graph $K_3 \otimes_{f_1} K_4$, where $V(K_3) = \{a, b, c\}, V(K_4) = \{1, 2, 3, 4\}, f_1(a) = 1, f_1(b) = 2, f_1(c) = 3$, and the graph $K_4 \otimes_{f_2} K_3$, where $f_2(1) = a, f_2(2) = b, f_2(3) = c, f_2(4) = c$. The inner edges are solid while the connecting edges are dashed. The solid subgraphs are aH, bH, cH for $H = K_4$ and $1H, 2H, 3H, 4H$ for $H = K_3$.

For any vertex g of G, the subgraph of $G \otimes_f H$ induced by the set of vertices $\{(g, h) \mid h \in V(H)\}$ is called the *subgraph associated with* g and is denoted by gH. We may view the graph $G \otimes_f H$ as partitioned into graphs gH, one for every vertex g of G, and connecting for every edge $\{g, g'\} \in E(G)$ the corresponding vertex $f(g)$ in $g'H$ to the vertex $f(g')$ in gH. The edges of $G \otimes_f H$ naturally fall into two classes. The edges connecting different subgraphs gH are called *connecting edges*, while the edges inside some subgraph gH are called *inner edges*. Given gH, we may add its neighbourhood, i.e. all connecting edges and their endvertices, to obtain a graph denoted by gH^N. If we identify all newly added vertices to a single vertex, denoted by g_H, we obtain a graph, denoted by $H + g$.

Example 2.2. Figure 3 (left) shows the Sierpiński product of K_3 and K_4 with respect to the following function f_1. The vertices of K_3 are labelled with letters a, b, c, the vertices of K_4 are labelled with numbers $1, 2, 3, 4$ and $f_1\colon V(K_3) \to V(K_4)$ is given by

$f_1(a) = 1, f_1(b) = 2, f_1(c) = 3$. Figure 3, right, shows the Sierpiński product of K_4 and K_3 with respect to $f_2\colon V(K_4) \to V(K_3)$ defined by $f_2(1) = a, f_2(2) = b, f_2(3) = c, f_2(4) = c$. This shows that the Sierpiński product is not commutative. Figure 4 depicts examples of the graphs gH^N and $H + g$ in $K_3 \otimes_{f_1} K_4$ and $K_4 \otimes_{f_2} K_3$.

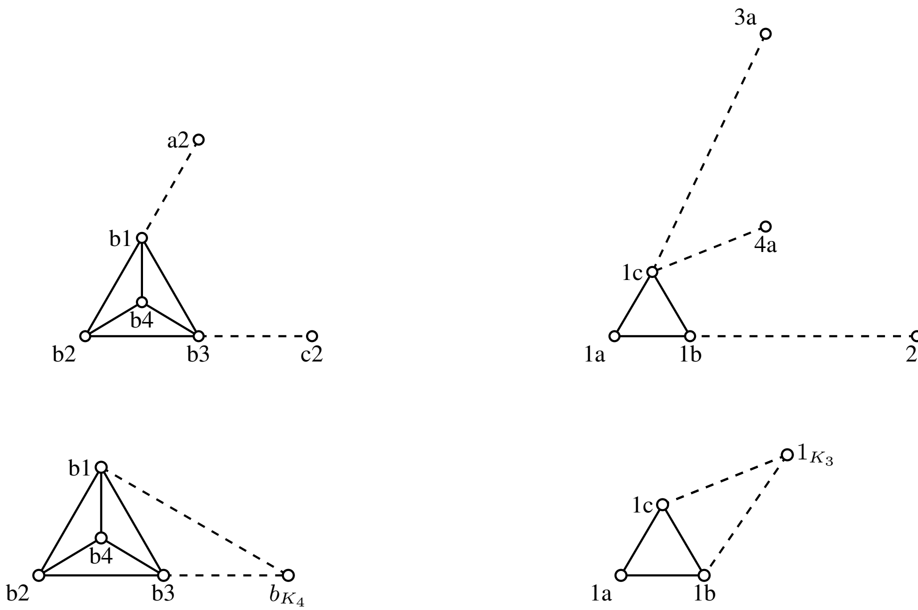

Figure 4: Top: the graphs bH^N for $H = K_4$ and $1H^N$ for $H = K_3$. Bottom: the graphs $H + b$ for $H = K_4$ and $H + 1$ for $H = K_3$.

2.1 Basic properties of the Sierpiński product of graphs

We now state some observations regarding the structure of the Sierpiński product of two graphs. We omit most of the proofs, since they follow straight from the definition.

Proposition 2.3. *Let G, H be graphs and let $f\colon V(G) \to V(H)$ be a function. Then the following statements hold.*

(i) *If $|G| = 1$, then $G \otimes_f H$ is isomorphic to H.*

(ii) *If $|H| = 1$, then $G \otimes_f H$ is isomorphic to G.*

Lemma 2.4. *Let G, H be graphs and let $f\colon V(G) \to V(H)$ be a function. Let G', H' be subgraphs of G, H, respectively, and let f' be the restriction of f to $V(G')$ such that $\mathrm{Im}(f') \subseteq V(H')$. Then the graph $G' \otimes_{f'} H'$ is a subgraph of the graph $G \otimes_f H$.*

The following result explains the role of the graphs G and gH in $G \otimes_f H$.

Lemma 2.5. *Let G, H be graphs, and let $f\colon V(G) \to V(H)$ be a function. Then the following statements hold.*

(i) *For every vertex g of G, the subgraph gH of $G \otimes_f H$ is isomorphic to H.*

(ii) *If, in addition, the graph H is connected, then the graph G is a minor of $G \otimes_f H$.*

(iii) *In particular, if the graph H is connected and the graph $G \otimes_f H$ is planar, then the graphs G and H are planar.*

Notice that planarity of both factors G and H in (iii) easily follows from the fact that H is a connected subgraph, and G is a minor of $G \otimes_f H$. However, if H is not connected, the conclusion in (ii) that G is a minor of $G \otimes_f H$ does not necessarily follow, since we cannot simply contract every subgraph gH of $G \otimes_f H$ to a single vertex. This is shown by the following example.

Example 2.6. Take G to be K_5 with vertices from 1 to 5 and H to be two isolated vertices, denoted by a and b, and then let f map 1 and 2 from K_5 to a of H and the remaining three vertices of K_5 to b of H. The resulting graph $G \otimes_f H$ is isomorphic to the disjoint union of K_2, K_3 and $K_{2,3}$ and is therefore planar. Hence, it cannot have K_5 as a minor.

Proposition 2.7. *Let G, H be graphs and let $f\colon V(G) \to V(H)$ be a function. Then the graph $G \otimes_f H$ has $|G| \cdot |H|$ vertices and $||H|| \cdot |G| + ||G||$ edges. In particular, $G \otimes_f H$ has $||H|| \cdot |G|$ inner edges and $||G||$ connecting edges.*

Lemma 2.8. *Let G and H be graphs and let $f\colon V(G) \to V(H)$ be a function. Then the following holds.*

(i) *For every $g, g' \in V(G)$ there exists at most one edge connecting graphs gH and $g'H$. In addition, graphs gH and $g'H$ are joined by an edge if and only if vertices g and g' are adjacent in graph G.*

(ii) *If a vertex $g \in V(G)$ has d neighbours in G, then there are d vertices and d edges belonging to gH^N that are not a part of gH. These d edges are connecting edges.*

(iii) *The function f is locally injective if and only if the connecting edges form a matching in $G \otimes_f H$ or, equivalently, if and only if every vertex of $G \otimes_f H$ is an endvertex of at most one connecting edge.*

Proof. (i) Only an edge of form $\{(g, f(g')), (g', f(g))\}$ can connect graphs gH and $g'H$ and this happens if and only if there exists an edge between g and g'. Hence, there exists a bijective correspondence between the connecting edges of $G \otimes_f H$ and the edges of G.
(ii) This follows from (i) and the fact that the connecting edges with an endvertex in gH are in bijective correspondence with the edges connecting g to its neighbours in G.
(iii) For every $g \in V(G)$, the connecting edges leading from gH to the rest of the graph $G \otimes_f H$ have distinct endvertices outside gH. If f is locally injective, they have distinct endvertices within gH. It follows that no two connecting edges share a common endvertex, so the connecting edges form a matching in $G \otimes_f H$. Conversely, if f fails to be locally injective, then at least two connecting edges share a common endvertex. □

The lexicographic product of two graphs G and H is the graph $G \circ H$ with vertex set $V(G) \times V(H)$ and two vertices (g, h) and (g', h') are adjacent in $G \circ H$ if and only if either g is adjacent with g' in G or $g = g'$ and h is adjacent with h' in H. For $g \in V(G)$, we denote by gH the subgraph of $G \circ H$ induced by the set $\{(g, h) | \, h \in V(H)\}$. Then the graph $G \circ H$ consists of $|G|$ copies of H, and for every edge $\{g, g'\}$ in G, every vertex of gH is connected to every vertex in $g'H$. Therefore, the next result follows straight from Definition 2.1.

Proposition 2.9. *Let G and H be graphs and let $f\colon V(G) \to V(H)$ be a function. Then the graph $G \otimes_f H$ is a spanning subgraph of the lexicographic product $G \circ H$.*

Note that for different functions f, f', the graphs $G \otimes_f H$ and $G \otimes_{f'} H$ may be isomorphic or nonisomorphic.

Proposition 2.10. *Let G, H be graphs and let $f\colon V(G) \to V(H)$ be a function. Let $\alpha \in \mathrm{Aut}(G)$, $\beta \in \mathrm{Aut}(H)$ and $f' = \beta \circ f \circ \alpha$. Then $G \otimes_f H$ is isomorphic to $G \otimes_{f'} H$.*

Proof. Define the function $\gamma\colon V(G \otimes_f H) \to V(G \otimes_{f'} H)$ by $\gamma(g, h) = (\alpha^{-1}(g), \beta(h))$ for $g \in V(G)$ and $h \in V(H)$. Since α, β are bijections, γ is also a bijection. Since β is an automorphism, γ maps inner edges to inner edges.

Take a connecting edge in $G \otimes_f H$, say $\{(g, f(g')), (g', f(g))\}$, where $\{g, g'\} \in E(G)$. Then $\gamma(g, f(g')) = (\alpha^{-1}(g), \beta(f(g'))$ and $\gamma(g', f(g)) = (\alpha^{-1}(g'), \beta(f(g))$. Since $f'(\alpha^{-1}(g)) = \beta(f(\alpha(\alpha^{-1}(g)))) = \beta(f(g))$ and $f'(\alpha^{-1}(g')) = \beta(f(\alpha(\alpha^{-1}(g')))) = \beta(f(g'))$, we see that γ also maps a connecting edge to a connecting edge. Therefore γ is an isomorphism. □

Corollary 2.11. *Let G be a graph and let $f \in \mathrm{Aut}(G)$. Then $G \otimes G$ is isomorphic to $G \otimes_f G$.*

In the remainder of this section we consider when the Sierpiński product of two graphs is connected.

Proposition 2.12. *Let G and H be graphs and let $f\colon V(G) \to V(H)$ be a function. Then $G \otimes_f H$ is connected if and only if G and H are connected.*

Proof. Suppose G and H are connected. Then $G \otimes_f H$ is connected by Definition 2.1 and Lemma 2.5.

Conversely, suppose $G \otimes_f H$ is connected. Pick two vertices g and g' from G. Then a path from gH to $g'H$ in $G \otimes_f H$ corresponds to a path in G from g to g'. Therefore, G is also connected. Suppose now that H is not connected. Denote by H_1 a connected component of H such that $V_1 = \{g \in G|\ f(g) \in V(H_1)\}$ is nonempty. Take any vertices $g \in V_1$ and $h \in V(H_1)$. If there exists an edge of form $\{(g, h), (g', f(g))\}$, then $h = f(g')$, so $g' \in V_1$. Note that $f(g) \in V(H_1)$. Therefore, all the neighbours of (g, h) belong either to gH_1 or to $g'H_1$ for some $g' \in V_1$. It follows that there are no edges between the set of vertices $\{(g, h) \in V(G \otimes_f H)|\ g \in V_1 \text{ and } h \in V(H_1)\}$ and the rest of the vertices of $G \otimes_f H$. So $G \otimes_f H$ is not connected. This is a contradiction, so H must be connected. □

In [19], Klavžar and Zemljič have characterized which generalized Sierpiński graphs are k-connected and k-edge connected. Unfortunately, their results do not carry over directly to the Sierpiński product of distinct graphs since the factors may have different connectivity properties. Moreover, the result does not depend only on the connectivity of its factors, but also on the choice of the function f. For instance, $C_5 \otimes C_5$ is 2-connected. However, for a constant function f, the product $C_5 \otimes_f C_5$ is only 1-connected.

2.2 Planarity

In this section we study planarity of the Sierpiński product $G \otimes_f H$ of graphs G and H with respect to a function $f\colon V(G) \to V(H)$. We have already mentioned in Lemma 2.5 that for a connected graph H, planarity of the graph $V(G \otimes_f H)$ implies planarity of both factors. The next theorem characterizes when a Sierpiński product $G \otimes_f H$ is planar when f is a locally injective function. Recall that the construction of a graph $H + g$ was introduced in the beginning of Section 2.

Theorem 2.13. *Let G be a 2-connected graph or $G = K_2$, let H be a connected graph and let $f\colon V(G) \to V(H)$ be a locally injective function. Then the graph $G \otimes_f H$ is planar if and only if the following three conditions are fulfilled:*

(i) *the graphs G and H are planar,*

(ii) *for every $g \in V(G)$ the graph $H + g$ is planar,*

(iii) *there exists an embedding of the graph G in the plane with a chosen orientation which has the following property: for every $g \in V(G)$, with $g_1, g_2, \dots, g_k$ being the cyclic order of vertices around the vertex g, there exists an embedding of the graph $H + g$ in the plane such that the cyclic order of vertices around the vertex g_H in graph $H + g$ is $f(g_k), f(g_{k-1}), \dots, f(g_1)$.*

Proof. The fact that f is a locally injective function simplifies the arguments. Namely, all the vertices in $N(g)$ are distinct for every $g \in V(G)$.

($\Rightarrow$) First, assume that the graph $G \otimes_f H$ is planar. Then, the graphs G, H are planar by Lemma 2.5 and (i) is established.

Suppose $G \otimes_f H$ is embedded in the plane. Note that for every $g \in V(G)$, the embedding of $G \otimes_f H$ induces a planar embedding of gH. For every $g \in V(G)$, let $N(gH) = \{g'H | g' \in N(g)\}$ denote the collection of graphs $g'H$ that are adjacent to gH. Since the graph G is 2-connected, or $G = K_2$, and the graph H is connected, all the graphs from $N(gH)$ are inside the same face F_g of the graph gH (otherwise the vertex g would be a cut vertex in G). For a fixed g_0 we may assume that F_{g_0} is the outer face of g_0H. If not, we take a different stereographic projection of $G \otimes_f H$. But then, for every $g \in V(G)$, the corresponding face F_g is the outer face of gH. Hence, we may contract every graph gH to a single vertex. From now on we assume that the graph $G \otimes_f H$ is embedded in the plane as we just explained and choose an orientation of the plane. If we contract every subgraph gH of $G \otimes_f H$ to a single vertex, we obtain a minor of $G \otimes_f H$ which is isomorphic to G. Its planar embedding is determined from the planar embedding of $G \otimes_f H$. Hence, for every vertex of G the cyclic order of its neighbours is defined.

Now again take the embedding of $G \otimes_f H$ as described above. Take an arbitrary $g \in V(G)$ and consider the subgraph gH^N that inherits the planar embedding and has all of its pending edges attached to the vertices in the outer face of gH. This establishes a planar embedding of $H + g$, which, in turn, is obtained from gH^N by a suitable vertex identification. This proves (ii).

Now it is easy to see that the embedding of G, obtained from $G \otimes_f H$ by contracting every copy of H to a single vertex, fulfills (iii). Namely, the cyclic order $g_1, g_2, \dots, g_k$ of vertices around a vertex g of G in this embedding corresponds to the ordering of the vertices $(g, f(g_1)), (g, f(g_2)), \dots, (g, f(g_k))$ around the outer face of gH in the planar embedding of $G \otimes_f H$. The cyclic order of vertices around g_H in the planar embedding of $gH + g$

described above is then $(g, f(g_k)), (g, f(g_{k-1})), \ldots, (g, f(g_1))$. Therefore, an appropriate embedding of $H + g$ exists for every $g \in V(g)$.

($\Leftarrow$) The converse is proved by construction. By (i), the graphs G and H are planar. Moreover, by (ii), for every $g \in V(G)$, the graph $H + g$ is planar. Using (iii), we embed the graph G in the plane. Then for each vertex g of G, we replace the vertex g by the planar embedding of the graph gH, induced by the embedding of $H + g$ from (iii). Again by (iii), it is possible to connect the copies of H among themselves in such a way that a planar embedding of the resulting graph, isomorphic to $G \otimes_f H$, is obtained. □

We believe that Theorem 2.13 holds also if the function f is not locally injective. However, the arguments in the proof become much more involved in this case.

Remark 2.14. Note that the condition in Theorem 2.13 that the graph G is 2-connected is essential. For example, take G to be the graph obtained from K_4 (whose vertices are denoted by $1, 2, 3, 4$) by adding a new vertex (named 5) to it and joining it to the vertex 4 only. The graph $G \otimes G$ is then planar, but $G + 4$ is not and the condition (ii) in Theorem 2.13 does not hold. See Figure 5.

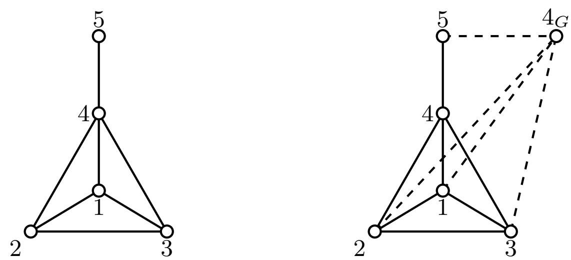

Figure 5: The graph G is planar but not 2-connected. The graph $G + 4$ contains a subdivision of K_5 and is not planar.

The next result is evident from the proof of Theorem 2.13.

Corollary 2.15. *Let G be a 2-connected graph or $G = K_2$, let H be a connected graph and let $f\colon V(G) \to V(H)$ be a locally injective function. If the graph $G \otimes_f H$ is planar, then for every $g \in V(G)$ there exists an embedding of the graph H in the plane such that the vertices $\{f(g');\ g' \in N(g)\}$ lie on the boundary of the same face.*

Using Theorem 2.13 we can now determine when the graph $G \otimes G$ is planar for a connected graph G. We also give a sufficient condition for the graph $G \otimes_f H$ to be planar when $G \neq H$. By a *block* we mean a maximal connected subgraph of a given graph that has no cut-vertices. Note that a block with more than two vertices is 2-connected.

Theorem 2.16. *Let G be a connected graph. Then the graph $G \otimes G$ is planar if and only if every block of G is outerplanar or equal to K_4.*

Proof. Note that for every block B of G, the identity function $V(B) \to V(B)$ is locally injective. So we may use Theorem 2.13 and Corollary 2.15 for every block of G.

First assume that the graph $G \otimes G$ is planar. Let B be a block of G. Suppose that B is planar but it is not outerplanar or equal to K_4. Then it contains a subdivision of $K_{2,3}$ or a subdivision of K_4 (with at least one additional vertex) as a subgraph, see [3]. Such a graph B always contains a vertex such that in every planar embedding of B, not all of its neighbours will be on the boundary of the same face. Therefore $B \otimes B$ is not planar by Corollary 2.15. On the other hand, $B \otimes B$ is a subgraph of $G \otimes G$ by Lemma 2.4, so it is planar. A contradiction. Therefore, the graph B must be outerplanar or equal to K_4.

We prove the converse by induction on the number of blocks of the graph G. If G has just one block that is outerplanar or equal to K_4, then the conditions (i), (ii), (iii) from Theorem 2.13 are fulfilled, so $G \otimes G$ is planar.

Suppose now that G has more than one block and that every block of G is outerplanar or equal to K_4. Let B be a block that corresponds to a leaf in the block-cut tree of G and let v be the only cut vertex of G contained in B. Let G' be the graph obtained from G by deleting all the vertices of B with the exception of v. Then the graph G' has one block less than G and, by the induction hypothesis, the graph $G' \otimes G'$ is planar. Take a planar embedding of $G' \otimes G'$. We obtain a planar embedding of $G \otimes G$ in the following way. We take a planar embedding of B in which v appears on the boundary of the outer face; moreover if B is outerplanar, we take an embedding of B such that every vertex of B lies on the boundary of the outer face. For every $g \in V(G')$, we insert a copy of B in a face of gG' that contains the vertex (g, v) and then identify the vertex (g, v) with the vertex v of B. We denote the graph obtained so far by K. Observe that K is embedded in the plane. To the graph K we still need to add a copy of G for every vertex of B except v. These copies of G are connected only to the copies of G in $G \otimes G$ corresponding to the vertices of B. Choose an orientation of the plane. Now we consider two cases.

First, let $B = K_4$, with vertices v, v_1, v_2, v_3. Any three vertices of B are on the boundary of a same face in every embedding of B in the plane. Therefore, in the subgraph vG of K there exists a face such that the vertices $(v, v_1), (v, v_2), (v, v_3)$ all lie on its boundary; without loss of generality we may assume that they are arranged in this order (with respect to the chosen orientation of the plane). We may embed the graph v_1G in the plane such that the vertices $(v_1, v), (v_1, v_3), (v_1, v_2)$ lie on the outer face in this order, likewise, we may embed the graph v_2G in the plane such that the vertices $(v_2, v), (v_2, v_1), (v_2, v_3)$ lie on the outer face in this order, and we may embed the graph v_3G in the plane such that the vertices $(v_3, v), (v_3, v_2), (v_3, v_1)$ lie on the outer face in this order. Now place v_iG in the face of the subgraph vG of K that contains the vertices $(v, v_1), (v, v_2), (v, v_3)$ close to vertex (v, v_i) and connect vertices (v, v_i) and (v_i, v), for $i = 1, 2, 3$. The graphs v_iG are embedded in the plane such that it is possible to also connect the pairs of vertices $(v_1, v_2), (v_2, v_1)$, $(v_1, v_3), (v_3, v_1)$ and $(v_2, v_3), (v_3, v_2)$ without crossings. This gives us a planar embedding of the graph $G \otimes G$.

Next, let B be outerplanar. The subgraph vG of K is embedded in the plane and it contains a copy of B such that all the vertices of B are on the boundary of the same face of K, say F. We consider a planar embedding of the graph G in which all the vertices of the block B are on the boundary of the outer face and in the reverse order as the corresponding vertices in vG. For every vertex u of B except v, we place such a copy of G with vertex set $\{u\} \times V(G)$ into face F close to the vertex (v, u). Then we connect vertices (v, u) and (u, v) with an edge if u is a neighbour of v in G. The graphs uG, $u \in V(B)\backslash\{v\}$ are now all in the same face of K. Moreover, the subgraphs uB, $u \in V(B)\backslash\{v\}$, are all in the same face of K with all their vertices on the boundary of the outer face (of the embedding of uG in the plane) and the order of these vertices is reversed compared to the order of the corresponding vertices in vB. Note that the cyclic order of the graphs uB, $u \in V(B)\backslash\{v\}$ is the same as the order of the corresponding vertices in vB. Therefore, it is possible to connect the vertices (u, u') and (u', u) for every edge $\{u, u'\}$ of B without crossings. Again we obtain a planar embedding of the graph $G \otimes G$. This completes the proof. □

Theorem 2.17. *Let G, H be connected graphs and let $f\colon V(G) \to V(H)$ be a function. Assume that G is planar, $\Delta(G) \leq 3$ and H is outerplanar. Then $G \otimes_f H$ is planar.*

Proof. Denote $K = G \otimes_f H$ for convenience. Suppose K is not planar. Then it contains a subdivision of $K_{3,3}$ or K_5 as a subgraph. First assume that K contains a subdivision of $K_{3,3}$. Denote the set vertices of degree 3 of the subdivision of $K_{3,3}$ in K by X. There are four cases to consider, depending on how many vertices from X are in the same copy of H in K.

1. Every vertex from X is in a separate copy of H. If no path connecting two vertices from X passes through any other copies of H containing vertices from X, and at most one such path passes through every copy of H not containing vertices from X, then by contracting gH to a single vertex, for every $g \in V(G)$, we see that $K_{3,3}$ is a minor in G, so G is not planar. Otherwise we need at least four edges connecting some copy of H to the other copies of H in K. This is not possible, since the maximal degree in G is at most three.

2. There are between two and four vertices from X in some gH. Then we need at least four edges connecting gH to the other copies of H in K. This is again not possible, since the maximal degree of G is at most 3.

3. There are five vertices from X in some gH and one vertex from X in some $g'H$ for $g \neq g'$. Since H is outerplanar, gH cannot contain a subdivision of $K_{2,3}$. Therefore, we need at least two edges going out of gH to obtain a subdivision of $K_{2,3}$ from the five vertices in gH. We also need three edges going out of gH to connect gH to the vertex of $K_{3,3}$ in $g'H$. This is again not possible, since the maximal degree of G is at most 3.

4. The only remaining possibility is that all six vertices from X are in the same copy gH of H. Since H is outerplanar, there can be at most seven edges (or paths) between pairs of vertices of $K_{3,3}$ in gH. The remaining two paths must go through the other copies of H, which means that we again need at least four edges connecting gH to the other copies of H in K. A contradiction.

Therefore, K does not contain a subdivision of $K_{3,3}$. Next assume that K contains a subdivision of K_5. Denote the set vertices of degree 4 of the subdivision of K_5 in K by Y. There are two cases to consider, depending on how many vertices from Y are in the same copy of H.

1. There are between one and four vertices from Y in some gH. Then we need at least four edges connecting gH to the other copies of H in K. This is not possible, since the maximal degree in G is at most three.

2. All five vertices from Y are in the same copy of H. Since H is outerplanar, it does not contain a subdivision of K_4 or $K_{2,3}$. Therefore, there can be at most eight edges (or paths) between pairs of these vertices in gH (in fact, there can be at most six such paths). The remaining two paths must go through other copies of H, which means that we need at least four edges connecting gH to other copies of H in K. A contradiction.

It follows that K does not contain a subdivision of $K_{3,3}$ or K_5, so it is planar. □

If a connected graph is not planar, it is natural to consider its genus. The genus of a graph G is denoted by $\gamma(G)$. Recall that by Lemma 2.5, if the graph H is connected, the graph G is a minor of $G \otimes_f H$ for any function $f: V(G) \to V(H)$, and the graph $G \otimes_f H$ contains $|G|$ copies of the graph H as induced subgraphs. Suppose G, H are connected and f is arbitrary. Then it is easy to see, cf. [21, Theorem 4.4.2], that

$$\gamma(G \otimes_f H) \geq \gamma(G) + |G| \cdot \gamma(H). \tag{2.1}$$

Note that the bound is not sharp even if the factors are planar. In the case of a planar Sierpiński product, we were able to settle the case in Theorem 2.13. It would be interesting to find some sufficient condition for the equality in (2.1) to hold also for non-planar Sierpiński products.

3 Symmetries of the Sierpiński product of graphs

Throughout this section let G, H be connected graphs and let $f: V(G) \to V(H)$ be a function. Recall that the edge set of $G \otimes_f H$ can be naturally partitioned into two subsets:

- *inner edges* $\{(g, h), (g, h')\}$ for every vertex $g \in V(G)$ and every edge $\{h, h'\} \in E(H)$, and
- *connecting edges* $\{(g, f(g')), (g', f(g))\}$ for every edge $\{g, g'\} \in E(G)$.

We call this partition of the edge set the *fundamental edge partition*. We will say that an automorphism of $G \otimes_f H$ *respects the fundamental edge partition* if it takes inner edges to inner edges and connecting edges to connecting edges. We denote the set of all automorphisms of $G \otimes_f H$ that respect the fundamental edge partition by $\tilde{A}(G, H, f)$. This set is a subgroup of the whole automorphism group of $G \otimes_f H$. For connected graphs G and H, the automorphisms that respect the fundamental edge partition have the following useful property.

Proposition 3.1. *Let G and H be connected graphs. Then every automorphism $\tilde{\gamma} \in \tilde{A}(G, H, f)$ permutes the subgraphs gH, $g \in G$. Moreover, the restriction $\tilde{\gamma}|_{V(gH)} : V(gH) \to V(g'H)$, where $g' \in V(G)$, is a graph isomorphism.*

In this section we first show that every automorphism of $G \otimes_f H$ that respects the fundamental edge partition induces automorphisms of G and H. And conversely, we define two families of automorphisms of $G \otimes_f H$ that respect the fundamental edge partition using automorphisms of G and H. As it turns out, one of them is a subfamily of the other one. Then we show that in several cases all the automorphisms of $G \otimes_f H$ respect the fundamental edge partition. Finally, we focus on the case when $G = H$ and f is an automorphism. In this case we can completely describe the group of automorphisms that respect the fundamental edge partition.

3.1 Automorphisms that respect the fundamental edge partition

Let $\tilde{\gamma}$ be an automorphism of $G \otimes_f H$ that respects the fundamental edge partition. Then, it permutes the subgraphs gH, $g \in G$. Define the function $\gamma\colon V(G) \to V(G)$ such that $\gamma(g) = g'$ if $\tilde{\gamma}$ maps gH to $g'H$. Obviously, γ is a bijection. Let $\{g, g_1\}$ be an edge of G. Then $\{(g, f(g_1)), (g_1, f(g))\}$ is a connecting edge of $G \otimes_f H$. Since $\tilde{\gamma}$ respects the fundamental edge partition, it maps this edge to another connecting edge, say $\{(g', f(g_1')), (g_1', f(g'))\}$, where g' and g_1' are adjacent in G. But then γ maps the edge $\{g, g_1\}$ to an edge (i.e. to $\{g', g_1'\}$) and γ is an automorphism. We will say that γ is the *projection* of $\tilde{\gamma}$ on G. Conversely, $\tilde{\gamma}$ is a *lift* of γ. Note that the projection of $\tilde{\gamma} \in \mathrm{Aut}(G \otimes_f H)$ on G is uniquely defined. However, given an automorphism of G, it can have a unique lift, more than one lift or none at all.

On the other hand, the application of $\tilde{\gamma}$ on every copy of gH in $G \otimes_f H$ induces an automorphism γ_g of H, defined by $\gamma_g(h) = h'$ if $\tilde{\gamma}$ sends (g, h) to (g_1, h') for some $g_1 \in V(G)$ and $h' \in V(H)$.

We will now introduce the first family of automorphisms of $G \otimes_f H$ that can be obtained from automorphisms of G and H. All such automorphisms respect the fundamental edge partition.

Definition 3.2. Let G, H be connected graphs and let $f\colon V(G) \to V(H)$ be a function. Let $\alpha \in \mathrm{Aut}(G)$ and let $\mathcal{B} : V(G) \to \mathrm{Aut}(H)$ be a function. For simplicity we will write β_g instead of $\mathcal{B}(g)$ for $g \in V(G)$. Define the function $\Psi(\alpha, \mathcal{B})\colon V(G \otimes_f H) \to V(G \otimes_f H)$ by

$$\Psi(\alpha, \mathcal{B})\colon (g, h) \mapsto (\alpha(g), \beta_g(h)).$$

If $\mathcal{B}$ is a constant function, say $\beta_g = \beta$ for all $g \in V(G)$, we denote $\Psi(\alpha, \mathcal{B})$ by $\Psi(\alpha, \beta)$.

By the discussion at the beginning of this section, we may conclude that the following holds.

Theorem 3.3. *Let G, H be connected graphs and let $f\colon V(G) \to V(H)$ be a function. Then, every automorphism of $G \otimes_f H$ that respects the fundamental edge partition is of form $\Psi(\alpha, \mathcal{B})$ for some $\alpha \in \mathrm{Aut}(G)$ and some function $\mathcal{B}\colon V(G) \to \mathrm{Aut}(H)$.*

We now determine when the function $\Psi(\alpha, \mathcal{B})$ from Definition 3.2 is an automorphism. In Propositions 3.4, 3.5, 3.6 and Corollaries 3.7, 3.8, and 3.9, the assumptions from Definition 3.2 hold.

Proposition 3.4. *The function $\Psi(\alpha, \mathcal{B})$ is a bijection.*

Proof. It is enough to prove that $\Psi(\alpha, \mathcal{B})$ is injective. This is straightforward since α and β_g, $g \in V(G)$, are all injective. □

Proposition 3.5. *The function $\Psi(\alpha, \mathcal{B})$ is an automorphism if and only if for every $g \in V(G)$ we have $f \circ \alpha = \beta_g \circ f$ on $N(g)$. Moreover, in this case $\Psi(\alpha, \mathcal{B})$ respects the fundamental edge partition.*

Proof. We first show that $\Psi(\alpha, \mathcal{B})$ always maps an inner edge to an inner edge. To see this, let $e = \{(g, h_1), (g, h_2)\}$ be an inner edge. Then $\Psi(\alpha, \mathcal{B})$ maps the edge e to $\{(\alpha(g), \beta_g(h_1)),(\alpha(g), \beta_g(h_2))\}$, which is an inner edge since β_g is an automorphism of H.

Suppose now that $\Psi(\alpha,\mathcal{B})$ is an automorphism. Since $\Psi(\alpha,\mathcal{B})$ maps inner edges to inner edges, it must map connecting edges to connecting edges. Let $e = \{(g, f(g_1)), (g_1, f(g))\}$ be a connecting edge. Then $\Psi(\alpha,\mathcal{B})(e) = \{(\alpha(g), \beta_g(f(g_1))), (\alpha(g_1), \beta_{g_1}(f(g)))\}$ is also a connecting edge. Therefore $f(\alpha(g_1)) = \beta_g(f(g_1))$. Since g_1 can be any neighbour of g in G, we have $f \circ \alpha = \beta_g \circ f$ on $N(g)$.

Conversely, let $f \circ \alpha = \beta_g \circ f$ on $N(g)$ for every $g \in V(G)$. Let $e = \{(g, f(g_1)), (g_1, f(g))\}$ be a connecting edge in $G \otimes_f H$. Then $\Psi(\alpha,\mathcal{B})(e) = \{(\alpha(g), \beta_g(f(g_1))), (\alpha(g_1), \beta_{g_1}(f(g)))\}$. Since $f(\alpha(g)) = \beta_{g_1}(f(g))$ and $f(\alpha(g_1)) = \beta_g(f(g_1))$, $\Psi(\alpha,\mathcal{B})(e)$ is a connecting edge. Therefore, $\Psi(\alpha,\mathcal{B})$ is an automorphism. □

Proposition 3.6. *The function $\Psi(\alpha,\beta)$ is an automorphism if and only if $f \circ \alpha = \beta \circ f$.*

Proof. Let $f \circ \alpha = \beta \circ f$ on $N(G)$ for every $g \in V(G)$. Since G is connected, it has no isolated vertices, and so $f \circ \alpha = \beta \circ f$ on $V(G)$. The claim then follows from Proposition 3.5. □

A few special cases now follow as simple corollaries. Recall that $\alpha, \beta, \Psi(\alpha,\beta)$ are defined in Definition 3.2.

Corollary 3.7. *Suppose $G = H$ and f is a bijective function. Then the function $\Psi(\alpha,\beta)$ is an automorphism if and only if $\beta = f \circ \alpha \circ f^{-1}$.*

Corollary 3.8. *Suppose $G = H$ and f is the identity function. Then the function $\Psi(\alpha,\beta)$ is an automorphism if and only if $\alpha = \beta$.*

Corollary 3.9. *Suppose $V(G) \subseteq V(H)$, f is the identity function on its domain and $\beta|_{V(G)} = \alpha$. Then the function $\Psi(\alpha,\beta)$ is an automorphism.*

Remark 3.10. If f is injective and $G \neq H$, we can always relabel the vertices of G, H such that f is the identity on its domain.

We now give some examples showing that f need not be injective or surjective and we can still have automorphisms of type $\Psi(\alpha,\mathcal{B})$.

Example 3.11. Let $G = K_3$ and $H = K_{3,3}$ with $V(G) = \{1,2,3\}$ and $V(H) = \{1,2,\ldots,6\}$, with adjacencies as in Figure 6. Let $f\colon V(G) \to V(H)$ map $1 \mapsto 1$, $2 \mapsto 3$, $3 \mapsto 5$. Let $\alpha = (1\ 2\ 3)$, $\beta_1 = (1\ 3\ 5)(2\ 4\ 6)$, $\beta_2 = (1\ 3\ 5)(2\ 6\ 4)$, $\beta_3 = (1\ 3\ 5)$ and let $\mathcal{B}\colon V(G) \to \mathrm{Aut}(G)$ be defined by $B(g) = \beta_g$. Then $f \circ \alpha = \beta_1 \circ f = \beta_2 \circ f = \beta_3 \circ f$ and

$$\Psi(\alpha,\mathcal{B}) = (11\ 23\ 35)(12\ 24\ 32)(13\ 25\ 31)(14\ 26\ 34)(15\ 21\ 33)(16\ 22\ 36)$$

is an automorphism of $G \otimes_f H$ that cyclically permutes the subgraphs gH, see Figure 6.

Example 3.12. Let $G = H = K_{1,3}$ with the edge set $\{\{1,2\},\{2,3\},\{2,4\}\}$, and let $f\colon V(G) \to V(G)$ be defined as $f = (1\ 2\ 3\ 4)$. Note that f is a bijection that is not an automorphism of G. If $\alpha = (3\ 4)$ and $\beta = f \circ \alpha \circ f^{-1} = (1\ 4)$, then $f \circ \alpha = \beta \circ f$ and

$$\Psi(\alpha,\beta) = (11\ 14)(21\ 24)(31\ 44)(32\ 42)(33\ 43)(34\ 41)$$

is an automorphism of $G \otimes_f G$, that swaps the copies $3G$ and $4G$, see Figure 7.

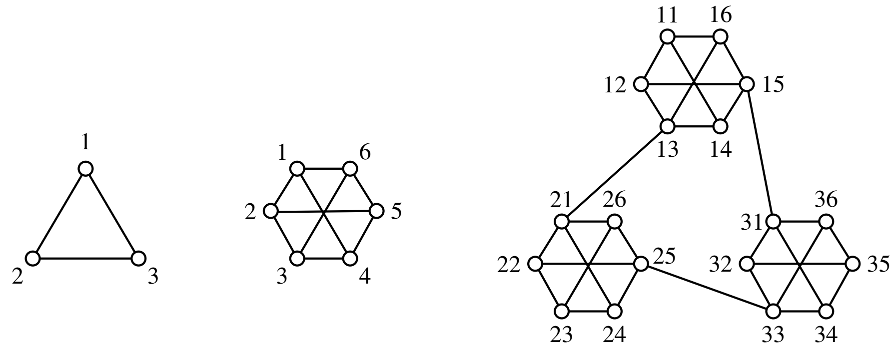

Figure 6: The graphs K_3, $K_{3,3}$ and their Sierpiński product with respect to $f : V(K_3) \to V(K_{3,3})$, $f : 1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 5$.

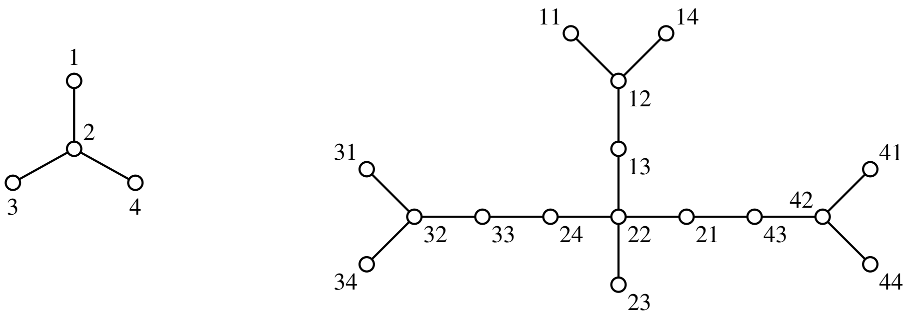

Figure 7: The graph $G = K_{1,3}$ and the Sierpiński product $G \otimes_f G$ with respect to $f = (1\,2\,3\,4)$.

Example 3.13. Let $G = C_4$ with $V(G) = \{1, 2, 3, 4\}$ and adjacencies as in Figure 8, and let H be a star $K_{1,3}$, with the edge set $\{\{1, 2\}, \{2, 3\}, \{2, 4\}\}$. Let $f\colon V(G) \to V(H)$ map $1 \mapsto 2$, $2 \mapsto 2$, $3 \mapsto 4$ and $4 \mapsto 3$. Note that the function f is neither injective nor surjective. If $\alpha = (1\,2)(3\,4)$ and $\beta = (3\,4)$, then $f \circ \alpha = \beta \circ f$ and

$$\Psi(\alpha, \beta) = (11\ 21)(12\ 22)(13\ 24)(14\ 23)(31\ 41)(32\ 42)(33\ 44)(34\ 43)$$

is a reflection automorphism of $G \otimes_f H$, swapping the copies $1H, 2H$ and $3H, 4H$, see Figure 8.

Now let us introduce the second family of automorphisms of $G \otimes_f H$. Let $g \in V(G)$ and $\beta \in \mathrm{Aut}(H)$. Define the function $\Phi(g, \beta) : V(G \otimes_f H) \to V(G \otimes_f H)$ by

$$\Phi(g, \beta)\colon (g_1, h_1) \mapsto \begin{cases} (g_1, h_1) & \text{if } g_1 \neq g, \\ (g_1, \beta(h_1)) & \text{if } g_1 = g. \end{cases}$$

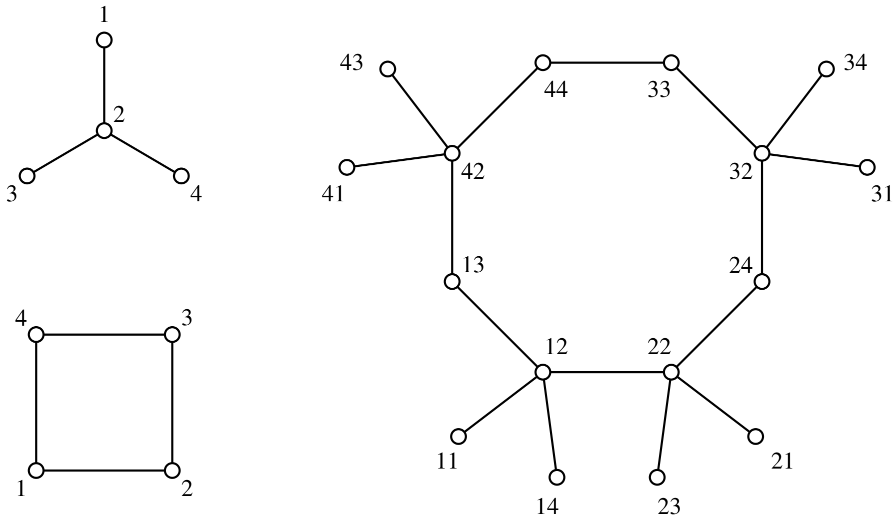

Figure 8: The graphs C_4, $K_{1,3}$ and their Sierpiński product with respect to $f\colon V(C_4) \to V(K_{1,3})$, $f\colon 1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 4, 4 \mapsto 3$.

Proposition 3.14. *Let $g \in V(G)$ and let $\beta \in \mathrm{Aut}(H)$. The function $\Phi(g,\beta)$ is an automorphism of $G \otimes_f H$ if and only if β is in the pointwise stabilizer of $f(N(g))$. Moreover, in this case $\Phi(g,\beta)$ respects the fundamental edge partition.*

Proof. The function $\Phi(g,\beta)$ is obviously a bijection since it fixes all the vertices of $G \otimes_f H$ not in gH and it permutes the vertices in gH. It also fixes the inner edges and connecting edges that do not have any endvertices in gH and it permutes the inner edges in gH.

Take a vertex $g' \in N(g)$. Then $\{(g, f(g')), (g', f(g))\}$ is a connecting edge. The function $\Phi(g,\beta)$ maps $\{(g, f(g')), (g', f(g))\}$ to the set $\{(g, \beta(f(g')), (g', f(g))\}$, which is an edge if and only if $\beta(f(g')) = f(g')$. So $\Phi(g,\beta)$ is an automorphism if and only if β is in the stabilizer of $f(g')$ for every $g' \in N(G)$. □

Remark 3.15. Note that by Theorem 3.3, the function $\Phi(g,\beta)$ is the same as $\Psi(\alpha,\mathcal{B})$ for some $\alpha \in \mathrm{Aut}(G)$ and $\mathcal{B}\colon V(G) \to \mathrm{Aut}(H)$. Indeed, it is easy to verify that for $\alpha = \mathrm{id}$ and $\mathcal{B}$ defined by the rules $\mathcal{B}\colon g_1 \to \mathrm{id}$ if $g_1 \neq g$ and $\mathcal{B}\colon g \to \beta$, we have $\Phi(g,\beta) = \Psi(\alpha,\mathcal{B})$.

Given a group X acting on a set Y, we denote by $X_{(Y)}$ the pointwise stabilizer of Y, i.e. the subgroup of X that fixes every element of Y. For $g \in G$ denote by $\hat{B}_g(G,H,f)$ the group generated by $\{\Phi(g,\beta_g) | \, \beta_g \in \mathrm{Aut}(H)_{(f(N(g)))}\}$. Denote by $\hat{B}(G,H,f)$ the group generated by $\{\hat{B}_g(G,H,f) | \; g \in V(G)\}$. We will now study the structure of the group $\hat{B}(G,H,f)$.

Proposition 3.16. *Let g, g' be distinct vertices of G and let $\beta_g \in \mathrm{Aut}(H)_{(f(N(g)))}$, $\beta_{g'} \in \mathrm{Aut}(H)_{(f(N(g')))}$. Then $\Phi(g,\beta_g)$ and $\Phi(g',\beta_{g'})$ commute.*

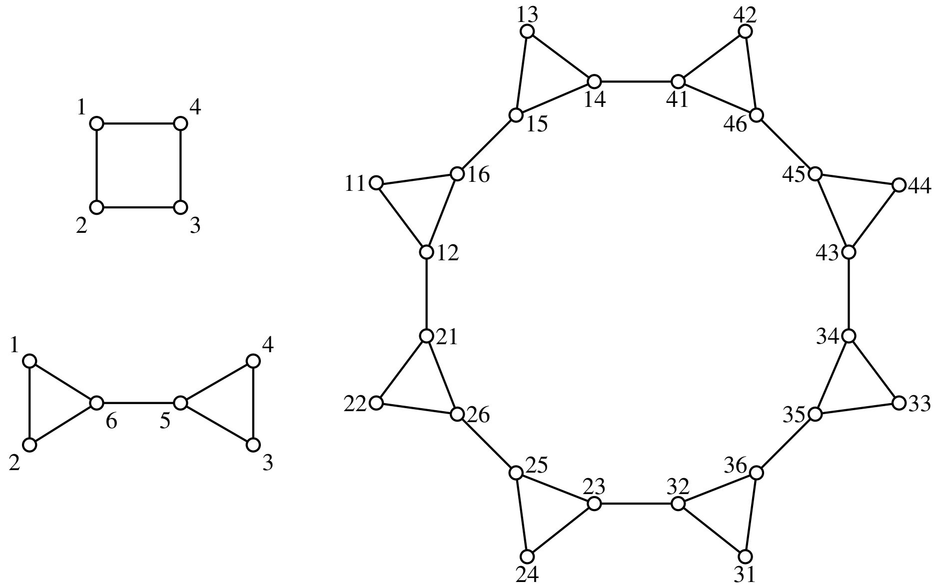

Figure 9: The graphs C_4, $2K_3 + e$ and their Sierpiński product with respect to $f = \mathrm{id}$.

Proof. The functions $\Phi(g, \beta_g)$ and $\Phi(g', \beta_{g'})$ commute since as permutations they have disjoint supports. □

Theorem 3.17. *The group $\hat{B}(G, H, f)$ is a subgroup of the group $\tilde{A}(G, H, f)$ and is a direct product*

$$\hat{B}(G, H, f) = \prod_{g \in V(G)} \hat{B}_g(G, H, f). \tag{3.1}$$

Moreover, the group $\hat{B}(G, H, f)$ is isomorphic to the group $\prod_{g \in V(G)} \mathrm{Aut}(H)_{(f(N(g)))}$.

Proof. The group $\hat{B}(G, H, f)$ is a subgroup of the group $\tilde{A}(G, H, f)$ by definition and Propositon 3.14. The groups $\hat{B}_g(G, H, f)$, $g \in V(G)$, have pairwise only the identity in common, they generate the group $\hat{B}(G, H, f)$, and the elements of two distinct groups $\hat{B}_g(G, H, f)$ commute, therefore equation (3.1) holds. The last claim is true since for every $g \in G$, the groups $\hat{B}_g(G, H, f)$ and $\mathrm{Aut}(H)_{(f(N(g)))}$ are isomorphic in the obvious way. □

3.2 When do all the automorphisms respect the fundamental edge partition

Given connected graphs G, H and a function $f\colon V(G) \to V(H)$, in general there can exist automorphisms of $G \otimes_f H$ that do not respect the fundamental edge partition. Figure 9 shows such an example. There, $G = C_4$, $H = 2K_3 + e$ and $f\colon V(G) \to V(H)$ is the identity function on its domain. One can easily observe that by rotating the graph $G \otimes_f H$, the inner edge $\{16, 15\}$ can be mapped to the connecting edge $\{14, 41\}$.

Note that in the example above, the graph H is not 2-connected. When two graphs G and H are both 2-connected, we have so far not been able to find an automorphism of

$G \otimes_f H$ that does not respect the fundamental edge partition. Therefore, we propose the following conjecture.

Conjecture 3.18. *Let G, H be 2-connected graphs and let $f\colon V(G) \to V(H)$ be a function. Then*

$$\tilde{A}(G, H, f) = \mathrm{Aut}(G \otimes_f H).$$

In this section we prove this conjecture for two special cases. In the first case $G = H, f \in \mathrm{Aut}(G)$ and G is a regular triangle-free graph. In the second case every edge of H is contained in a short cycle. Note that in these two cases the assumption that G, H are 2-connected is not needed.

Theorem 3.19. *Let G be a connected regular triangle-free graph and let $f\colon V(G) \to V(G)$ be an automorphism of G. Then every automorphism of $G \otimes_f G$ respects the fundamental edge partition. In other words,*

$$\tilde{A}(G, G, f) = \mathrm{Aut}(G \otimes_f G).$$

Proof. Let k denote the valency of G. Then the endvertices of every connecting edge in $G \otimes_f G$ have valency $k + 1$ by Lemma 2.8. An endvertex of an inner edge may have valency k or $k + 1$. Clearly, if at least one endvertex of an inner edge has valency k, this edge cannot be mapped to a connecting edge by any automorphism.

Suppose now that both endvertices of an inner edge $\{(g, g_1), (g, g_2)\}$ have degree $k+1$. This is only possible if (g, g_1) and (g, g_2) are the endvertices of some connecting edges, say $\{(g, g_1), (g_1', f(g))\}$ and $\{(g, g_2), (g_2', f(g))\}$ where $g_1 = f(g_1')$ and $g_2 = f(g_2')$. But then g_1' and g_2' are adjacent to g in G. Since g_1 and g_2 are adjacent in G and f is an automorphism, g_1' and g_2' are also adjacent. But then g, g_1', g_2' form a triangle in G, a contradiction. Therefore no inner edge can be mapped to a connecting edge, so every automorphism of $G \otimes_f G$ respects the fundamental edge partition. □

Lemma 3.20. *Let G and H be graphs and let $f\colon V(G) \to V(H)$ be a function. Let $\{g, g'\}$ be an edge of G.*

(i) *If $\{g, g'\}$ is not contained in any cycle of G, then the edge $\{(g, f(g')), (g', f(g))\}$ is not contained in any cycle of $G \otimes_f H$.*

(ii) *Let c be the length of the shortest cycle that contains $\{g, g'\}$. Then the shortest cycle that contains the edge $\{(g, f(g')), (g', f(g))\}$ in $G \otimes_f H$ has length at least c.*

(iii) *Suppose that f is locally injective and let c be the length of the shortest cycle that contains $\{g, g'\}$. Then the shortest cycle that contains the edge $\{(g, f(g')), (g', f(g))\}$ in $G \otimes_f H$ has length at least $2c$.*

Proof. Let C be a cycle in $G \otimes_f H$ that contains $\{(g, f(g')), (g', f(g))\}$. Suppose that $\{(g, f(g')), (g', f(g))\}, \{(g', f(g_1)), (g_1, f(g'))\}, \dots, \{(g_k, f(g)), (g, f(g_k))\}$ are the connecting edges in C in that order. Then $gg'g_1g_2 \dots g_kg$ is a cycle of length k in G that contains the edge $\{g, g'\}$, so $k \geq c$. Furthermore, if $\{g, g'\}$ is not contained in any cycle of G, then the edge $\{(g, f(g')), (g', f(g))\}$ cannot be contained in any cycle of $G \otimes_f H$. Recall that if f is locally injective, any vertex of $G \otimes_f H$ is an endvertex of at most one connecting edge by Lemma 2.8. Therefore, in this case the shortest cycle that contains $\{(g, f(g')), (g', f(g))\}$ has length at least $2c$. □

Theorem 3.21. *Let G and H be connected graphs, let $f\colon V(G) \to V(H)$ be a function and let the girth of G be equal to c. In any of the following cases, every automorphism of $G \otimes_f H$ respects the fundamental edge partition, i.e.*

$$\tilde{A}(G, G, f) = \mathrm{Aut}(G \otimes_f G).$$

(i) *G is a tree and H is a bridgeless graph;*

(ii) *every edge of H is contained in a cycle of length at most $c - 1$;*

(iii) *the function f is locally injective and every edge of H is contained in a cycle of length at most $2c - 1$.*

Proof. By Lemma 3.20, the shortest cycle that contains a connecting edge has length at least c in case (ii), length at least $2c$ in case (iii) and is not contained in any cycle in case (i). Since every inner edge is contained in a cycle in case (i), in a cycle of length at most $c - 1$ in case (ii), and in a cycle of length at most $2c - 1$ in case (iii), a connecting edge cannot be mapped to an inner edge by any automorphism. □

3.3 Group of automorphisms of $G \otimes_f G$

We now consider the group of automorphisms that respect the fundamental edge partition in the special case when $G = H$ and $f\colon V(G) \to V(G)$ is an automorphism. Since in this case $G \otimes_f G$ is isomorphic to $G \otimes G$, we could restrict ourselves to the case where f is the identity. Note that in that case, the structure of the automorphism group was sketched in the paper [8], but the proofs were never published.

Recall that by Corollary 3.7, every automorphism α of G has a lift, $\Psi(\alpha, f \circ \alpha \circ f^{-1})$. We call this automorphism the *diagonal automorphism* of $G \otimes_f G$ corresponding to α, and denote it by $\bar{\alpha}$. Denote by $\bar{A}(G, f)$ the set of all diagonal automorphisms. The following proposition is straightforward to prove.

Proposition 3.22. *The set $\bar{A}(G, f)$ is a subgroup of $\tilde{A}(G, G, f)$, isomorphic to $\mathrm{Aut}(G)$.*

To determine the structure of the group $\tilde{A}(G, G, f)$, we first show that every element of $\tilde{A}(G, G, f)$ can be written as a product of an element from $\hat{B}(G, G, f)$ and an element of $\bar{A}(G, f)$. Furthermore, we show that $\hat{B}(G, G, f)$ is a normal subgroup of $\tilde{A}(G, G, f)$.

Proposition 3.23. *Let G be a connected graph and let $f\colon V(G) \to V(G)$ be an automorphism. Let $\tilde{\gamma}$ be an automorphism of $G \otimes_f G$ that preserves the fundamental edge partition. Then there exist $\alpha \in \mathrm{Aut}(G)$ and $\beta_g \in \mathrm{Aut}(G)_{(f(N(g)))}$ for every $g \in V(G)$ such that $\tilde{\gamma} = \bar{\alpha} \left(\prod_{g \in V(G)} \Phi(g, \beta_g)\right)$.*

Proof. Let α be the projection of $\tilde{\gamma}$ to $\mathrm{Aut}(G)$. Then $\bar{\alpha} = \Psi(\alpha, f \circ \alpha \circ f^{-1})$ permutes the copies gG in the right way, such as $\tilde{\gamma}$ does. Observe that $\bar{\alpha}$ already agrees with $\tilde{\gamma}$ on the endvertices of all the connecting edges. To obtain $\tilde{\gamma}$ from $\bar{\alpha}$, we only need to adjust, for every $g \in V(G)$, the action of $\bar{\alpha}$ on the vertices from $f(N(g))$ that are not endvertices of connecting edges. We can do this on every copy gG separately, by acting with $\Phi(g, \beta_g)$, where $\beta_g \in \mathrm{Aut}(G)$ is induced by $\bar{\alpha}^{-1}\tilde{\gamma}$. Also, $\beta_g \in \mathrm{Aut}(G)_{(f(N(g)))}$ since the vertices from $f(N(g))$ have the right image already and are fixed by β_g. □

Proposition 3.24. *Let G be a connected graph and let $f\colon V(G) \to V(G)$ be an automorphism. Then the group $\hat{B}(G,G,f)$ is a normal subgroup of the group $\tilde{A}(G,G,f)$.*

Proof. Observe that the function $\lambda\colon \tilde{A}(G,G,f) \to \bar{A}(G,f)$ defined by $\lambda\colon \Psi(\alpha,\mathcal{B}) \to \Psi(\alpha, f\circ\alpha\circ f^{-1})$ is a homomorphism of groups, with $\hat{B}(G,G,f)$ being its kernel. Therefore, $\hat{B}(G,G,f)$ is a normal subgroup of $\tilde{A}(G,G,f)$. □

Theorem 3.25. *Let G be a connected graph and let $f\colon V(G) \to V(G)$ be an automorphism. Then the group $\tilde{A}(G,G,f)$ is a semidirect product,*

$$\tilde{A}(G,G,f) = \bar{A}(G,f) \ltimes \hat{B}(G,G,f).$$

Proof. The group $\hat{B}(G,G,f)$ is a normal subgroup of $\tilde{A}(G,G,f)$ by Proposition 3.24. By Proposition 3.23, every element of $\tilde{A}(G,G,f)$ can be written as a product of a diagonal automorphism and an element from $\hat{B}(G,G,f)$. Moreover, only the identity is in both $\bar{A}(G,f)$ and $\hat{B}(G,G,f)$. This proves that $\tilde{A}(G,G,f)$ is a semidirect product of $\bar{A}(G,f)$ and $\hat{B}(G,G,f)$. □

4 Sierpiński product with multiple factors

To form a Sierpiński product $G_3 \otimes_f (G_2 \otimes_{f_1} G_1)$ of graphs G_3, G_2 and G_1, one needs functions $f_1\colon V(G_2) \to V(G_1)$ and $f\colon V(G_3) \to G_2 \otimes_{f_1} G_1$, which is rather impractical. Suppose a function $f_2\colon V(G_3) \to V(G_2)$ is given. Then a function $f\colon V(G_3) \to V(G_2 \otimes_{f_1} G_1)$ can be defined in a natural way as $f(g) = (f_2(g), f_1(f_2(g))$ for $g \in V(G_3)$. In other words, let $\varphi\colon V(G_2) \to V(G_2 \otimes_{f_1} G_1)$ be the function that maps every vertex $g \in V(G_2)$ to the vertex $(g, f_1(g)) \in V(G_2 \otimes_{f_1} G_1)$. Then $f = \varphi \circ f_2$. Now we can define the Sierpiński product of the graphs G_3, G_2 and G_1 with respect to f_2 and f_1 in the following way:

$$G_3 \otimes_{f_2} G_2 \otimes_{f_1} G_1 = G_3 \otimes_{\varphi\circ f_2} (G_2 \otimes_{f_1} G_1).$$

Note that with given functions f_2 and f_1, we cannot form this product in any other way, therefore, the Sierpiński product is not associative.

In Figure 10 it is shown how the product $C_3 \otimes_{f_2} C_4 \otimes_{f_1} C_3$ is formed in two steps (with $f_1\colon V(C_4) \to V(C_3)$, $f_1\colon i \mapsto i \pmod 3$ and $f_2\colon V(C_3) \to V(C_4)$ being the identity function on its domain).

It is now easy to see that Sierpiński products possess a nice recursive structure, similar to Sierpiński graphs and generalized Sierpiński graphs. By the same reasoning as above, the product $G_m \otimes_{f_{m-1}} \cdots \otimes_{f_2} G_2 \otimes_{f_1} G_1$, where $V(G_\ell) = \{0,1,\ldots,|G_\ell|-1\}$, and $f_\ell\colon V(G_{\ell+1}) \to V(G_\ell)$, $\ell = 1,\ldots,m-1$, are arbitrary functions, can be constructed as follows.

- First, take $|G_2|$ copies of the graph G_1 and label them iG_1, $i \in \{0,\ldots,|G_2|-1\}$. Vertices of these graphs have labels of form g_2g_1, where $g_2 \in V(G_2)$ and $g_1 \in V(G_1)$.

- Connect any two copies iG_1 and jG_1 if there is an edge $\{i,j\}$ in G_2. More precisely, if $\{i,j\} \in E(G_2)$, we add an edge $\{if_1(j), jf_1(i)\}$ between iG_1 and jG_1. The resulting graph is then indeed the Sierpiński product $G_2 \otimes_{f_1} G_1$.

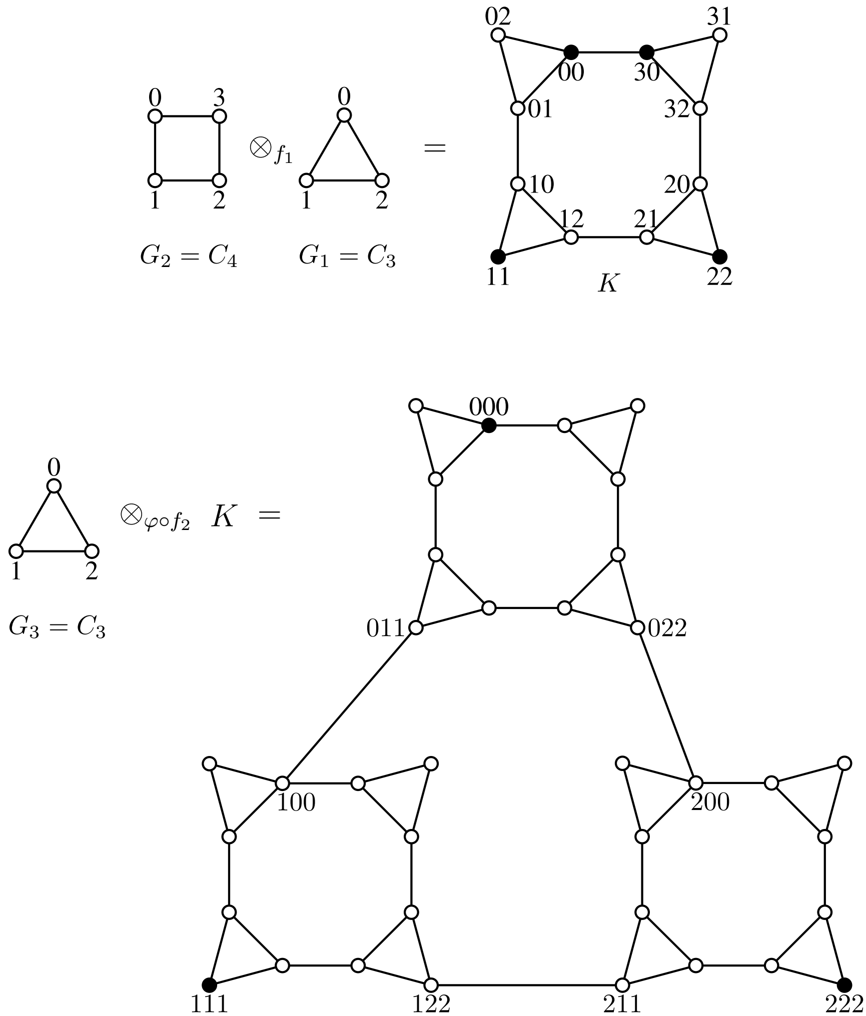

Figure 10: Construction of the graph $C_3 \otimes_{f_2} C_4 \otimes_{f_1} C_3$, where $f_1 \colon i \mapsto i \pmod 3$ and $f_2 = \mathrm{id}$.

- Next, we form the Sierpiński product of the graphs G_3 and $K(2) := G_2 \otimes_{f_1} G_1$. To do so we take $|G_3|$ copies of the graph $K(2)$, label them $iK(2)$, $i \in \{0, \dots, |G_3| - 1\}$, and connect $iK(2)$ and $jK(2)$ whenever $\{i, j\}$ is an edge in G_3. Such an edge then has the form $\{if_2(j)f_1(f_2(j)), jf_2(i)f_1(f_2(i))\}$.

- The final step is to form the Sierpiński product of the graphs G_m and $K(m-1)$ in the same way as we formed all the products so far: make $|G_m|$ copies of $K(m-1)$ and label them $iK(m-1)$; then for every edge $\{i, j\}$ in G_m we add an edge between copies $iK(m-1)$ and $jK(m-1)$. Such an edge has then the following form

$$\{if_{m-1}(j) \dots f_1(f_2 \dots (f_{m-1}(j)) \dots),\ jf_{m-1}(i) \dots f_1(f_2 \dots (f_{m-1}(i)) \dots)\}.$$

The resulting graph is the product $G_m \otimes_{f_{m-1}} \cdots \otimes_{f_2} G_2 \otimes_{f_1} G_1$.

If $G_1 = \cdots = G_m = G$ and functions $f_1, \ldots, f_{m-1}$ are all the identity function, then $G_m \otimes_{f_{m-1}} \cdots \otimes_{f_2} G_2 \otimes_{f_1} G_1$ is the generalized Sierpiński graph S_G^n; see also [8].

We can calculate the order and the size of the Sierpiński product of multiple factors directly from the above construction.

Proposition 4.1. *Let $m \geq 2$, and let $G_1, \ldots, G_m$ be arbitrary graphs. Further, let $f_1 \colon V(G_2) \to V(G_1), \ldots, f_{m-1} \colon V(G_m) \to V(G_{m-1})$ be arbitrary functions. Then the order and the size of the Sierpiński product $G_m \otimes_{f_{m-1}} \cdots \otimes_{f_1} G_1$ are as follows*

$$|G_m \otimes_{f_{m-1}} \cdots \otimes_{f_1} G_1| = \prod_{\ell=1}^{m} |G_\ell| \,,$$

$$||G_m \otimes_{f_{m-1}} \cdots \otimes_{f_1} G_1|| = \sum_{\ell=1}^{m} \left(\prod_{j=\ell+1}^{m} |G_j| \right) ||G_\ell|| \,.$$

Note that neither the order nor the size of the Sierpiński product depends on the functions f_ℓ. It would also be interesting to study some properties of the Sierpiński product with multiple factors, such as diameter and girth.

5 Conclusion

This paper generalizes Sierpiński graphs even further than generalized Sierpiński graphs, where the whole structure is based only on one graph. Here we create a product-like structure of two (or more) factors. Some basic graph theoretical properties are studied in detail, and planar Sierpiński products are completely characterized. Apart from this, the symmetries of Sierpiński products are studied as well. In general, these are not fully understood. In several cases we are able to determine the automorphism group of the Sierpiński product of two graphs exactly.

In [14] an algorithm is given for recognizing generalized Sierpiński graphs. Given a graph it is also natural to ask whether it can be represented as a Sierpiński product of two or more graphs. Moreover, one can ask if such a representation is unique. The latter question has a negative answer. Consider the Sierpiński product of C_4 and $2K_3 + e$ with function f as in Figure 9. It can be easily verified that it is isomorphic to $C_8 \otimes_{f'} K_3$ where $f' \colon V(C_8) \to V(K_3)$ is defined by $f'(1) = f'(2) = f'(5) = f'(6) = 1$ and $f'(3) = f'(4) = f'(7) = f'(8) = 2$. However, in this case not all the factors are prime with respect to the Sierpiński product: C_8 can be represented as a Sierpiński product of C_4 and K_2 while $2K_3 + e$ can be represented as a Sierpiński product of K_2 and K_3. It would be interesting to see whether there exist prime graphs with respect to the Sierpiński product G, H, G', H' and functions $f \colon V(G) \to V(H)$, $f' \colon V(G') \to V(H')$ such that G, H are not isomorphic to G', H' while $G \otimes_f H$ is isomorphic to $G' \otimes_{f'} H'$.

The Sierpiński product can also be defined in a similar way for graphs with loops and multiple edges. In this case, a loop in G, say $\{g, g\}$, would correspond to a loop $\{(g, f(g)), (g, f(g))\}$ in $G \otimes_f H$ and a multiple edge in G would correspond to a multiple edge in $G \otimes_f H$. Finally, as with other products, one could also study the Sierpiński product of infinite graphs.

ORCID iDs

Jurij Kovič https://orcid.org/0000-0003-0567-2626
Tomaž Pisanski https://orcid.org/0000-0002-1257-5376
Sara Sabrina Zemljič https://orcid.org/0000-0003-4026-0854
Arjana Žitnik https://orcid.org/0000-0001-7737-1836

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