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Logarithms of a binomial series: A Stirling number approach

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Abstract

The p-th power of the logarithm of the Catalan generating function is computed using the Stirling cycle numbers. Instead of Stirling numbers, one may write this generating function in terms of higher order harmonic numbers.

Keywords: Catalan numbers, logarithm, generating function, Stirling number.

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1 Introduction

Knuth [6, 7] proposed the exciting formula

$$(\log C(z))^2 = \sum_{n\ge1} \binom{2n}{n} (H_{2n-1} - H_n) \frac{z^n}{n},$$

where

$$C(z) = \frac{1-\sqrt{1-4z}}{2z} = \sum_{n\ge0} \frac{1}{n+1}\binom{2n}{n} z^n \tag{1.1}$$

and

$$H_n = \sum_{1\le k\le n} \frac{1}{k}$$

with the generating function of Catalan numbers and harmonic numbers.

This formula was recently extended by Chu [1] to general exponents p. Chu's approach is based on the use of (exponential) Bell polynomials. Note that Knuth talked about the exponent 1 in his Christmas lecture from 2014 [5].

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We present here a very simple approach to this question using Stirling cycle numbers; recall [3] that they transform falling powers into ordinary powers viz.

$$x^{\underline{n}} = \sum_{0\le k\le n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} x^k.$$

For the readers' convenience it is mentioned that the numbers $\left[{n \atop k}\right](-1)^{n-k}$ appear often in the older literature as $s(n,k)$ and are then denoted as Stirling numbers of the first kind.

2 The expansion of the p-th power

The substitution $z = \frac{u}{(1+u)^2}$ was presented in [2] and it is extremely useful when dealing with Catalan numbers and Catalan statistics. Using it in (1.1), we get $C(z) = 1 + u$, and, by the Lagrange inversion formula [8],

$$u^m = \sum_{n\ge m} \frac{m}{n} \binom{2n}{n-m} z^n$$

for $m \ge 1$. For $m = 0$ the formula is still true when taking a limit. We now consider the bivariate generating function

$$\begin{aligned} F(z,\alpha) &= \sum_{p\ge 0} \frac{\alpha^p}{p!} (\log C(z))^p = \exp(\alpha \log C(z)) \\ &= C^\alpha(z) = (1+u)^\alpha = \sum_{m\ge 0} \binom{\alpha}{m} u^m. \end{aligned}$$

But

$$\binom{\alpha}{m} = \frac{1}{m!}\alpha^{\underline{m}} = \frac{1}{m!} \sum_{0\le k\le m} (-1)^{m-k} \begin{bmatrix} m \\ k \end{bmatrix} \alpha^k.$$

Therefore

$$F(z,\alpha) = \sum_{0\le k\le m\le n} \frac{1}{m!}(-1)^{m-k} \begin{bmatrix} m \\ k \end{bmatrix} \alpha^k \frac{m}{n} \binom{2n}{n-m} z^n.$$

The desired formula follows from reading off coefficients of α^p:

$$(\log C(z))^p = p![\alpha^p]F(z,\alpha) = \sum_{p\le m\le n} \frac{p!}{m!}(-1)^{m-p} \begin{bmatrix} m \\ p \end{bmatrix} \frac{m}{n} \binom{2n}{n-m} z^n. \tag{2.1}$$

3 Special cases

For $p = 1$ in equation (2.1), we get the instance of the Christmas lecture:

$$\log C(z) = [\alpha^1]F(z,\alpha) = \sum_{1\le m\le n} \frac{1}{m!}(-1)^{m-1} \begin{bmatrix} m \\ 1 \end{bmatrix} \frac{m}{n} \binom{2n}{n-m} z^n.$$

Since $\left[{m \atop 1}\right] = (m-1)!$, this leads to

$$\log C(z) = [\alpha^1]F(z,\alpha) = \frac{1}{2} \sum_{n\ge 1} \frac{1}{n} \binom{2n}{n} z^n.$$

Now we turn to the instance $p = 2$ from [6, 7]. (Note that $\left[{m \atop 2}\right] = (m-1)!H_{m-1}$.) Equation (2.1) leads to

$$\begin{aligned}2[\alpha^2]F(z,\alpha) &= \sum_{2\le m\le n}\frac{2}{m!}(-1)^m\left[{m \atop 2}\right]\frac{m}{n}\binom{2n}{n-m}z^n\\ &= 2\sum_{2\le m\le n}H_{m-1}(-1)^m\frac{1}{n}\binom{2n}{n-m}z^n\\ &= 2\sum_{1\le j<m\le n}\frac{1}{j}(-1)^m\frac{1}{n}\binom{2n}{n-m}z^n\\ &= 2\sum_{1\le j<n}\frac{1}{j}(-1)^{j-1}\frac{1}{n}\binom{2n-1}{n-j-1}z^n.\end{aligned}$$

In the last step we used the formula

$$\sum_{j<m\le n}(-1)^m\binom{2n}{n-m} = (-1)^{j-1}\binom{2n-1}{n-j-1},$$

which is a standard summation for binomial coefficients [3].

To obtain the form proposed by Knuth, we still need to prove that

$$\binom{2n}{n}(H_{2n-1}-H_n) = 2\sum_{1\le j<n}\frac{(-1)^{j-1}}{j}\binom{2n-1}{n-j-1}.$$

Modern computer algebra systems readily simplify the difference of these two sides to 0, as expected.

4 Connection with harmonic numbers — the general case

In [4], there is the general formula

$$\frac{1}{n!}\left[{n+1 \atop r+1}\right] = (-1)^r\sum_{\{r\}}\prod_{j=1}^{l}\frac{(-1)^{i_j}}{i_j!}\left(\frac{H_n^{(r_j)}}{r_j}\right)^{i_j}.$$

Here, the sum is over all partitions of r:

$$r = i_1r_1 + \cdots + i_lr_l,$$

with parts $r_1 > \cdots > r_l \ge 1$ and positive integers $i_1, \ldots, i_l$. As an example, the partitions of $r = 4$ are 4, $3+1$, $2+2$, $2+1+1$, $1+1+1+1$, written alternatively as $1\cdot 4$, $1\cdot 3+1\cdot 1$, $2\cdot 2$, $1\cdot 2+2\cdot 1$, $4\cdot 1$.

There appear higher order harmonic numbers as well:

$$H_n^{(i)} = \sum_{1\le k\le n}\frac{1}{k^i}.$$

Here are the first few instances:

$$\begin{aligned}
\frac{1}{n!}\begin{bmatrix} n+1 \\ 2 \end{bmatrix} &= H_n, \\
\frac{1}{n!}\begin{bmatrix} n+1 \\ 3 \end{bmatrix} &= -\frac{1}{2}H_n^{(2)} + \frac{1}{2}H_n^2, \\
\frac{1}{n!}\begin{bmatrix} n+1 \\ 4 \end{bmatrix} &= \frac{1}{3}H_n^{(3)} - \frac{1}{2}H_n^{(2)}H_n + \frac{1}{6}H_n^3, \\
\frac{1}{n!}\begin{bmatrix} n+1 \\ 5 \end{bmatrix} &= -\frac{1}{4}H_n^{(4)} + \frac{1}{3}H_n^{(3)}H_n + \frac{1}{8}\big(H_n^{(2)}\big)^2 - \frac{1}{4}H_n^{(2)}H_n^2 + \frac{1}{24}H_n^4.
\end{aligned}$$

This allows to replace $\frac{1}{(m-1)!}\left[\begin{smallmatrix} m \\ p \end{smallmatrix}\right]$ in

$$(\log C(z))^p = \sum_{p\le m\le n} \frac{1}{(m-1)!}\begin{bmatrix} m \\ p \end{bmatrix}(-1)^{m-p}\frac{p!}{n}\binom{2n}{n-m}z^n$$

by an expression involving $H_{m-1}^{(1)}, \dots, H_{m-1}^{(p-1)}$.

5 Extension

If instead of $u = z(1+u)^2$ we work with $u = z(1+u)^\lambda$, then we deal with the generating function of extended (generalized) Catalan numbers

$$C_\lambda(z) = \sum_{n\ge 0}\binom{1+n\lambda}{n}\frac{z^n}{1+n\lambda}.$$

From [3], we infer that

$$u^m = \sum_{n\ge m}\binom{\lambda n+m}{n}\frac{m}{\lambda n+m}z^n.$$

So

$$\begin{aligned}
F(z,\alpha) &= \sum_{p\ge 0}\frac{\alpha^p}{p!}(\log C_\lambda(z))^p = \exp(\alpha\log C_\lambda(z)) = C_\lambda^\alpha(z) \\
&= (1+u)^\alpha = \sum_{m\ge 0}\binom{\alpha}{m}u^m \\
&= \sum_{0\le k\le m\le n}\frac{1}{m!}(-1)^{m-k}\begin{bmatrix} m \\ k \end{bmatrix}\alpha^k\binom{\lambda n+m}{n}\frac{m}{\lambda n+m}z^n.
\end{aligned}$$

The desired formula follows from reading off coefficients of α^p:

$$(\log C_\lambda(z))^p = p![\alpha^p]F(z,\alpha) = \sum_{p\le m\le n}\frac{p!}{m!}(-1)^{m-p}\begin{bmatrix} m \\ p \end{bmatrix}\binom{\lambda n+m}{n}\frac{m}{\lambda n+m}z^n.$$

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