Arc-transitive graphs of valency twice a prime admit a semiregular automorphism

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Received 1 January 2019, accepted 16 February 2020, published online 15 October 2020

Abstract

We prove that every finite arc-transitive graph of valency twice a prime admits a non-trivial semiregular automorphism, that is, a non-identity automorphism whose cycles all have the same length. This is a special case of the Polycirculant Conjecture, which states that all finite vertex-transitive digraphs admit such automorphisms.

Keywords: Arc-transitive graphs, polycirculant conjecture, semiregular automorphism.


1 Introduction

All graphs in this paper are finite. In 1981, Marušič asked if every vertex-transitive digraph admits a nontrivial semiregular automorphism [13], that is, an automorphism whose cycles all have the same length. This question has attracted considerable interest and a generalisation of the affirmative answer is now referred to as the Polycirculant Conjecture [4]. See [1] for a recent survey on this problem.

One line of investigation of this question has been according to the valency of the graph or digraph. Every vertex-transitive graph of valency at most four admits such an automorphism [7, 14], and so does every vertex-transitive digraph of out-valency at most three [9].

* Authors are grateful to the anonymous referees for their helpful suggestions.
† The research of the first author was supported by the ARC Discovery Project DP160102323.
‡ Gabriel Verret is grateful to the N.Z. Marsden Fund for its support (via grant UOA1824).

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Every arc-transitive graph of prime valency has a nontrivial semiregular automorphism [10] and so does every arc-transitive graph of valency 8 [17]. Partial results were also obtained for arc-transitive graphs of valency a product of two primes [18]. We continue this theme by proving the following theorem.

**Theorem 1.1.** Arc-transitive graphs of valency twice a prime admit a nontrivial semiregular automorphism.

### 2 Preliminaries

If \( G \) is a group of automorphisms of a graph \( \Gamma \) and \( v \) is a vertex of \( \Gamma \), we denote by \( G_v \) the stabiliser in \( G \) of \( v \), by \( \Gamma(v) \) the neighbourhood of \( v \), and by \( G_{\Gamma(v)}^v \) the permutation group induced by \( G_v \) on \( \Gamma(v) \). We will need the following well-known results.

**Lemma 2.1.** Let \( \Gamma \) be a connected graph and \( G \leq \text{Aut}(\Gamma) \). If a prime \( p \) divides \( |G_v| \) for some \( v \in V(\Gamma) \), then there exists \( u \in V(\Gamma) \) such that \( p \) divides \( |G_{\Gamma(u)}^u| \).

**Proof.** Since \( p \) divides \( |G_v| \), there exists an element \( g \) of order \( p \) in \( G_v \). As \( g \neq 1 \), there are vertices not fixed by \( g \). Among these vertices, let \( w \) be one at minimal distance from \( v \). Let \( P \) be a path of minimal length from \( v \) to \( w \) and let \( u \) be the vertex preceding \( w \) on \( P \). By the definition of \( w \), we have that \( u \) is fixed by \( g \), so \( g \in G_u \). On the other hand, \( g \) does not fix the neighbour \( w \) of \( u \), so \( g^{\Gamma(u)} \neq 1 \) hence \( |g^{\Gamma(u)}| = p \) and the result follows. \( \square \)

**Lemma 2.2.** Let \( G \) be a permutation group and let \( K \) be a normal subgroup of \( G \) such that \( G/K \) acts faithfully on the set of \( K \)-orbits. If \( G/K \) has a semiregular element \( Kg \) of order \( r \) coprime to \( |K| \), then \( G \) has a semiregular element of order \( r \).

**Proof.** See for example [17, Lemma 2.3]. \( \square \)

**Lemma 2.3.** A transitive group of degree a power of a prime \( p \) contains a semiregular element of order \( p \).

**Proof.** In a transitive group of degree a power of a prime \( p \), every Sylow \( p \)-subgroup is transitive. A non-trivial element of the center of this subgroup must be semiregular. \( \square \)

Recall that a permutation group is *quasiprimitive* if every non-trivial normal subgroup is transitive, and *biquasiprimitive* if it is transitive but not quasiprimitive and every non-trivial normal subgroup has at most two orbits.

### 3 Arc-transitive graphs of prime valency

In the most difficult part of our proof, the arc-transitive graph of valency twice a prime will have a normal quotient with prime valency. We will thus need a lot of information about arc-transitive graphs of prime valency, which we collect in this section. We start with the following result, which is [3, Theorem 5]:

**Theorem 3.1.** Let \( \Gamma \) be a connected \( G \)-arc-transitive graph of prime valency \( p \) such that the action of \( G \) on \( V(\Gamma) \) is either quasiprimitive or biquasiprimitive. Then one of the following holds:

1. \( G \) contains a semiregular element of odd prime order;
(2) $|V(\Gamma)|$ is a power of 2;

(3) $\Gamma = K_{12}$, $G = M_{11}$ and $p = 11$;

(4) $|V(\Gamma)| = (p^2 - 1)/2s$ and $G = \text{PSL}_2(p)$ or $\text{PGL}_2(p)$, where $p$ is a Mersenne prime and $s$ is a proper divisor of $(p - 1)/2$ but also a multiple of the product of the distinct prime divisors of $(p - 1)/2$;

(5) $|V(\Gamma)| = (p^2 - 1)/s$ and $G = \text{PGL}_2(p)$, where $p$ and $s$ are as in part (4), and $\Gamma$ is the canonical double cover of the graph given in (4).

(Recall that the canonical double cover of a graph $\Gamma$ is $\Gamma \times K_2$.) We note that in cases (4) and (5), we must have $p \geq 127$, since this is the smallest Mersenne prime $p$ such that $(p - 1)/2$ is not squarefree. This fact will be used at the end of Section 4.

**Corollary 3.2.** Let $\Gamma$ be a connected $G$-arc-transitive graph of prime valency. Then one of the following holds:

1. $G$ contains a semiregular element of odd prime order;
2. $|V(\Gamma)|$ is a power of 2;
3. $G$ contains a normal 2-subgroup $P$ such that $(\Gamma/P, G/P)$ is one of the graph-group pairs in (3–5) of Theorem 3.1.

**Proof.** Suppose that $|V(\Gamma)|$ is not a power of 2. If $G$ is quasiprimitive or biquasiprimitive on $V(\Gamma)$, then the result follows immediately from Theorem 3.1 (with $P = 1$ in case (3)). We thus assume that this is not the case and let $P$ be a normal subgroup of $G$ that is maximal subject to having at least three orbits on $V(\Gamma)$. In particular, $P$ is the kernel of the action of $G$ on the set of $P$-orbits. Hence $G/P$ acts faithfully, and quasiprimitively or biquasiprimitively on $V(\Gamma/P)$. Since $\Gamma$ has prime valency, is connected and $G$-arc-transitive, [12, Theorem 9] implies that $P$ is semiregular. We may thus assume that $P$ is a 2-group. (Otherwise $P$ contains a semiregular element of odd prime order.) If $G/P$ contains a semiregular element of odd prime order, then Lemma 2.2 implies that so does $G$. We may assume that this is not the case. Similarly, we may assume that $|V(\Gamma/P)|$ is not a power of 2. (Otherwise, $|V(\Gamma)|$ is a power of 2.) It follows that $\Gamma/P$ and $G/P$ are as is (3–5) of Theorem 3.1.

We will now prove some more results about the graphs that appear in (3–5) of Theorem 3.1. Let us first recall the notion of coset graphs. Let $G$ be a group with a subgroup $H$ and let $g \in G$ such that $g^2 \in H$ but $g \notin N_G(H)$. The graph $\text{Cos}(G, H, HgH)$ has vertices the right cosets of $H$ in $G$, with two cosets $Hx$ and $Hy$ adjacent if and only if $xy^{-1} \in HgH$. Observe that the action of $G$ on the set of vertices by right multiplication induces an arc-transitive group of automorphisms such that $H$ is the stabiliser of a vertex. Moreover, every arc-transitive graph can be constructed in this way [16, Theorem 2].

**Lemma 3.3.** The graphs in (3) and (4) of Theorem 3.1 have a 3-cycle.

**Proof.** Clearly $K_{12}$ has a 3-cycle so suppose that $\Gamma$ is one of the graphs given in (4). Let $G$ be as in Theorem 3.1 and let $v \in V(\Gamma)$. Then $G$ is one of $\text{PSL}_2(p)$ or $\text{PGL}_2(p)$ and acts arc-transitively on $\Gamma$. In both cases, let $X = \text{PSL}_2(p)$, so $G = X$ or $|G : X| = 2$. By [3, Lemma 5.3], we have that $G_v \cong C_p \rtimes C_s$ if $G = \text{PSL}_2(p)$, and $C_p \rtimes C_{2s}$ if
Let $\Gamma$ be a graph and let $S_0$ be a subset of $V(\Gamma)$. Let $S = S_0$. If a vertex $u$ outside $S$ has at least two neighbours in $S$, add $u$ to $S$. Repeat this procedure until no more vertices outside $S$ have this property. If at the end of the procedure, we have $S = V(\Gamma)$, then we say that $\Gamma$ is dense with respect to $S_0$.

It is an easy exercise to check that denseness is well-defined.

**Corollary 3.5.** Let $\Gamma$ be a graph in (3) or (4) of Theorem 3.1 and let $S_0 = \{u, v\}$ be an edge of $\Gamma$. Then $\Gamma$ is dense with respect to $S_0$.
Proof. Since $\Gamma$ is arc-transitive of prime valency $p$, the local graph at $v$ (that is, the subgraph induced on $\Gamma(v)$) is a vertex-transitive graph of order $p$ and thus vertex-primitive. By Lemma 3.3, $\Gamma$ has a 3-cycle so the local graph has at least one edge and thus must be connected. It follows that, running the process described in Definition 3.4 starting at $S = S_0$, eventually $S$ will contain all neighbours of $v$. Repeating this argument and using connectedness of $\Gamma$ yields the desired conclusion. \hfill \square

The following is immediate from Definition 3.4.

Lemma 3.6. Let $\Gamma$ be a graph and $S_0$ be a set of vertices such that $\Gamma$ is dense with respect to $S_0$. Then the canonical double cover of $\Gamma$, with vertex-set $V(\Gamma) \times \{0, 1\}$, is dense with respect to $S_0 \times \{0, 1\}$.

Proof. Let $S_i$ be the sequence of subsets of $V(\Gamma)$ obtained when running the procedure from Definition 3.4 starting with $S_0$ and ending with $S_n$ for some $n$. Since $\Gamma$ is dense with respect to $S_0$, we have $S_n = V(\Gamma)$. For $i \in \{1, \ldots, n\}$, let $v_i = S_i \setminus S_{i-1}$. (In other words, $v_1$ is the first vertex added to $S_0$ to get $S_1$, then $v_2$ is added to $S_1$ to get $S_2$, etc.)

Now, let $\Gamma' = \Gamma \times K_2$ be the canonical double cover of $\Gamma$ and let $S'_0 = S_0 \times \{0, 1\} \subseteq V(\Gamma')$. We now run the procedure from Definition 3.4 starting at $S'_0$. At the first step, we note that, since $v_1$ was added to $S_0$, it must have at least two neighbours in $S_0$, say $u_1$ and $w_1$. It follows that both $(v_1, 0)$ and $(v_1, 1)$ also have at least two neighbours in $S'_0$ (for example, $(u_1, 1)$ and $(w_1, 1)$, and $(u_1, 0)$ and $(w_1, 0)$, respectively). We thus add $(v_1, 0)$ and $(v_1, 1)$ to $S'_0$ to get $S'_1 = S'_0 \cup \{(v_1, 0), (v_1, 1)\}$. Note that $S'_1 = S_1 \times \{0, 1\}$. We then repeat this procedure, preserving the condition $S'_i = S_i \times \{0, 1\}$ at each iteration. At the end of this process, we have $S'_n = S_n \times \{0, 1\} = V(\Gamma) \times \{0, 1\} = V(\Gamma')$ and so $\Gamma'$ is dense with respect to $S_0 \times \{0, 1\}$.

\hfill \square

4 Proof of Theorem 1.1

Let $p$ be a prime, let $\Gamma$ be an arc-transitive graph of valency $2p$ and let $G = \text{Aut}(\Gamma)$. We may assume that $\Gamma$ is connected. If $G$ is quasiprimitive or bi-quasiprimitive, then $G$ contains a nontrivial semiregular element, by [8, Theorem 1.2] and [10, Theorem 1.4]. We may thus assume that $G$ has a minimal normal subgroup $N$ such that $N$ has at least three orbits. In particular, $\Gamma/N$ has valency at least 2.

If $N$ is nonabelian, then $G$ has a nontrivial semiregular element by [18, Theorem 1.1].

We may therefore assume that $N$ is abelian and, in particular, $N$ is an elementary abelian $q$-group for some prime $q$.

We may also assume that $N$ is not semiregular that is, $N_v \neq 1$ for some vertex $v$. It follows from Lemma 2.1 that $1 \not\in N_v^{\Gamma(v)} \subseteq G_v^{\Gamma(v)}$. As $\Gamma$ is $G$-arc-transitive, we have that $G_v^{\Gamma(v)}$ is transitive and so the orbits of $N_v^{\Gamma(v)}$ all have the same size, either 2 or $p$. Since $N$ is a $q$-group, this size is equal to $q$. Writing $d$ for the valency of $\Gamma/N$, we have that either $(d, q) = (2, p)$ or $(d, q) = (p, 2)$.

If $d = 2$ and $q = p$, then it follows from [15, Theorem 1] that $\Gamma$ is isomorphic to the graph denoted by $C(p, r, s)$ in [15]. By [15, Theorem 2.13], $\text{Aut}(C(p, r, s))$ contains the nontrivial semiregular automorphism $\varrho$ defined in [15, Lemma 2.5].

We may thus assume that $d = p$ and $q = 2$. In this case, if $u$ is adjacent to $v$, then $u$ has exactly $2 = 2p/d$ neighbours in $v^N$. Let $K$ be the kernel of the action of $G$ on $N$-orbits. By the previous observation, the orbits of $K_v^{\Gamma(v)}$ have size 2 and thus it is a 2-group. It follows from Lemma 2.1 that $K_v$ is a 2-group and thus so is $K = NK_v$. 183
Now, \( G/K \) is an arc-transitive group of automorphisms of \( \Gamma/N \), so we may apply Corollary 3.2. If \( G/K \) has a semiregular element of odd prime order, then so does \( G \), by Lemma 2.2. If \(|V(\Gamma/N)|\) is a power of 2, then so is \(|V(\Gamma)|\) and, in this case, \( G \) contains a semiregular involution by Lemma 2.3. We may thus assume that we are in case (3) of Corollary 3.2, that is, \( G/K \) contains a normal 2-subgroup \( P/K \) such that \((\Gamma/P,G/P)\) is one of the graph-group pairs in (3–5) of Theorem 3.1. Note that \( P \) is a 2-group. Let \( M \) be a minimal normal subgroup of \( G \) contained in the centre of \( P \). Note that \( M \) is an elementary abelian 2-group. We may assume that \( M \) is not semiregular hence \( M_v \neq 1 \) and so by Lemma 2.1, \( M_v^{\Gamma(v)} \neq 1 \). Moreover, \(|M| \neq 2 \) as otherwise \( M_v = M \) and we would deduce that \( M \) fixes each element of \( V(\Gamma) \), a contradiction. Since \( M \) is central in \( P \), \( M_v \) fixes every vertex in \( v^P \).

Note that the \( G \)-conjugates of \( M_v \) must cover \( M \), otherwise \( M \) contains a nontrivial semiregular element. By the previous paragraph, the number of conjugates of \( M_v \) is bounded above by the number of \( P \)-orbits, that is \(|V(\Gamma/P)|\), so we have

\[ |M| \leq |M_v||V(\Gamma/P)|. \]

Since \( \Gamma \) is connected and \( G \)-arc-transitive, there are no edges within \( P \)-orbits. As \( M_v^{\Gamma(v)} \neq 1 \), there exists \( g \in M_v \) such that \( w \) and \( w^g \) are distinct neighbours of \( v \). Let \( u \) be the other neighbour of \( w \) in \( v^P \). Since \( M_v \) fixes every element of \( v^P \) it follows that \( u \) is also a neighbour of \( w \) and \( w^g \) and so \( \{v, w, u, w^g\} \) is a 4-cycle in \( \Gamma \). Thus the graph induced between adjacent \( P \)-orbits is a union of \( C_4 \)'s.

If \( x \) is a vertex and \( y^P \) is a \( P \)-orbit adjacent to \( x \), then there is a unique \( C_4 \) containing \( x \) between \( x^P \) and \( y^P \), and thus a unique vertex \( z \) antipodal to \( x \) in this \( C_4 \). We say that \( z \) is the buddy of \( x \) with respect to \( y^P \). The set of buddies of \( v \) is equal to \( \Gamma_2(v) \cap v^P \), which is clearly fixed setwise by \( G_v \). Moreover, each vertex has the same number of buddies. Furthermore, since \( G_v \) transitively permutes the set of \( p \) \( P \)-orbits adjacent to \( v^P \), either \( v \) has a unique buddy or it has exactly \( p \) buddies.

If \( v \) has a unique buddy \( z \), then \( \Gamma(v) = \Gamma(z) \), and so swapping every vertex with its unique buddy is a nontrivial semiregular automorphism. Thus it remains to consider the case where \( v \) has \( p \) buddies. We first prove the following.

**Claim.** If \( X \) is a subgroup of \( M \) that fixes pointwise both \( a^P \) and \( b^P \), and \( c^P \) is a \( P \)-orbit adjacent to both \( a^P \) and \( b^P \), then \( X \) fixes \( c^P \) pointwise.

**Proof.** Suppose that some \( x \in X \) does not fix \( c \). Now \( x \) fixes \( a^P \) pointwise, so \( c^x \) must be the buddy of \( c \) with respect to \( a^P \). Similarly, \( c^x \) must be the buddy of \( c \) with respect to \( b^P \). These are distinct, which is a contradiction. It follows that \( X \) fixes \( c \) and, since \( X \subseteq M \), also \( c^P \). \( \square \)

Let \( s \geq 1 \), let \( \alpha = (v_0, \ldots, v_s) \) be an \( s \)-arc of \( \Gamma \) and let \( \alpha' = (v_0, \ldots, v_{s-1}) \). Now \(|v_s M_{v_{s-1}}| = 2|, so |M_{v_{s-1}} : M_{v_{s-1}} v_s| = 2| and |M_{\alpha'} : M_{\alpha}| \leq 2| \). Applying induction yields that

\[ |M_{v_0} : M_{\alpha}| \leq 2^s. \] (4.1)

We first assume that \( \Gamma/P \) and \( G/P \) are as in (3) or (4) of Theorem 3.1. Let \( \{u, v\} \) be an edge of \( \Gamma \). By the previous paragraph, we have \(|M_{\alpha} : M_{\alpha} v| \leq 2| \). Recall that \( M_v \) fixes all vertices in \( v^P \), so \( M_{\alpha} v \) fixes all vertices in \( v^P \cup u^P \). Combining the claim with Corollary 3.5
yields that $M_{uv} = 1$ and thus $|M_v| = 2$. It follows that $|M| \leq |M_v||V(\Gamma/P)|$ so $|M| \leq 2|V(\Gamma/P)|$. Since $M$ is minimal normal in $G$, it is an irreducible $G$-module over $GF(2)$, of dimension at least two. In fact, since $M$ is central in $P$, it is also an irreducible $(G/P)$-module. Since $G/P$ is nonabelian simple or has a nonabelian simple group as an index two subgroup, this implies that $M$ is a faithful irreducible $(G/P)$-module over $GF(2)$. If $G/P = M_{11}$, then $|M| \geq 2^{10}$ [11, Theorem 8.1], contradicting $|M| \leq 2 \cdot 12 = 24$. If $G/P = PSL_2(p)$ or $PGL_2(p)$ then by [2, Section VIII], $|M| \geq 2^{(p-1)/2}$. Recall that $p \geq 127$ and so this contradicts $|M| \leq 2(p^2 - 1)/2s < p^2 - 1$.

Finally, we assume that $\Gamma/P$ is in (5) of Theorem 3.1, that is, $\Gamma/P$ is the canonical double cover of a graph $\Gamma'$ which appears in (4) of Theorem 3.1. In particular, $V(\Gamma/P) = V(\Gamma') \times \{0, 1\}$. By Lemma 3.3, $\Gamma'$ has a 3-cycle, say $(u, v, w)$. By Corollary 3.5 and Lemma 3.6, $\Gamma/P$ is dense with respect to $\{u, v\} \times \{0, 1\}$. Now, let

$$\overline{\alpha} = ((u, 0), (v, 1), (w, 0), (u, 1), (v, 0)).$$

Since $\overline{\alpha}$ contains $\{u, v\} \times \{0, 1\}$, $\Gamma/P$ is dense with respect to $\overline{\alpha}$. Note that $\overline{\alpha}$ is a 4-arc of $\Gamma/P$. Let $\alpha$ be a 4-arc of $\Gamma$ that projects to $\overline{\alpha}$. Since $\Gamma/P$ is dense with respect to $\overline{\alpha}$, arguing as in the last paragraph yields $M_\alpha = 1$. On the other hand, if $v \in V(\Gamma')$ is the initial vertex of $\alpha$, then by (4.1), we have $|M_v : M_\alpha| \leq 2^4$ and thus $|M_v| \leq 2^4$. Since $|M| \leq |M_v||V(\Gamma/P)|$ it follows that $|M| \leq 2^4(p^2 - 1)/s$. As above, $M$ is a faithful irreducible $(G/P)$-module over $GF(2)$ of dimension at least two. Since $G/P = PGL_2(p)$ we have from [2] that $|M| \geq 2^{(p-1)/2}$, which again contradicts $|M| \leq 2^4(p^2 - 1)/s < 2^4(p^2 - 1)$.

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