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Bipartite edge-transitive bi-p-metacirculants*

Yan-Quan Feng †, Yi Wang

Department of Mathematics, Beijing Jiaotong University, Beijing, P.R. China

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Abstract

A graph is a bi-Cayley graph over a group if the group acts semiregularly on the vertex set of the graph with two orbits. Let G be a non-abelian metacyclic p-group for an odd prime p. In this paper, we prove that if G is a Sylow p-subgroup in the full automorphism group $\mathrm{Aut}(\Gamma)$ of a graph Γ, then G is normal in $\mathrm{Aut}(\Gamma)$. As an application, we classify the half-arc-transitive bipartite bi-Cayley graphs over G of valency less than $2p$, while the case for valency 4 was given by Zhang and Zhou in 2019. It is further shown that there are no semisymmetric or arc-transitive bipartite bi-Cayley graphs over G of valency less than p.

Keywords: Bi-Cayley graph, half-arc-transitive graph, metacyclic group.

Math. Subj. Class.: 05C10, 05C25, 20B25

1 Introduction

All graphs considered in this paper are finite, connected, simple and undirected. For a graph Γ, we use $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. A graph Γ is said to be *vertex-transitive*, *edge-transitive* or *arc-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$, $E(\Gamma)$ or $A(\Gamma)$ respectively, *semisymmetric* if it is edge-transitive but not vertex-transitive, and *half-arc-transitive* if it is vertex-transitive, edge-transitive, but not arc-transitive.

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. A group G is *metacyclic* if it has a normal subgroup N such that both N and G/N are cyclic.

Let Γ be a graph with $G \leq \mathrm{Aut}(\Gamma)$. Then Γ is called a *Cayley graph* over G if G is regular on $V(\Gamma)$ and a *bi-Cayley graph* over G if G is semiregular on $V(\Gamma)$ with two orbits.

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†Corresponding author.

E-mail addresses: yqfeng@bjtu.edu.cn (Yan-Quan Feng), yiwang@bjtu.edu.cn (Yi Wang)

In particular, if G is normal in $\mathrm{Aut}(\Gamma)$, the Cayley graph or the bi-Cayley graph Γ is called *a normal Cayley graph* or *a normal bi-Cayley graph* over G, respectively.

Determining the automorphism group of a graph is fundamental in algebraic graph theory, but very difficult in general. If Γ is a connected normal Cayley graph over a group G, then $\mathrm{Aut}(\Gamma)$ is determined by Godsil [27], and if Γ is a connected normal bi-Cayley graph over G, then $\mathrm{Aut}(\Gamma)$ is also determined by Zhou and Feng [55]. Thus a natural problem is to determine normality of Cayley graphs or bi-Cayley graphs over groups.

The normality of Cayley graphs over cyclic group of order a prime and over group of order twice a prime was solved by Alspach [1] and Du et al. [19], respectively. Dobson [14] determined all non-normal Cayley graphs over group of order a product of two distinct primes, and Dobson and Witte [16] determined all non-normal Cayley graphs over group of order a prime square. Dobson and Kovács [15] determined the full automorphism groups of Cayley graphs over elementary abelian group of rank 3. However, it seems still very difficult to obtain normality of Cayley graphs for general valencies. On the other hand, many results on the normality of Cayley graphs with small valencies were obtained, and for example, one may refer to [20, 21, 22] for finite non-abelian simple groups and to [4, 23, 26, 51, 54] for solvable groups. Due to nice properties on automorphism groups of non-abelian p-groups, the normality of Cayley graphs with general valencies over certain non-abelian p-groups was obtained. A connected Cayley graph or bi-Cayley graph over a non-abelian metacyclic p-group, for an odd prime p, is called a *p-metacirculant* or a *bi-p-metacirculant*, respectively. Li and Sim [34] proved that a p-metacirculant Γ is normal except a special case when the non-abelian metacyclic p-group is a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$, and Wang and Feng [50] proved that this special case cannot occur. In this paper we prove the following theorem.

Theorem 1.1. *Let Γ be a connected bipartite bi-p-metacirculant over a non-abelian metacyclic p-group G. If G is a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$, then G is normal in $\mathrm{Aut}(\Gamma)$.*

It is well-known that Cayley graphs play an important role in the study of symmetry of graphs. However, graphs with various symmetries can be constructed by bi-Cayley graphs. The smallest trivalent semisymmetric graph is the Gray graph [6], which is a bi-Cayley graph over a non-abelian metacyclic group of order 27, and infinite semisymmetric graphs were constructed in [17, 18, 37]. Boben et al. [5] studied properties of cubic bi-Cayley graphs over cyclic groups and the configurations arising from these graphs. Kovács et al. [31] gave a description of arc-transitive one-matching bi-Cayley graphs over abelian groups. All cubic vertex-transitive bi-Cayley graphs over cyclic groups, abelian groups or dihedral groups were determined in [39, 52, 54]. Recently, Conder et al. [11] investigated bi-Cayley graphs over abelian groups, dihedral groups and metacyclic p-groups, and using these results, a complete classification of connected trivalent edge-transitive graphs of girth at most 6 was obtained. Furthermore, Qin et al. [41] classified connected edge-transitive bi-p-metacirculants of valency p, and as an application of Theorem 1.1, we prove that there are no such graphs with valency less than p.

Theorem 1.2. *For any odd prime p, there are no connected arc-transitive or semisymmetric bipartite bi-p-metacirculants of valency less than p.*

In 1966, Tutte [46] initiated an investigation of half-arc-transitive graphs by showing that a vertex- and edge-transitive graph with odd valency must be arc-transitive. A few years later, in order to answer Tutte's question on the existence of half-arc-transitive graphs of even valency, Bouwer [7] constructed a $2k$-valent half-arc-transitive graph for every $k \geq 2$. One of the standard problems in the study of half-arc-transitive graphs is to classify such graphs for certain orders. Let p be a prime. It is well known that there are no half-arc-transitive graphs of order p or p^2, and no such graphs of order $2p$ by Cheng and Oxley [8]. Alspach and Xu [2] classified half-arc-transitive graphs of order $3p$ and Kutnar et al. [33] classified such graphs of order $4p$. Despite all of these efforts, however, further classifications of half-arc-transitive graphs with general valencies seem to be very difficult, and special attention has been paid to the study of half-arc-transitive graphs with small valencies, which were extensively studied from different perspectives over decades by many authors; see [3, 9, 10, 24, 25, 29, 32, 35, 38, 40, 43, 47, 48, 49] for example.

The smallest half-arc-transitive graph constructed in Bouwer [7] is a bi-Cayley graph over the non-abelian metacyclic group of order 27 with exponent 9. Zhang and Zhou [56] proved that a half-arc-transitive bi-Cayley graph over cyclic group has valency at least 6, and this was extended to abelian groups by Conder et al. [11]. In fact, half-arc-regular bi-Cayley graphs of valency 6, over cyclic groups, were classified in [56], and two infinite families of bipartite tetravalent half-arc-transitive bi-p-metacirculants of order p^3 were constructed in [11], of which one is Cayley and the other is not Cayley. Furthermore, Zhang and Zhou [53] classified tetravalent half-arc-transitive bi-p-metacirculants, and all these graphs are bipartite. This was the main motivation for the research leading to Theorem 1.3, namely the classification of bipartite half-arc-transitive bi-p-metacirculants of valency less than $2p$. It was also motivated in part by the classification of half-arc-transitive p-metacirculants of valency less than $2p$, given by Li and Sim [35].

For a positive integer n, denote by $\mathbb{Z}_n$ the cyclic group of order n, as well as the ring of integers modulo n, and by $\mathbb{Z}_n^*$ the multiplicative group of the ring $\mathbb{Z}_n$ consisting of numbers coprime to n.

Theorem 1.3. *Let p be an odd prime and let Γ be a connected bipartite bi-p-metacirculant of valency $2k$ with $k < p$ over a non-abelian metacyclic p-group G. Then Γ is half-arc-transitive if and only if*

$$k \geq 2, \quad k \mid (p-1), \quad G \cong G_{\alpha,\beta,\gamma} \quad \textit{and} \quad \Gamma \cong \Gamma^{\pm}_{m,k,\ell},$$

where $0 < \gamma < \alpha \leq \beta + \gamma$, $m \in \mathbb{Z}^*_{p^{\alpha-\gamma}}$, $0 \leq \ell < k$ with $\frac{k}{(k,\ell)} \mid \frac{(p-1)}{2}$, and $\mathrm{Aut}(\Gamma^{\pm}_{m,k,\ell}) \cong (G_{\alpha,\beta,\gamma} \rtimes \mathbb{Z}_k).\mathbb{Z}_2$.

The groups $G_{\alpha,\beta,\gamma}$ and graphs $\Gamma^{\pm}_{m,k,\ell}$ above are defined in Equation (2.1) and Equation (4.3). By Zhang and Zhou [53], the graphs $\Gamma^{\pm}_{m,2,\ell}$ can be Cayley or non-Cayley for certain values m and ℓ, and this implies that the extensions $(G_{\alpha,\beta,\gamma} \rtimes \mathbb{Z}_2).\mathbb{Z}_2$ above can be split or non-split.

2 Background results

Let G be a finite metacyclic p-group. Lindenberg [36] proved that the automorphism group of G is a p-group when G is non-split. The following proposition describes the automorphism group of the remaining case when G is split. It is easy to show that every non-abelian split metacyclic p-group G for an odd prime p has the following presentation:

$$G_{\alpha,\beta,\gamma} = \langle a, b \mid a^{p^{\alpha}} = 1,\ b^{p^{\beta}} = 1,\ b^{-1}ab = a^{1+p^{\gamma}} \rangle, \tag{2.1}$$

where α, β, γ are positive integers such that $0 < \gamma < \alpha \leq \beta + \gamma$. Li and Sim characterized the automorphism group $\mathrm{Aut}(G_{\alpha,\beta,\gamma})$ of the group $G_{\alpha,\beta,\gamma}$.

Proposition 2.1 ([35, Theorem 2.8]). *For an odd prime p, we have*

$$|\mathrm{Aut}(G_{\alpha,\beta,\gamma})| = (p-1)p^{\min(\alpha,\beta)+\min(\beta,\gamma)+\beta+\gamma-1}.$$

Moreover, all Hall p'-subgroups of $\mathrm{Aut}(G_{\alpha,\beta,\gamma})$ are conjugate and isomorphic to $\mathbb{Z}_{p-1}$. In particular, the map $\theta\colon a \mapsto a^{\varepsilon},\ b \mapsto b$ induces an automorphism of $G_{\alpha,\beta,\gamma}$ of order $p-1$, where ε is an element of order $p-1$ in $\mathbb{Z}^*_{p^{\alpha}}$.

A p-group G is said to be *regular* if for any $x, y \in G$ there exist $d_i \in \langle x, y\rangle'$, $1 \leq i \leq n$, for some positive integer n such that $x^p y^p = (xy)^p \prod_{i=1}^{n} d_i^p$. If G is metacyclic, then the derived subgroup G' is cyclic, and hence G is regular by [30, Kapitel III, 10.2 Satz]. For regular p-groups, the following proposition holds by [30, Kapitel III, 10.8 Satz].

Proposition 2.2. *Let G be a metacyclic p-group for an odd prime p. If $|G'| = p^n$, then for any $m \geq n$, we have*

$$(xy)^{p^m} = x^{p^m} y^{p^m},$$

for any $x, y \in G$.

Remark 2.3. For the non-abelian split metacyclic group $G_{\alpha,\beta,\gamma}$ given in Equation (2.1), we have $|G'_{\alpha,\beta,\gamma}| = p^{\alpha-\gamma}$ and $\alpha - \gamma \leq \beta$, and by Proposition 2.2, if $(p, m) = 1$ then $o(b^m a^n) = \max\{o(a^n), p^{\beta}\}$, and if $\beta < \alpha$ and $p \mid n$ then $o(b^m a^n) \leq p^{\alpha-1}$.

For a finite group G, $N \leq G$ means that N is a subgroup of G, and $N < G$ means that N is a proper subgroup of G. The following proposition lists non-abelian simple groups having a proper subgroup of index prime-power order.

Proposition 2.4 ([28, Theorem 1]). *Let T be a non-abelian simple group with $H < T$, and let $|T : H| = p^a$ for a prime p. Then one of the following holds.*

(1) $T = \mathrm{PSL}(n, q)$ and H is the stabilizer of a line or hyperplane. Furthermore, $|T : H| = (q^n - 1)/(q - 1) = p^a$ and n must be a prime.

(2) $T = \mathrm{A}_n$ and $H \cong \mathrm{A}_{n-1}$ with $n = p^a$.

(3) $T = \mathrm{PSL}(2, 11)$ and $H \cong \mathrm{A}_5$.

(4) $T = \mathrm{M}_{23}$ and $H \cong \mathrm{M}_{22}$ or $T = \mathrm{M}_{11}$ and $H \cong \mathrm{M}_{10}$.

(5) $T = \mathrm{PSU}(4, 2) \cong \mathrm{PSp}(4, 3)$ and H is the parabolic subgroup of index 27.

For a group G and a prime p, denote by $\mathrm{O}_p(G)$ the largest normal p-subgroup of G, and by $\mathrm{O}_{p'}(G)$ the largest normal subgroup of G whose order is not divisible by p. The next proposition is about transitive permutation groups of prime-power degree.

Proposition 2.5 ([34, Lemma 2.5]). *Let p be a prime, and let A be a transitive permutation group with p-power degree. Let B be a nontrivial subnormal subgroup of A. Then B has a proper subgroup of p-power index, and $\mathrm{O}_{p'}(B) = 1$. In particular, $\mathrm{O}_{p'}(A) = 1$.*

It is well-known that $\mathrm{GL}(d, q)$ has a cyclic group of order $q^d - 1$, the so called Singer-cycle subgroup, which also induces a cyclic subgroup of $\mathrm{PSL}(d, q)$.

Proposition 2.6 ([30, Kapitel II, 7.3 Satz]). *The group $G = \mathrm{GL}(d, q)$ contains a cyclic subgroup of order $q^d - 1$, and it induces a cyclic subgroup of order $\frac{q^d-1}{(q-1)(q-1,d)}$ of $\mathrm{PSL}(d, q)$.*

Let G and E be two groups. We call an extension E of G by N a *central extension* if N lies in the center of E and $E/N \cong G$, and if E is further perfect, that is, the derived group $E' = E$, we call E a *covering group* of G. Schur [42] proved that for every non-abelian simple group G there is a unique maximal covering group M such that every covering group of G is a factor group of M (also see [30, Chapter 5, Section 23]). This group M is called the *full covering group* of G, and the center of M is the *Schur multiplier* of G, denoted by $\mathrm{M}(G)$. For a group G, we denote by $\mathrm{Out}(G)$ the outer automorphism group of G, that is, $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, where $\mathrm{Inn}(G)$ is the inner automorphism group of G induced by conjugation.

The following proposition is about outer automorphism group and Schur multiplier of a non-abelian simple group having a proper subgroup of prime-power index.

Proposition 2.7 ([34, Lemma 2.3]). *Let p be an odd prime and let T be a non-abelian simple group that has a subgroup H of index $p^\ell > 1$. Then*

(1) $p \nmid |\mathrm{M}(T)|$;

(2) either $p \nmid |\mathrm{Out}(T)|$, *or* $T \cong \mathrm{PSL}(2, 8)$ *and* $p^\ell = 3^2$.

A group G is said to be *a central product* of its subgroups $H_1, \ldots, H_n$ $(n \geq 2)$ if $G = H_1 \cdots H_n$ and for any $i \neq j$, H_i and H_j commute elementwise. A group G is called *quasisimple* if $G' = G$ and $G/Z(G)$ is a non-abelian simple group, where $Z(G)$ is the centralizer of G. A group G is called *semisimple* if $G' = G$ and $G/Z(G)$ is a direct product of non-abelian simple groups. Clearly, a quasisimple group is semisimple.

Proposition 2.8 ([45, Theorem 6.4]). *A central product of two semisimple groups is also semisimple. Any semisimple group can be decomposed into a central product of quasisimple groups, and this set of quasisimple groups is uniquely determined.*

A subnormal quasisimple subgroup of a group G is called a *component* of G. By [45, 6.9(iv), p. 450], any two distinct components of G commute elementwise, and by Proposition 2.8, the product of all components of G is semisimple, denoted by $E(G)$, which is characteristic in G. We use $F(G)$ to denote the *Fitting subgroup* of G, that is, $F(G) = \mathrm{O}_{p_1}(G) \times \mathrm{O}_{p_2}(G) \times \cdots \times \mathrm{O}_{p_t}(G)$, where $p_1, p_2, \ldots, p_t$ are the distinct prime factors of $|G|$. Set $F^*(G) = F(G)E(G)$ and call $F^*(G)$ the *generalized Fitting subgroup* of G. The following is one of the most significant properties of $F^*(G)$. For a group G and a subgroup H of G, denote by $\mathrm{C}_G(H)$ the centralizer of H in G.

Proposition 2.9 ([45, Theorem 6.11]). *For any finite group G, we have*

$$\mathrm{C}_G(F^*(G)) \leq F^*(G).$$

An *action* of a group G on a set Ω is a homomorphism from G to the symmetric group S_Ω on Ω. We denote by $\Phi(G)$ the *Frattini subgroup* of G, that is, the intersection of all maximal subgroups of G. Note that for a prime p, $\mathrm{O}_p(G)$ is a p-group and $\mathrm{O}_p(G)/\Phi(\mathrm{O}_p(G))$ is an elementary abelian p-group. Thus, $\mathrm{O}_p(G)/\Phi(\mathrm{O}_p(G))$ can be viewed as a vector space over the field $\mathbb{Z}_p$. The following lemma considers a natural action of a group G on the vector space $\mathrm{O}_p(G)/\Phi(\mathrm{O}_p(G))$.

Proposition 2.10 ([50, Lemma 2.9]). *For a finite group G and a prime p, let $H = \mathrm{O}_p(G)$ and $V = H/\Phi(H)$. Then G has a natural action on V, induced by conjugation via elements of G on H. If $\mathrm{C}_G(H) \leq H$, then H is the kernel of this action of G on V.*

Let a group T act on two sets Ω and Σ, and these two actions are *equivalent* if there is a bijection $\lambda\colon \Omega \mapsto \Sigma$ such that

$$(\alpha^t)^\lambda = (\alpha^\lambda)^t \text{ for all } \alpha \in \Omega \text{ and } t \in T.$$

When the two actions above are transitive, there is a simple criterion on whether or not they are equivalent.

Proposition 2.11 ([13, Lemma 1.6B]). *Assume that a group T acts transitively on the two sets Ω and Σ, and let W be a stabilizer of a point in the first action. Then the actions are equivalent if and only if W is the stabilizer of some point in the second action.*

For a group G and two subgroups H and K of G, we consider the actions of G on the right cosets of H and K by right multiplication. The stabilizers of Hx and Ky are H^x and K^y, respectively. By Proposition 2.11, these two right multiplication actions are equivalent if and only if H and K are conjugate in G.

3 Automorphisms of bipartite bi-p-metacirculants

Let Γ_N be the *quotient graph* of a graph Γ with respect to $N \leq \mathrm{Aut}(\Gamma)$, that is, the graph having the orbits of N as vertices with two orbits O_1, O_2 adjacent in Γ_N if and only if there exist some $u \in O_1$ and $v \in O_2$ such that $\{u, v\}$ is an edge in Γ. Denote by $[O_1]$ the induced subgraph of Γ by O_1, and by $[O_1, O_2]$ the subgraph of $[O_1 \cup O_2]$ with edge set $\{\{u, v\} \in E(\Gamma) \mid u \in O_1, v \in O_2\}$.

Proof of Theorem 1.1. Let G a non-abelian metacyclic p-group of order p^n for an odd prime p and a positive integer n, and let Γ be a connected bipartite bi-p-metacirculant over G. Set $A = \mathrm{Aut}(\Gamma)$, and let $G \in \mathrm{Syl}_p(A)$, where $\mathrm{Syl}_p(A)$ is the set of Sylow p-subgroups of A. To finish the proof, it suffices to show that $G \trianglelefteq A$.

Let W_0 and W_1 be the two parts of the bipartite graph Γ. Then $\{W_0, W_1\}$ is a complete block system of A on $V(\Gamma)$ with $|W_0| = |W_1| = |G| = p^n$. Let A^* be the kernel of A on $\{W_0, W_1\}$, that is, the subgroup of A fixing W_0 and W_1 setwise. Then $A^* \trianglelefteq A$, $A/A^* \leq \mathbb{Z}_2$ and $\mathrm{Syl}_p(A) = \mathrm{Syl}_p(A^*)$. It follows that $G \in \mathrm{Syl}_p(A^*)$. Noting that $|G| = p^n$, we have $p^n \mid |A|$ and $p^{n+1} \nmid |A|$, that is, $p^n \parallel |A|$. The group G has exactly two orbits, that is, W_0 and W_1, and G is regular on both W_0 and W_1. By Frattini argument [30, Kapitel I, 7.8 Satz], $A^* = GA^*_u$ for any $u \in V(\Gamma)$, implying that A^*_u is a p'-group. Clearly, $A_u = A^*_u$, and so A_u is also a p'-group.

Let K be the kernel of A^* acting on W_0. Then $K \leq A^*_v$ for any $v \in W_0$, and $K \trianglelefteq A^*$. The orbits of K on W_1 have the same length, and so it is a divisor of p^n. It follows that if $K \neq 1$ then $p \bigm| |K|$, which is impossible because A^*_v is a p'-group. Thus, A^* acts faithfully on W_0 (resp. W_1). Since Sylow p-subgroups of A are conjugate, every p-subgroup of A is semiregular on both W_0 and W_1.

Claim 1. Any minimal normal subgroup N of A^* is abelian.

We argue by contradiction and we suppose that N is non-abelian. Then $N \cong T_1 \times \cdots \times T_k$ with $k \geq 1$, where $T_i \cong T$ is a non-abelian simple group. By Proposition 2.5, $p \mid |N|$ and so $p \mid |T_i|$ for each $1 \leq i \leq k$. Since $G \in \mathrm{Syl}_p(A^*)$, we have $G \cap N \in \mathrm{Syl}_p(N)$, and hence $G \cap N = P_1 \times \cdots \times P_k$ for some $P_i \in \mathrm{Syl}_p(T_i)$, where $P_i \neq 1$ for each $1 \leq i \leq k$. Since G is metacyclic and $G \cap N \trianglelefteq G$, $G \cap N$ is metacyclic and this implies $k \leq 2$.

Set $\Omega = \{T_1, \ldots, T_k\}$ and write $B = \mathrm{N}_{A^*}(T_1)$. Considering the conjugation action of A^* on Ω, we have $B \trianglelefteq A^*$ as $k \leq 2$, and hence $A^*/B \lesssim \mathrm{S}_2$, forcing $B \trianglelefteq A^*$. Thus, $\mathrm{Syl}_p(B) = \mathrm{Syl}_p(A^*)$ and so B is transitive on both W_0 and W_1.

Let Γ_{T_1} be the quotient graph of Γ with respect to T_1. Since $T_1 \trianglelefteq B$, all orbits of T_1 on W_0 have the same length, and the length must be a p-power as $|W_0| = p^n$. Since each p-subgroup is semiregular, this length is the order of a Sylow p-subgroup of T_1. Similarly, all orbits of T_1 on W_1 have the same length and it is also the order of a Sylow p-subgroup of T_1. Thus, $V(\Gamma_{T_1}) = \{\Delta_1, \ldots, \Delta_s, \Delta'_1, \ldots, \Delta'_s\}$, the set of all orbits of T_1, with $W_0 = \Delta_1 \cup \cdots \cup \Delta_s$ and $W_1 = \Delta'_1 \cup \cdots \cup \Delta'_s$. Furthermore, for any $1 \leq i, j \leq s$ we have $|\Delta_i| = |\Delta'_j| = p^m$ for some $1 \leq m \leq n$, and hence $s = p^{n-m}$. Since $T_1 \trianglelefteq B$, B has a natural action on $V(\Gamma_{T_1})$ and let K be the kernel of this action. Clearly, $T_1 \leq K$. Recall that $p \nmid |A_u|$ for any $u \in V(\Gamma)$. Then $p \nmid |(T_1)_u|$, and by Guralnick [28, Corollary 2], T_1 is 2-transitive on each Δ_i or Δ'_i. Since $p^n \parallel |A^*|$, we have $p^n \parallel |B|$, implying that $p^m \parallel |K|$. Since $(T_1)_u$ is a proper subgroup of T_1 of index p-power, Proposition 2.7 implies that either $T_1 = \mathrm{PSL}(2,8)$ with $p^m = 3^2$, or $p \nmid |\mathrm{Out}(T_1)|$. To finish the proof of Claim 1, we will obtain a contradiction for both cases.

Case 1. $T_1 = \mathrm{PSL}(2,8)$ with $p^m = 3^2$.

In this case, $|\Delta_i| = |\Delta'_j| = 9$. If $s = 1$ then $|G| = p^m = 3^2$, contradicting that G is non-abelian. Thus $s \geq 2$ and $s = 3^{n-2}$. By Atlas [12], $\mathrm{PSL}(2,8)$ has only one conjugate class of subgroups of index 9, and by Proposition 2.11, T_1 acts equivalently on Δ_i and Δ'_j.

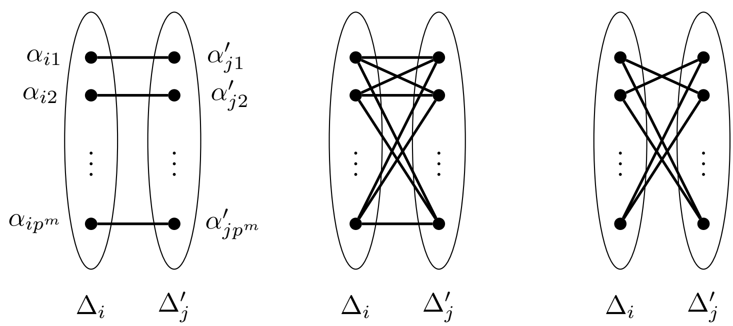

Figure 1: The subgraphs $[\Delta_i, \Delta'_j]$.

Set $\Delta_i = \{\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{i9}\}$ and $\Delta'_j = \{\alpha'_{j1}, \alpha'_{j2}, \ldots, \alpha'_{j9}\}$ for $1 \leq i, j \leq 3^{n-2}$. Recall that T_1 is 2-transitive on Δ_i and Δ'_j. Since T_1 acts equivalently on Δ_i and Δ'_j, by Proposition 2.11, we may assume that $(T_1)_{\alpha_{i\ell}} = (T_1)_{\alpha'_{j\ell}}$ for any $1 \leq i, j \leq 3^{n-2}$ and $1 \leq \ell \leq 3^2$. The subgraph $[\Delta_i, \Delta'_j]$ is either a null graph, or one of the three graphs in Figure 3 because $(T_1)_{\alpha_{i\ell}} = (T_1)_{\alpha'_{j\ell}}$ acts transitively on both $\Delta_i \setminus \{\alpha_{i\ell}\}$ and $\Delta'_j \setminus \{\alpha'_{j\ell}\}$.

The three graphs have edge sets $\{\{\alpha_{i\ell}, \alpha'_{j\ell}\} \mid 1 \leq \ell \leq 3^2\}$, $\{\{\alpha_{ik}, \alpha'_{j\ell}\} \mid 1 \leq k, \ell \leq 3^2\}$ or $\{\{\alpha_{ik}, \alpha'_{j\ell}\} \mid 1 \leq k, \ell \leq 3^2, k \neq \ell\}$, respectively.

For any $g \in \mathrm{S}_9$, define a permutation σ_g on $V(\Gamma)$ by $(\alpha_{i\ell})^{\sigma_g} = \alpha_{i\ell^g}$ and $(\alpha'_{j\ell})^{\sigma_g} = \alpha'_{j\ell^g}$ for any $1 \leq i, j \leq 3^{n-2}$ and $1 \leq \ell \leq 3^2$. Then σ_g fixes each Δ_i and Δ'_j, and permutes the elements of Δ_i and Δ'_j in the 'same way' for each $1 \leq i, j \leq 3^{n-2}$. Since $[\Delta_i, \Delta'_j]$ is either a null graph, or one graph in Figure 3, σ_g induces an automorphism of $[\Delta_i, \Delta'_j]$, for all $1 \leq i, j \leq 3^{n-2}$. Also σ_g induces automorphisms of $[\Delta_i]$ and $[\Delta'_j]$ for all $1 \leq i, j \leq 3^{n-2}$ because $[\Delta_i]$ and $[\Delta'_j]$ have no edges (Γ is bipartite). It follows that $\sigma_g \in \mathrm{Aut}(\Gamma)$. Thus, $L := \{\sigma_g \mid g \in \mathrm{S}_9\} \leq \mathrm{Aut}(\Gamma)$ and $L \cong \mathrm{S}_9$.

Clearly, $L \leq A^*$. If $L \not\leq B$, there exists $x \in L$ such that $T_1^x \neq T_1$, and hence $N \cong T_1 \times T_2$ with $k = 2$ and $T_1^x = T_2$, which implies that T_1 and T_2 have the same orbits because x fixes each orbit of T_1, contradicting that Sylow p-subgroups of N are semiregular. Thus $L \leq B$. Recall that K is the kernel of B acting on $V(\Gamma_{T_1})$ and $3^2 = p^m \parallel |K|$. Since L fixes each orbit of T_1 and $3^3 \bigm| |L|$, we have $L \leq K$ and $3^3 \bigm| |K|$, a contradiction.

Case 2. $p \nmid |\mathrm{Out}(T_1)|$.

Since $B/T_1\mathrm{C}_B(T_1) \lesssim \mathrm{Out}(T_1)$, we have $p^n \parallel |T_1\mathrm{C}_B(T_1)|$. Since T_1 is non-abelian simple, $T_1 \cap \mathrm{C}_B(T_1) = 1$ and hence $T_1\mathrm{C}_B(T_1) = T_1 \times \mathrm{C}_B(T_1)$. If $p \bigm| |\mathrm{C}_B(T_1)|$, then G is conjugate to $Q_1 \times Q_2$, where $Q_1 \in \mathrm{Syl}_p(T_1)$ and $Q_2 \in \mathrm{Syl}_p(\mathrm{C}_B(T_1))$. Since G is metacyclic, G can be generated by two elements, and since G is a p-group, any minimal generating set of G has cardinality 2. It follows that both Q_1 and Q_2 are cyclic, and so G is abelian, a contradiction. Thus, $p \nmid |\mathrm{C}_B(T_1)|$ and hence $p^n \parallel |T_1|$, forcing $s = 1$. Furthermore, $W_0 = \Delta_1$, $W_1 = \Delta'_1$ and T_1 is 2-transitive on both W_0 and W_1. Note that $(T_1)_u$ is proper subgroup of T_1 of index p^n. Since G is a Sylow p-subgroup of A of order p^n, all Sylow p-subgroups of T_1 are also Sylow p-subgroups of A, and so they are isomorphic to G. Without loss of generality, we may assume $G \leq T_1$. By Proposition 2.4, $T_1 = \mathrm{PSL}(2, 11)$, M_{11}, M_{23}, $\mathrm{PSU}(4, 2)$, A_{p^n}, or $\mathrm{PSL}(d, q)$ with $\frac{q^d-1}{q-1} = p^n$ and d a prime.

Suppose $T_1 = \mathrm{PSL}(2, 11)$, M_{11} or M_{23}. By Proposition 2.4, $|W_0| = |W_1| = 11$, 11 or 23 respectively, and hence $|G| = 11$, 11 or 23, contradicting that G is non-abelian.

Suppose $T_1 = \mathrm{PSU}(4, 2)$ or A_{p^n}. For the former, T_1 has one conjugate class of subgroups of index 27 by Atlas [12], and for the latter, T_1 has one conjugate class of subgroups of index p^n. By Proposition 2.11, T_1 acts equivalently on W_0 and W_1, and since Γ is connected, the 2-transitivity of T_1 on W_0 and W_1 implies that $\Gamma \cong K_{p^n,p^n}$ or $K_{p^n,p^n} - p^nK_2$. Then $A = \mathrm{S}_{p^n} \wr \mathrm{S}_2$ or $\mathrm{S}_{p^n} \times \mathbb{Z}_2$ respectively. Since G is non-abelian, we have $n \geq 3$, and so $p^{n+1} \bigm| |A|$, a contradiction.

Suppose $T_1 = \mathrm{PSL}(d, q)$ with $\frac{q^d-1}{q-1} = p^n$ and d a prime. By Proposition 2.6, T_1 has a cyclic subgroup of order $\frac{q^d-1}{(q-1)(q-1,d)}$. Since d is a prime, either $(q-1, d) = 1$ or $(q-1, d) = d$. Note that $(q-1, d) \bigm| \frac{q^d-1}{q-1}$. If $(q-1, d) = d$ then $d = p$ and $p \mid (q-1)$. Since $p \geq 3$ and $p^2 \mid (q^2-1)(q-1)$, we have $p^{n+1} \bigm| \frac{(q^p-1)(q^p-q)\cdots(q^p-q^{p-1})}{(q-1)(q,d)}$, that is, $p^{n+1} \bigm| |T_1|$, a contradiction. If $(q-1, d) = 1$ then T_1 has a cyclic subgroup of order $\frac{q^d-1}{q-1} = p^n$, contradicting that G is non-abelian. This completes the proof of Claim 1.

Claim 2. $\mathrm{C}_{A^*}(\mathrm{O}_p(A^*)) \leq \mathrm{O}_p(A^*)$.

Suppose that D is a component of A^*, that is, a subnormal quasisimple subgroup of A^*. Then $D = D'$ and $D/Z(D) \cong T$, a non-abelian simple group. By Proposition 2.5, D has a proper subgroup C of p-power index and $Z(D)$ is a p-group. Since $|D : C| = |D : CZ(D)| \cdot |CZ(D) : C|$, we have that $|D : CZ(D)|$ is a p-power. If $D = CZ(D)$ then $D = D' = C'$, contradicting that C is a proper subgroup of D. Thus, $CZ(D) \neq D$. Since $|D/Z(D) : CZ(D)/Z(D)| = |D : CZ(D)|$, we have that $D/Z(D)$ has a proper subgroup $CZ(D)/Z(D)$ of p-power index. By Proposition 2.7(1), $p \nmid |M(D/Z(D))|$ and hence $p \nmid |Z(D)|$. Since $Z(D)$ is a p-group, we have $Z(D) = 1$ and so $D \cong T$. Recall that $E(A^*)$ is the product of all components of A^*. Then $D \leq E(A^*)$ and since $D \cong T$, D is a direct factor of $E(A^*)$. Clearly, D^a is also a direct factor of $E(A^*)$ for any $a \in A^*$. It follows that A^* contains a minimal normal subgroup which is isomorphic to T^ϵ with $\epsilon \geq 1$, contradicting Claim 1. Thus, A^* has no component and $E(A^*) = 1$. It follows that the generalized Fitting subgroup $F^*(A^*) = F(A^*)$. By Proposition 2.5, $\mathrm{O}_{p'}(A^*) = 1$ and hence $F^*(A^*) = \mathrm{O}_p(A^*)$. By Proposition 2.9, $\mathrm{C}_{A^*}(\mathrm{O}_p(A^*)) \leq \mathrm{O}_p(A^*)$, as claimed.

Now we are ready to finish the proof. Since $|A : A^*| \leq 2$ and G has no subgroups of index 2, we only need to show $G \trianglelefteq A^*$.

Let $H = \mathrm{O}_p(A^*)$. By Claim 1, $H \neq 1$. Write $\overline{H} = H/\Phi(H)$ and $\overline{A^*} = A^*/\Phi(H)$. Then $\mathrm{O}_p(A^*/H) = 1$ and $H \leq G$ as $G \in \mathrm{Syl}_p(A^*)$. By Claim 2 and Proposition 2.10, $A^*/H \leq \mathrm{Aut}(\overline{H})$. Since G is metacyclic, $\overline{H} = \mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.

Assume $\overline{H} = \mathbb{Z}_p$. Then $A^*/H \leq \mathbb{Z}_{p-1}$, and $G = H \trianglelefteq A^*$, as required.

Assume $\overline{H} = \mathbb{Z}_p \times \mathbb{Z}_p$. Then $A^*/H \leq \mathrm{GL}(2,p)$. If $p \nmid |A^*/H|$ then $G = H \trianglelefteq A^*$, as required. To finish the proof, we suppose $p \bigm| |A^*/H|$ and will obtain a contradiction.

Since $p \parallel |\mathrm{GL}(2,p)|$, we have $p \parallel |A^*/H|$, and since $\mathrm{Syl}_p(\mathrm{SL}(2,p)) = \mathrm{Syl}_p(\mathrm{GL}(2,p))$, we have $\mathrm{Syl}_p(A^*/H) \subseteq \mathrm{Syl}_p(\mathrm{SL}(2,p))$. Note that $A^*/H \cdot \mathrm{SL}(2,p) \leq \mathrm{GL}(2,p)$. Then $p \parallel |A^*/H \cdot \mathrm{SL}(2,p)|$ and so $p \bigm| |(A^*/H) \cap \mathrm{SL}(2,p)|$. Since $\mathrm{O}_p(A^*/H) = 1$, A^*/H has at least two Sylow p-subgroups, and hence $(A^*/H) \cap \mathrm{SL}(2,p)$ has at least two Sylow p-subgroups, that is, $(A^*/H) \cap \mathrm{SL}(2,p)$ has no normal Sylow p-subgroups. By [44, Theorem 6.17], $(A^*/H) \cap \mathrm{SL}(2,p)$ contains $\mathrm{SL}(2,p)$, that is, $\mathrm{SL}(2,p) \leq A^*/H \leq \mathrm{GL}(2,p)$. In particular, the induced faithful representation of A^*/H on the linear space $\overline{H}$ is irreducible, and hence $\overline{H}$ is a minimal normal subgroup of $\overline{A^*}$.

Recall that $A^* \trianglelefteq A$ and $H = \mathrm{O}_p(A^*)$, which is characteristic in A^*. Then $H \trianglelefteq A$, and since $\Phi(H)$ is characteristic in H, we have $\Phi(H) \trianglelefteq A$. Let $\Gamma_{\Phi(H)}$ be the quotient digraph of Γ relative to $\Phi(H)$, and let L be the kernel of A acting on $V(\Gamma_{\Phi(H)})$. Clearly, $\Gamma_{\Phi(H)}$ is bipartite. Furthermore, $L \trianglelefteq A$, $L \leq A^*$, $\Phi(H) \leq L$ and $L = \Phi(H)L_u$ for any $u \in V(\Gamma)$ because both $\Phi(H)$ and L are transitive on the orbit of $\Phi(H)$ containing u. Since $\Phi(H) \leq G$, $\Phi(H)$ is semiregular on $V(\Gamma)$, and hence $\Phi(H) \cap L_u = 1$. Since $p \nmid |A_u|$, L_u is a Hall p'-subgroup of L. Since $\Phi(H) \trianglelefteq L$ and $\Phi(H) \in \mathrm{Syl}_p(L)$, the Schur-Zassenhaus Theorem implies that all Hall p'-subgroup of L are conjugate. By Frattini argument [30, Kapitel I, 7.8 Satz], $A = L\mathrm{N}_A(L_u) = \Phi(H)L_u\mathrm{N}_A(L_u) = \Phi(H)\mathrm{N}_A(L_u)$ and $H = H \cap A = H \cap (\Phi(H)\mathrm{N}_A(L_u)) = \Phi(H)(H \cap \mathrm{N}_A(L_u))$. Since Frattini subgroup is generated by nongenerators (see [30, Kapitel III, 3.2 Satz]), $H = \Phi(H)(H \cap N_A(L_u))$ if and only if $H = H \cap N_A(L_u)$, that is, $H \leq N_A(L_u)$. It follows that $A = \mathrm{N}_A(L_u)$, that is, $L_u \trianglelefteq A$. By taking $u \in W_0$, we have $L_u = L_v$ for any $v \in W_0$ because A is transitive on W_0, and since A^* acts faithfully on W_0, we have $L_u = 1$. It follows that $L = \Phi(H)$, that is, the kernel of A acting on $V(\Gamma_{\Phi(H)})$ is $\Phi(H)$. Thus $\overline{A} = A/\Phi(H)$ is faithful on $V(\Gamma_{\Phi(H)})$, and then $\overline{A^*}$ is faithful on each of the parts of $V(\Gamma_{\Phi(H)})$, that is,

$\overline{A^*}$ is a transitive permutation group with p-power degree (the cardinality of each part of $V(\Gamma_{\Phi(H)})$).

Since $\overline{A^*}/\overline{H} \cong A^*/H$, we have $\mathrm{SL}(2,p) \leq \overline{A^*}/\overline{H} \leq \mathrm{GL}(2,p)$. Write $\overline{R}/\overline{H} = Z(\overline{A^*}/\overline{H})$. Then $\overline{R} \trianglelefteq \overline{A^*}$ and $1 \neq \overline{R}/\overline{H}$ is a p'-group. Since $\overline{H} \trianglelefteq \overline{R}$ and $\overline{H} \in \mathrm{Syl}_p(\overline{R})$, the Schur-Zassenhaus Theorem [44, Theorem 8.10] implies that there is a p'-group $\overline{V} \leq \overline{R}$ such that $\overline{R} = \overline{H}\overline{V}$ and all Hall p'-subgroup of $\overline{R}$ are conjugate. Note that $\overline{V} \neq 1$. By Frattini argument [30, Kapitel I, 7.8 Satz], $\overline{A^*} = \overline{R}N_{\overline{A^*}}(\overline{V}) = \overline{H}N_{\overline{A^*}}(\overline{V})$. Since $\overline{H}$ is abelian, $\overline{H} \cap N_{\overline{A^*}}(\overline{V}) \trianglelefteq \overline{A^*}$, and by the minimality of $\overline{H}$, we have $\overline{H} \cap N_{\overline{A^*}}(\overline{V}) = \overline{H}$ or 1. If $\overline{H} \cap N_{\overline{A^*}}(\overline{V}) = \overline{H}$ then $\overline{H} \leq N_{\overline{A^*}}(\overline{V})$ and $\overline{A^*} = \overline{H}N_{\overline{A^*}}(\overline{V}) = N_{\overline{A^*}}(\overline{V})$, that is, $\overline{V} \trianglelefteq \overline{A^*}$. This implies that $\mathrm{O}_{p'}(\overline{A^*}) \neq 1$, contradicting Proposition 2.5. If $\overline{H} \cap N_{\overline{A^*}}(\overline{V}) = 1$ then $\overline{A^*} = \overline{H}N_{\overline{A^*}}(\overline{V})$ implies $\mathrm{ASL}(2,p) \leq \overline{A^*} \leq \mathrm{AGL}(2,p)$ as $\mathrm{SL}(2,p) \leq \overline{A^*}/\overline{H} \leq \mathrm{GL}(2,p)$. It follows that a Sylow p-subgroup of $\overline{A^*}$ is not metacyclic. On the other hand, since both normal subgroups and quotient groups of a metacyclic group are metacyclic, any Sylow p-subgroup of $\overline{A^*}$ is metacyclic because each Sylow p-subgroup of A^* is metacyclic, a contradiction. This completes the proof. □

4 Edge-transitive bipartite bi-p-metacirculants

A connected edge-transitive graph should be semisymmetric, arc-transitive or half-arc-transitive. In this section, as an application of Theorem 1.1, we prove that there are no connected arc-transitive or semisymmetric bipartite bi-p-metacirculants with valency less than p. Furthermore, we classify the connected half-arc-transitive bipartite bi-p-metacirculants with valency less than $2p$.

Let G be a group and let R, L and S be subsets of G such that $R = R^{-1}$, $L = L^{-1}$, $1 \notin R \cup L$ and $1 \in S$, where 1 is the identity of G. Let $\mathrm{BiCay}(G,R,L,S)$ be the graph having vertex set the union of the *right part* $W_0 = \{g_0 \mid g \in G\}$ and the *left part* $W_1 = \{g_1 \mid g \in G\}$, and edge set the union of the *right edges* $\{\{h_0, g_0\} \mid gh^{-1} \in R\}$, the *left edges* $\{\{h_1, g_1\} \mid gh^{-1} \in L\}$ and the *spokes* $\{\{h_0, g_1\} \mid gh^{-1} \in S\}$. For $g \in G$, define a permutation $\hat{g}$ on $V(\Gamma) = W_0 \cup W_1$ by the rule

$$h_i^{\hat{g}} = (hg)_i, \ \forall i \in \mathbb{Z}_2, \ h, g \in G.$$

It is easy to check that $\hat{g}$ is an automorphism of $\mathrm{BiCay}(G,R,L,S)$ and $\hat{G} = \{\hat{g} \mid g \in G\}$ is a semiregular group of automorphisms of $\mathrm{BiCay}(G,R,L,S)$ with two orbits W_0 and W_1. Thus, $\mathrm{BiCay}(G,R,L,S)$ is a bi-Cayley graph over $\hat{G}$, and $\mathrm{BiCay}(G,R,L,S)$ is also called a *bi-Cayley graph* over G relative to R, L and S. Furthermore, $\mathrm{BiCay}(G,R,L,S)$ is connected if and only if $G = \langle R \cup L \cup S\rangle$, and $\mathrm{BiCay}(G,R,L,S) \cong \mathrm{BiCay}(G,R^\theta, L^\theta, S^\theta)$ for any $\theta \in \mathrm{Aut}(G)$.

On the other hand, if Γ is a Bi-Cayley graph over G then $\Gamma \cong \mathrm{BiCay}(G,R,L,S)$ for some subsets R, L and S of G satisfying $R = R^{-1}$, $L = L^{-1}$, $1 \notin R \cup L$ and $1 \in S$.

For $\theta \in \mathrm{Aut}(G)$ and $x, y, g \in G$, define two permutations on $V(\mathrm{BiCay}(G,R,L,S)) = W_0 \cup W_1$ as following:

$$\delta_{\theta,x,y} \colon h_0 \mapsto (xh^\theta)_1, \ h_1 \mapsto (yh^\theta)_0, \ \forall h \in G,$$
$$\sigma_{\theta,g} \colon h_0 \mapsto (h^\theta)_0, \ h_1 \mapsto (gh^\theta)_1, \ \forall h \in G.$$

Set

$$I := \{\delta_{\theta,x,y} \mid \theta \in \mathrm{Aut}(G) \text{ s.t. } R^\theta = x^{-1}Lx,\ L^\theta = y^{-1}Ry,\ S^\theta = y^{-1}S^{-1}x\},$$
$$F := \{\sigma_{\theta,g} \mid \theta \in \mathrm{Aut}(G) \text{ s.t. } R^\theta = R,\ L^\theta = g^{-1}Lg,\ S^\theta = g^{-1}S\}.$$

The following proposition characterizes the normalizer of $\hat{G}$ in $\mathrm{Aut}(\Gamma)$.

Proposition 4.1 ([55, Theorem 1.1])**.** *Let $\Gamma = \mathrm{BiCay}(G, R, L, S)$ be a connected bi-Cayley graph over a group G, where R, L and S are subsets of G with $R = R^{-1}$, $L = L^{-1}$, $1 \notin R \cup L$ and $1 \in S$. If $I = \emptyset$ then $\mathrm{N}_{\mathrm{Aut}(\Gamma)}(\hat{G}) = \hat{G} \rtimes F$, and if $I \neq \emptyset$, then $\mathrm{N}_{\mathrm{Aut}(\Gamma)}(\hat{G}) = \hat{G}\langle F, \delta_{\theta,x,y}\rangle$ for some $\delta_{\theta,x,y} \in I$.*

Write $N = \mathrm{N}_{\mathrm{Aut}(\Gamma)}(\hat{G})$. By Proposition 4.1, $N_{1_0} = F$ and $N_{1_01_1} = \{\sigma_{\theta,1} \mid \theta \in \mathrm{Aut}(G) \text{ s.t. } R^\theta = R,\ L^\theta = L,\ S^\theta = S\}$. In particular, F is a group. For the special case $R = L = \emptyset$, it is easy to see that $F = \{\sigma_{\theta,s} \mid \theta \in \mathrm{Aut}(G), s \in S, S^\theta = s^{-1}S\}$ as $1 \in S$.

Lemma 4.2. *Let $\Gamma = \mathrm{BiCay}(G, \emptyset, \emptyset, S)$ be a connected bipartite bi-Cayley graph over G relative to S with $1 \in S$. Then $F = \{\sigma_{\theta,s} \mid \theta \in \mathrm{Aut}(G), s \in S, S^\theta = s^{-1}S\}$ is faithful on $S_1 = \{s_1 \mid s \in S\}$. If G is a p-group and F is a p'-group, then $F \cong \{\theta \mid \sigma_{\theta,s} \in F\}$.*

Proof. Set $L = \{\theta \mid \sigma_{\theta,s}\}$. Since Γ is connected, $G = \langle S\rangle$, and since $F_{1_1} = \{\sigma_{\theta,1} \mid \theta \in \mathrm{Aut}(G) \text{ s.t. } S^\theta = S\}$, F is faithful on S_1. The group F has operation $\sigma_{\theta,x}\sigma_{\delta,y} = \sigma_{\theta\delta,yx^\delta}$ for any $\sigma_{\theta,x}, \sigma_{\delta,y} \in F$, and so the map $\varphi\colon \sigma_{\theta,s} \mapsto \theta$ is an epimorphism from F to L. Let K be the kernel of φ. Then $\sigma_{\theta,s} \in K$ if and only if $\theta = 1$.

Let G be a p-group and F a p'-group. If $\sigma_{1,s} \in K$ for some $1 \neq s \in S$, then $s_1^{\langle\sigma_{1,s}\rangle} = \{s_1, s_1^2, \ldots, s^{o(s)-1}, 1_1\}$ because $1_1^{\sigma_{1,s}} = s_1$ and $(s_1^{l-1})^{\sigma_{1,s}} = s_1^l$ for any positive integer l. Since G is a p-group, $o(s)$ is a p-power and hence $p \mid o(\sigma_{1,s})$, which is impossible because F is a p'-group. Thus, $s = 1$ and hence $K = 1$. Since φ is an epimorphism from F to L, we have $F \cong L$. $\square$

By Equation (2.1), $G_{\alpha,\beta,\gamma} = \langle a, b \mid a^{p^\alpha} = 1,\ b^{p^\beta} = 1,\ b^{-1}ab = a^{1+p^\gamma}\rangle$ with $0 < \gamma < \alpha \leq \beta + \gamma$.

Lemma 4.3. *In $G_{\alpha,\beta,\gamma}$, the following properties hold:*

(1) For any non-negative integers i, j, we have $a^ib^j = b^ja^{i(1+p^\gamma)^j}$;

(2) Let $\theta \in \mathrm{Aut}(G_{\alpha,\beta,\gamma})$ such that $a^\theta = b^ma^n$ with $(m,p) = 1$. Then $\beta < \alpha$.

Proof. From $b^{-1}ab = a^{1+p^\gamma}$, we have $b^{-1}a^ib = a^{i(1+p^\gamma)}$ and hence $b^{-j}a^ib^j = a^{i(1+p^\gamma)^j}$. Part (1) follows. Since $a^\theta = b^ma^n$, we have $o(b^ma^n) = o(a) = p^\alpha$, and since $\langle a\rangle \trianglelefteq G_{\alpha,\beta,\gamma}$, we have $\langle b^ma^n\rangle \trianglelefteq G_{\alpha,\beta,\gamma}$. Then $(p, m) = 1$ implies $G_{\alpha,\beta,\gamma} = \langle a, b^ma^n\rangle = \langle a\rangle\langle b^ma^n\rangle$, and hence

$$p^{\alpha+\beta} = |G_{\alpha,\beta,\gamma}| = \frac{|\langle a\rangle| \cdot |\langle b^ma^n\rangle|}{|\langle a\rangle \cap \langle b^ma^n\rangle|} = \frac{p^\alpha \cdot p^\alpha}{|\langle a\rangle \cap \langle b^ma^n\rangle|} \leq p^\alpha \cdot p^\alpha,$$

that is, $\beta \leq \alpha$. If $\beta = \alpha$, then $|\langle a\rangle \cap \langle b^ma^n\rangle| = 1$ and hence $G_{\alpha,\beta,\gamma} = \langle a\rangle \times \langle b^ma^n\rangle$, contradicting that $G_{\alpha,\beta,\gamma}$ is non-abelian. Thus, $\beta < \alpha$ and part (2) follows. $\square$

A graph Γ is called *locally-transitive* if the stabilizer $\mathrm{Aut}(\Gamma)_u$, for any $u \in V(\Gamma)$, is transitive on the neighborhood of u in $V(\Gamma)$.

Theorem 4.4. *There are no connected locally-transitive bipartite bi-p-metacirculants of valency less than p for any odd prime p.*

Proof. Suppose to the contrary that Γ is a connected locally-transitive bipartite bi-Cayley graph of valency less than p over a non-abelian metacirculant p-group G. Since p is odd, the two orbits of G are exactly the parts of Γ, and we may assume that $\Gamma = \mathrm{BiCay}(G, \emptyset, \emptyset, S)$, where $1 \in S$, $|S| < p$ and $G = \langle S\rangle$. Let $A = \mathrm{Aut}(\Gamma)$. Since Γ has valency less than p, A_{1_0} is a p'-group, and by Theorem 1.1, $\hat{G} \trianglelefteq A$. Write $F = \{\sigma_{\theta,g} \mid \theta \in \mathrm{Aut}(G), S^\theta = g^{-1}S\}$ and $L = \{\theta \mid \sigma_{\theta,s} \in F\}$. By Proposition 4.1, $A_{1_0} = F$, and by Lemma 4.2, $F \cong L$.

Assume that G is non-split. By Lindenberg [36], the automorphism group of G is a p-group. Thus, $p \mid |L|$ and hence $p \mid |A_{1_0}|$, a contradiction.

Assume that G is split. Then $G = G_{\alpha,\beta,\gamma}$, as defined in Equation (2.1). Since F is a p'-group and $F \cong L$, Proposition 2.1 implies that F is cyclic and $|F| \mid (p-1)$. Let $|S_1| = k$ and $F = \langle\sigma_{\theta,s}\rangle$, where $\theta \in \mathrm{Aut}(G)$, $s \in S$ and $S^\theta = s^{-1}S$. Since F is transitive on S_1, $\sigma_{\theta,s}$ permutes all elements in S_1 cyclically, and so $\sigma_{\theta,s}^k$ fixes all elements in S_1. By Proposition 4.1, F is faithful on S_1, implying that $\sigma_{\theta,s}^k = 1$. It follows that $\sigma_{\theta,s}$ has order k and is regular on S_1. Since $F \cong L$, θ also has order k. Furthermore, $S_1 = 1_1^{\langle\sigma_{\theta,s}\rangle} = \{1_1, s_1, (ss^\theta)_1, \dots, (ss^\theta \cdots s^{\theta^{k-2}})_1\}$ and $1_1 = (ss^\theta \cdots s^{\theta^{k-1}})_1$, that is, $S = \{1, s, ss^\theta, \dots, ss^\theta \cdots s^{\theta^{k-2}}\}$ and

$$ss^\theta \cdots s^{\theta^{k-1}} = 1. \tag{4.1}$$

Note that for any $\tau \in \mathrm{Aut}(G)$, we have $\Gamma = \mathrm{BiCay}(G, \emptyset, \emptyset, S) \cong \mathrm{BiCay}(G, \emptyset, \emptyset, S^\tau)$, where $S^\tau = \{1, t, tt^{\theta^\tau}, \dots, tt^{\theta^\tau} \cdots t^{(\theta^\tau)^{k-2}}\}$ and $tt^{\theta^\tau} \cdots t^{(\theta^\tau)^{k-1}} = 1$ with $t = s^\tau$. By Proposition 2.1, all cyclic groups of order k in $\mathrm{Aut}(G)$ are conjugate, and so we may assume that θ is the automorphism induced by $a \mapsto a^e$, $b \mapsto b$, where $e \in \mathbb{Z}_{p^\alpha}^*$ has order k.

Let $s = b^ia^j \in G_{\alpha,\beta,\gamma}$. By Lemma 4.3, $a^ib^j = b^ja^{i(1+p^\gamma)^j}$, and since $a^\theta = a^e$ and $b^\theta = b$, we have $ss^\theta \cdots s^{\theta^{k-1}} = b^{ki}a^\epsilon$ for some $\epsilon \in \mathbb{Z}_{p^\alpha}$. By Equation (4.1), $b^{ki} = 1$, that is, $ki \equiv 0 \pmod{p^\beta}$. Since $k < p$, we have $i \equiv 0 \pmod{p^\beta}$, and hence $G = \langle S\rangle = \langle 1, a^j, a^ja^{je}, \dots, a^ja^{je} \cdots a^{je^{k-2}}\rangle \leq \langle a\rangle$, a contradiction. This completes the proof. $\square$

Theorem 1.2 is a direct corollary of Theorem 4.4.

To prove Theorem 1.3, we need two technical lemmas on integer numbers.

Lemma 4.5. *Let p be an odd prime and α a positive integer. Let e be an element of order k ($k \geq 2$) in $\mathbb{Z}_{p^\alpha}^*$ with $k \mid (p-1)$. Then $e^i - 1 \in \mathbb{Z}_{p^\alpha}^*$ for any $1 \leq i < k$, and $1 + e + \cdots + e^{k-1} \equiv 0 \pmod{p^\alpha}$.*

For $i \in \mathbb{Z}_k$, let $t_i = (e-1)^{-1}(e^i - 1)$ and $T = \{t_i \mid i \in \mathbb{Z}_k\}$, where $(e-1)^{-1}$ is the inverse of $e - 1$ in $\mathbb{Z}_{p^\alpha}^*$. For $x, y \in \mathbb{Z}_{p^\alpha}$, let $Tx + y = \{tx + y \mid t \in T\}$. Then $Tx + y = T$ in $\mathbb{Z}_{p^\alpha}$ if and only if $x \equiv e^l \pmod{p^\alpha}$ and $y \equiv (e-1)^{-1}(e^l - 1) \pmod{p^\alpha}$ for some $l \in \mathbb{Z}_k$. In particular, $Tx = T$ in $\mathbb{Z}_{p^\alpha}$ if and only if $x \equiv 1 \pmod{p^\alpha}$.

Proof. Suppose $e^i - 1 \notin \mathbb{Z}_{p^\alpha}^*$ for some $1 \leq i < k$. Then $p \mid (e^i - 1)$, and since e has order k, we have $e^i \not\equiv 1 \pmod{p^\alpha}$ and $(e^i)^k \equiv 1 \pmod{p^\alpha}$. Furthermore, $p \mid (e^i - 1)$ implies that there exist $l \in \mathbb{Z}_{p^\alpha}^*$ ($p \nmid l$) and $1 \leq s < \alpha$ such that $e^i = 1 + lp^s$. Note that

$$(e^i)^k - 1 = (1 + lp^s)^k - 1 = klp^s + C_k^2(lp^s)^2 + \cdots + C_k^{k-1}(lp^s)^{k-1} + (lp^s)^k.$$

Since $(e^i)^k \equiv 1 \pmod{p^\alpha}$, we have $p \mid kl$, and since $2 \le k < p$, we have $p \mid l$, a contradiction. Thus, $p \nmid (e^i - 1)$, that is, $e^i - 1 \in \mathbb{Z}_{p^\alpha}^*$. The equation $1 + e + \cdots + e^{k-1} \equiv 0 \pmod{p^\alpha}$ follows from $(e-1)(1+e+\cdots+e^{k-1}) = e^k - 1 \equiv 0 \pmod{p^\alpha}$ and $e - 1 \in \mathbb{Z}_{p^\alpha}^*$.

Note that $T \subseteq \mathbb{Z}_{p^\alpha}$ and $Tx + y \subseteq \mathbb{Z}_{p^\alpha}$. Since $1 + e + \cdots + e^{k-1} \equiv 0 \pmod{p^\alpha}$, we have

$$\begin{aligned}\sum_{t \in T} t &= (e-1)^{-1} \sum_{i \in \mathbb{Z}_k} (e^i - 1)\\ &= (e-1)^{-1}[(e-1) + \cdots + (e^{k-1} - 1)]\\ &= -k(e-1)^{-1} \in \mathbb{Z}_{p^\alpha}^*.\end{aligned}$$

Assume $Tx + y = T$ in $\mathbb{Z}_{p^\alpha}$. Then $\sum_{t \in T}(tx + y) = \sum_{t \in T} t$, and hence $ky = (1-x)\sum_{t \in T} t = -(1-x)k(e-1)^{-1}$ in $\mathbb{Z}_{p^\alpha}$. It follows $y = (e-1)^{-1}(x-1)$ because $k \in \mathbb{Z}_{p^\alpha}^*$. Then $Tx + (e-1)^{-1}(x-1) = T$ implies $x[T(e-1)+1] = T(e-1)+1$. Since $T(e-1) + 1 = \{e^i \mid i \in \mathbb{Z}_k\} = \langle e \rangle$, we have $x\langle e \rangle = \langle e \rangle$ in $\mathbb{Z}_{p^\alpha}^*$, that is, $x \equiv e^l \pmod{p^\alpha}$ for some $l \in \mathbb{Z}_k$. Furthermore, $y \equiv (e-1)^{-1}(e^l - 1) \pmod{p^\alpha}$.

On the other hand, let $x \equiv e^l \pmod{p^\alpha}$ and $y \equiv (e-1)^{-1}(e^l - 1) \pmod{p^\alpha}$ for some $l \in \mathbb{Z}_k$. Then in $\mathbb{Z}_{p^\alpha}$, we have

$$\begin{aligned}Tx + y &= \{e^l(e-1)^{-1}(e^i - 1) + (e-1)^{-1}(e^l - 1) \mid i \in \mathbb{Z}_k\}\\ &= (e-1)^{-1}\{e^l(e^i - 1) + (e^l - 1) \mid i \in \mathbb{Z}_k\}\\ &= (e-1)^{-1}\{e^{i+l} - 1 \mid i \in \mathbb{Z}_k\} = \{(e-1)^{-1}(e^i - 1) \mid i \in \mathbb{Z}_k\} = T.\end{aligned}$$

Thus $Tx + y = T$ in $\mathbb{Z}_{p^\alpha}$ if and only if $x \equiv e^l \pmod{p^\alpha}$ and $y \equiv (e-1)^{-1}(e^l - 1) \pmod{p^\alpha}$ for some $l \in \mathbb{Z}_k$. Applying this with $y = 0$, we obtain that $Tx = T$ in $\mathbb{Z}_{p^\alpha}$ if and only if $x \equiv 1 \pmod{p^\alpha}$. $\square$

Lemma 4.6. *Let p be an odd prime and let α, γ be positive integers with $0 < \gamma < \alpha$. Let e be an element of order k $(k \ge 2)$ in $\mathbb{Z}_{p^\alpha}^*$ with $k \mid (p-1)$. Then for any $m \in \mathbb{Z}_{p^{\alpha-\gamma}}^*$ and any $0 \le \ell \le k-1$, the following equation in $\mathbb{Z}_{p^\alpha}$*

$$e^\ell(1 + p^\gamma)^m = [(1 + p^\gamma)^m - x(1-e)]^2 \tag{4.2}$$

has a solution if and only if $\frac{k}{(k,\ell)} \mid \frac{(p-1)}{2}$, and in this case, there are exactly two solutions.

Proof. Since $e^\ell(1+p^\gamma)^m \in \mathbb{Z}_{p^\alpha}^*$, Equation (4.2) has a solution if and only if $e^\ell(1+p^\gamma)^m$ is a square in $\mathbb{Z}_{p^\alpha}^*$. Since $\mathbb{Z}_{p^\alpha}^* \cong \mathbb{Z}_{p^{\alpha-1}(p-1)}$, squares in $\mathbb{Z}_{p^\alpha}^*$ consists of the unique subgroup of order $\frac{(p-1)}{2}p^{\alpha-1}$ in $\mathbb{Z}_{p^\alpha}^*$, and so Equation (4.2) has a solution if and only if the order of $e^\ell(1+p^\gamma)^m$ in $\mathbb{Z}_{p^\alpha}^*$ is a divisor of $\frac{(p-1)}{2}p^{\alpha-1}$. Clearly, $(1+p^\gamma)^m$ has order $p^{\alpha-\gamma}$, and e^ℓ has order $\frac{k}{(k,\ell)}$. Thus, Equation (4.2) has a solution if and only if $\frac{k}{(k,\ell)} \mid \frac{(p-1)}{2}$. If $e^\ell(1+p^\gamma)^m = u^2$ for some $u \in \mathbb{Z}_{p^\alpha}^*$ then $(1-e)^{-1}[(1+p^\gamma)^m \pm u]$ are the only two solutions of Equation (4.2) in $\mathbb{Z}_{p^\alpha}$. $\square$

Now we construct the half-arc-transitive graphs in Theorem 1.3. Let p be an odd prime, and let α, β, γ be positive integers such that $0 < \gamma < \alpha \le \beta + \gamma$. Let e be an element of order k ($k \geq 2$) in $\mathbb{Z}_{p^\alpha}^*$ with $k \mid (p-1)$. Choose $0 \leq \ell < k$ such that $\frac{k}{(k,\ell)} \mid \frac{(p-1)}{2}$. Recall that

$$G_{\alpha,\beta,\gamma} = \langle a, b \mid a^{p^\alpha} = 1,\ b^{p^\beta} = 1,\ b^{-1}ab = a^{1+p^\gamma}\rangle.$$

Let

$$U = \{a^t \mid t \in \{(e-1)^{-1}(e^i - 1) \mid i \in \mathbb{Z}_k\}\} \text{ and}$$
$$V = \{b^m a^i \mid i \in \{(e-1)^{-1}(e^i - 1)(1+p^\gamma)^m + e^i n \mid i \in \mathbb{Z}_k\}\},$$

where $m \in \mathbb{Z}_{p^{\alpha-\gamma}}^*$ and n is a solution of $e^\ell(1+p^\gamma)^m = [(1+p^\gamma)^m - x(1-e)]^2$. Define

$$\Gamma_{m,k,\ell}^n = \mathrm{BiCay}(G_{\alpha,\beta,\gamma}, \emptyset, \emptyset, U \cup V). \tag{4.3}$$

By Lemma 4.6, there are exactly two solutions n of equation $e^\ell(1 + p^\gamma)^m = [(1 + p^\gamma)^m - x(1 - e)]^2$ in $\mathbb{Z}_{p^\alpha}$, and so the notation $\Gamma_{m,k,\ell}^n$ is also written as $\Gamma_{m,k,\ell}^\pm$, as used in Theorem 1.3. We first prove the sufficiency of Theorem 1.3.

Lemma 4.7. *The graphs $\Gamma_{m,k,\ell}^\pm$ are independent from the choice of element e of order k in $\mathbb{Z}_{p^\alpha}^*$ and half-arc-transitive, and $\mathrm{Aut}(\Gamma_{m,k,\ell}^\pm) \cong (G_{\alpha,\beta,\gamma} \rtimes \mathbb{Z}_k).\mathbb{Z}_2$.*

Proof. Write $\Gamma = \Gamma_{m,k,\ell}^n$ and $A = \mathrm{Aut}(\Gamma)$. Let $T = \{(e-1)^{-1}(e^i - 1) \mid i \in \mathbb{Z}_k\}$ and $T' = \{(e-1)^{-1}(e^i - 1)(1+p^\gamma)^m + e^i n \mid i \in \mathbb{Z}_k\}$. Then $U = \{a^\eta \mid \eta \in T\}$ and $V = \{b^m a^\eta \mid \eta \in T'\}$. Furthermore, $\Gamma = \mathrm{BiCay}(G, \emptyset, \emptyset, S)$ with $G = G_{\alpha,\beta,\gamma}$ and $S = U \cup V$. Clearly, $1 \in U$ and $G_{\alpha,\beta,\gamma} = \langle S\rangle$, implying that Γ is connected. Note that $T' = T[(1+p^\gamma)^m + n(e-1)] + n$.

Since $e \in \mathbb{Z}_{p^\alpha}^*$, any element of order k in $\mathbb{Z}_{p^\alpha}^*$ can be written as e^q with $(q,k) = 1$ and hence $\{e^i \mid i \in \mathbb{Z}_k\} = \langle e\rangle = \langle e^q\rangle = \{(e^q)^i \mid i \in \mathbb{Z}_k\}$. By Lemma 4.5, $e - 1 \in \mathbb{Z}_{p^\alpha}^*$ and $e^q - 1 \in \mathbb{Z}_{p^\alpha}^*$. Let

$$\overline{T} = \{(e^q - 1)^{-1}((e^q)^i - 1) \mid i \in \mathbb{Z}_k\},$$
$$\overline{T'} = \{(e^q - 1)^{-1}((e^q)^i - 1)(1+p^\gamma)^m + (e^q)^i n \mid i \in \mathbb{Z}_k\},$$
$$\overline{U} = \{a^\eta \mid \eta \in \overline{T}\} \text{ and}$$
$$\overline{V} = \{b^m a^\eta \mid \eta \in \overline{T'}\}.$$

It is easy to see that $a \mapsto a^{(e-1)(e^q-1)^{-1}}$ and $b \mapsto b$ induce an automorphism of G, say ρ. Then

$$U^\rho = \{a^{(e-1)(e^q-1)^{-1}(e-1)^{-1}(e^i-1)} \mid i \in \mathbb{Z}_k\}$$
$$= \{a^{(e^q-1)^{-1}((e^q)^i-1)} \mid i \in \mathbb{Z}_k\} = \{a^\eta \mid \eta \in \overline{T}\} = \overline{U},$$

and similarly, $V^\rho = \overline{V}$. Thus, $\mathrm{BiCay}(G, \emptyset, \emptyset, U \cup V) \cong \mathrm{BiCay}(G, \emptyset, \emptyset, \overline{U} \cup \overline{V})$, that is, Γ is independent from the choice of element e of order k in $\mathbb{Z}_{p^\alpha}^*$. To finish the proof, it suffices to prove that Γ is half-arc-transitive with $\mathrm{Aut}(\Gamma) \cong (G_{\alpha,\beta,\gamma} \rtimes \mathbb{Z}_k).\mathbb{Z}_2$.

Claim 1. $p \nmid |A_{1_0}|$.

We argue by contradiction and we suppose $p \mid |A_{1_0}|$. Let P is a Sylow p-subgroup of A containing $\hat{G}$ and let $X = \mathrm{N}_A(\hat{G})$. Then $\hat{G} < P$, and hence $\hat{G} < \mathrm{N}_P(\hat{G}) \leq X$. In particular, $p \mid |X : \hat{G}|$, and so $p \mid |X_{1_0}|$. Let τ be the automorphism of G induced by $a \mapsto a^e$ and $b \mapsto b$.

First we prove $\sigma_{\tau,a} \in X_{1_0}$. By Proposition 4.1, it is enough to show $S^\tau = a^{-1}S$. Clearly,

$$U^\tau = \{a^{e\eta} \mid \eta \in T\} = \{a^\eta \mid \eta \in Te\} \text{ and}$$
$$a^{-1}U = \{a^{\eta-1} \mid \eta \in T\} = \{a^\eta \mid \eta \in T-1\}.$$

By taking $\ell = 1$ in Lemma 4.5, we have $Te = T - 1$ and hence $U^\tau = a^{-1}U$. Similarly,

$$\begin{aligned} V^\tau &= \{b^m a^{e\eta} \mid \eta \in T'\} = \{b^m a^\eta \mid \eta \in T'e\} \text{ and} \\ a^{-1}V &= \{a^{-1}b^m a^\eta \mid \eta \in T'\} = \{b^m a^{-(1+p^\gamma)^m} a^\eta \mid \eta \in T'\} \\ &= \{b^m a^\eta \mid \eta \in T' - (1+p^\gamma)^m\}. \end{aligned}$$

By Equation (4.2), $(1+p^\gamma)^m + n(e-1) \in \mathbb{Z}^*_{p^\alpha}$, and hence $Te = T - 1$ implies

$$T[(1+p^\gamma)^m + n(e-1)]e + ne = T[(1+p^\gamma)^m + n(e-1)] + n - (1+p^\gamma)^m.$$

Since $T' = T[(1+p^\gamma)^m + n(e-1)] + n$, we have $T'e = T' - (1+p^\gamma)^m$, that is, $V^\tau = a^{-1}V$. It follows that $S^\tau = a^{-1}S$, as required.

Set $U_1 = \{u_1 \mid u \in U\}$, $V_1 = \{v_1 \mid v \in V\}$ and $S_1 = \{s_1 \mid s \in S\}$. Then $U_1 = 1_1^{\langle\sigma_{\tau,a}\rangle}$ and $V_1 = (b^m a^n)_1^{\langle\sigma_{\tau,a}\rangle}$. Since $\sigma_{\tau,a} \in X_{1_0}$, either X_{1_0} has two orbits of length k on S_1, or is transitive on S_1. By Lemma 4.2, X_{1_0} acts faithfully on S_1, and since $p \mid |X_{1_0}|$, any element of order p of X_{1_0} has an orbit of length p on S_1, implying that X_{1_0} is transitive on S_1 as $k < p$. From $|X_{1_0}| = |X_{1_0 1_1}| \cdot |1_1^{X_{1_0}}| = |X_{1_0 1_1}| \cdot 2k$, we have $p \mid |X_{1_0 1_1}|$. By Proposition 4.1, $X_{1_0 1_1} = \{\sigma_{\theta,1} \mid \theta \in \mathrm{Aut}(G) \text{ s.t. } S^\theta = S\}$. Let $\sigma_{\theta,1} \in X_{1_0 1_1}$ be of order p with $\theta \in \mathrm{Aut}(G)$. Then θ has order p and $S^\theta = S$. Recall that $k \geq 2$.

Assume $k > 2$. Since $a \in S$, we have $a^\theta \in S^\theta = S = U \cup V$. If $a^\theta \in V$ then $a^\theta = b^m a^i$ for some $i \in T'$. Note that $a^{1+e} \in S$ as $k > 2$. Since $m \in \mathbb{Z}^*_{p^{\alpha-\gamma}}$, we have $(m,p) = 1$, and by Lemma 4.5, $(p, 1+e) = 1$. Then $(a^{1+e})^\theta = (b^m a^i)^{1+e} \in V$, and considering the powers of b, we have $m(1+e) \equiv m \pmod{p^\beta}$, that is, $e \equiv 0 \pmod{p^\beta}$. It follows that $p \mid e$, contradicting that $e \in \mathbb{Z}^*_{p^\alpha}$. Thus, $a^\theta \in U$, and hence, $a^\theta = a^j$ for some $j \in T$. If $a^\theta \neq a$ then $a_1^{\sigma_{\theta,1}} = \{a_1, a_1^\theta, \ldots, a_1^{\theta^{p-1}}\}$ is an orbit of length p of $\sigma_{\theta,1}$ on S_1, which is impossible because there are exactly $k < p$ elements of type a^j in S. Thus, $a^\theta = a$ and θ fixes U pointwise. Furthermore, θ also fixes V pointwise because $|V| = k < p$. It follows that $\theta = 1$ as $G = \langle S\rangle$, and so $\sigma_{\theta,1} = 1$, a contradiction.

Assume $k = 2$. Then $e \equiv -1 \pmod{p^\alpha}$ and

$$S_1 = \{1_1, a_1, (b^m a^n)_1, (b^m a^{(1+p^\gamma)^m - n})_1\}.$$

Since $1_1^{\sigma_{\theta,1}} = 1_1$ and $p \geq 3$, $\sigma_{\theta,1}$ has order 3 and we may assume that $a_1^{\sigma_{\theta,1}} = (b^m a^n)_1$, $(b^m a^n)_1^{\sigma_{\theta,1}} = (b^m a^{(1+p^\gamma)^m - n})_1$ and $(b^m a^{(1+p^\gamma)^m - n})_1^{\sigma_{\theta,1}} = a_1$ (replace $\sigma_{\theta,1}$ by $\sigma_{\theta,1}^2$ if necessary), that is, $a^\theta = b^m a^n$, $(b^m a^n)^\theta = b^m a^{(1+p^\gamma)^m - n}$ and $(b^m a^{(1+p^\gamma)^m - n})^\theta = a$.

By Lemma 4.3, $\beta < \alpha$. It follows that

$$\begin{aligned} a &= (b^m a^{(1+p^\gamma)^m - n})^\theta = [(b^m a^n) a^{(1+p^\gamma)^m - 2n}]^\theta \\ &= b^m a^{(1+p^\gamma)^m - n} (b^m a^n)^{(1+p^\gamma)^m - 2n}, \end{aligned}$$

and so $0 \equiv m + m[(1+p^\gamma)^m - 2n] \pmod{p^\beta}$. Thus, $p \mid (1-n)$, which is impossible because otherwise $p^\alpha = o(a^{\theta^2}) = o(b^m a^{(1+p^\gamma)^m - n}) < p^\alpha$.

Summing up, we have proved $p \nmid |A_{1_0}|$, and this completes the proof of Claim 1.

By Claim 1, A_{1_0} is a p'-group, and by Theorem 1.1, Γ is normal. By Proposition 4.1, $A_{1_0} = X_{1_0} = F = \{\sigma_{\theta,s} \mid \sigma \in \mathrm{Aut}(G), s \in S, S^\theta = s^{-1}S\}$, and by Lemma 4.2, $F \cong L \leq \mathrm{Aut}(G)$, where $L = \{\theta \mid \sigma_{\theta,s} \in F\}$. By Proposition 2.1, F is cyclic, and since $\sigma_{\tau,a} \in A_{1_0}$, F is transitive on S_1 or has two orbits. By Lemma 4.2, F is faithful on S_1, and since F is cyclic, either F is regular on S_1, or $F = \langle \sigma_{\tau,a} \rangle$.

We suppose that F is regular on S_1 and will obtain a contradiction. Note that $F \cong \mathbb{Z}_{2k}$ and $|F : \langle \sigma_{\tau,a} \rangle| = 2$. Then $\langle \sigma_{\tau,a} \rangle \trianglelefteq F$, and the two orbits U_1 and V_1 of $\langle \sigma_{\tau,a} \rangle$ consist of an imprimitive block system of F on S_1. By the regularity of F, there exists $\sigma_{\theta,s} \in F$ such that $1_1^{\sigma_{\theta,s}} = (b^m a^n)_1$, implying that $s = b^m a^n$ and $S^\theta = s^{-1}S = (b^m a^n)^{-1}S$, and hence $U_1^{\sigma_{\theta,s}} = V_1$ because $1_1 \in U_1$ and $(b^m a^n)_1 \in V_1$. It follows that $a^\theta \in (b^m a^n)^{-1}V$. It is easy to see that

$$(b^m a^n)^{-1}S = (b^m a^n)^{-1}U \cup (b^m a^n)^{-1}V,$$

where

$$\begin{aligned} (b^m a^n)^{-1}U &= \{(b^m a^n)^{-1} a^\eta \mid \eta \in T\} = \{b^{-m} a^{-n(1+p^\gamma)^{-m} + \eta} \mid \eta \in T\} \\ &= \{b^{-m} a^\eta \mid \eta \in T - n(1+p^\gamma)^{-m}\} \text{ and} \\ (b^m a^n)^{-1}V &= \{(b^m a^n)^{-1} b^m a^\eta \mid \eta \in T'\} \\ &= \{a^{-n+\eta} \mid \eta \in T'\} = \{a^\eta \mid \eta \in T' - n\}. \end{aligned}$$

Since $T' = T[(1+p^\gamma)^m + n(e-1)] + n$, we have

$$(b^m a^n)^{-1}V = \{a^\eta \mid \eta \in T[(1+p^\gamma)^m + n(e-1)]\}.$$

Let $a^\theta = a^r \in (b^m a^n)^{-1}V$ for some $r \in T[(1+p^\gamma)^m + n(e-1)]$. Since $p^\alpha = o(a^\theta) = o(a^r)$, we have $r \in \mathbb{Z}_{p^\alpha}^*$. Note that

$$U^\theta = \{a^{\eta r} \mid \eta \in T\} = \{a^\eta \mid \eta \in Tr\} \subseteq (b^m a^n)^{-1}V.$$

Then

$$U^\theta = (b^m a^n)^{-1}V = \{a^\eta \mid \eta \in T[(1+p^\gamma)^m + n(e-1)]\},$$

and so $Tr = T[(1+p^\gamma)^m + n(e-1)]$ in $\mathbb{Z}_{p^\alpha}$. By Lemma 4.5, $r = (1+p^\gamma)^m + n(e-1)$.

Since $S^\theta = (b^m a^n)^{-1}S = (b^m a^n)^{-1}U \cup (b^m a^n)^{-1}V$, we have $V^\theta = (b^m a^n)^{-1}U$. In particular, $(b^m a^n)^\theta = b^{-m} a^t$ for some $t \in T - n(1+p^\gamma)^{-m}$. For $\eta \in T'$, since

$$(b^m a^\eta)^\theta = [(b^m a^n) a^{\eta - n}]^\theta = b^{-m} a^t a^{r(\eta - n)} = b^{-m} a^{r\eta - rn + t},$$

we have

$$\begin{aligned}\{b^{-m}a^{\eta} \mid \eta \in T - n(1+p^{\gamma})^{-m}\} \\ &= (b^m a^n)^{-1}U = V^{\theta} = \{(b^m a^{\eta})^{\theta} \mid \eta \in T'\} \\ &= \{b^{-m}a^{r\eta - rn + t} \mid \eta \in T'\} = \{b^{-m}a^{\eta} \mid \eta \in T'r - rn + t\}.\end{aligned}$$

This implies that

$$\begin{aligned}T - n(1+p^{\gamma})^{-m} \\ &= T'r - rn + t = Tr[(1+p^{\gamma})^m + n(e-1)] + rn - rn + t \\ &= Tr[(1+p^{\gamma})^m + n(e-1)] + t = T[(1+p^{\gamma})^m + n(e-1)]^2 + t\end{aligned}$$

in $\mathbb{Z}_{p^{\alpha}}$. By Equation (4.2), $e^{\ell}(1+p^{\gamma})^m = [(1+p^{\gamma})^m + n(e-1)]^2$. It follows that

$$T - n(1+p^{\gamma})^{-m} = Te^{\ell}(1+p^{\gamma})^m + t,$$

and hence

$$T = Te^{\ell}(1+p^{\gamma})^m + t + n(1+p^{\gamma})^{-m}.$$

By Lemma 4.5, there exists $\ell' \in \mathbb{Z}_k$ such that $e^{\ell'} \equiv e^{\ell}(1+p^{\gamma})^m \pmod{p^{\alpha}}$, that is, $e^{\ell'-\ell} = (1+p^{\gamma})^m \pmod{p^{\alpha}}$. Since e is an element of order k, we have $(1+p^{\gamma})^{mk} \equiv 1 \pmod{p^{\alpha}}$ and since $(mk,p)=1$, we have $p^{\gamma} \equiv 0 \pmod{p^{\alpha}}$, implying that $\gamma \geq \alpha$, which is impossible because $0 < \gamma < \alpha$.

Thus, $A_{1_0} = F = \langle \sigma_{\tau,a} \rangle \cong \mathbb{Z}_k$. Since A_{1_0} has two orbits on S_1, that is U_1 and V_1, Γ is not arc-transitive. To prove the half-arc-transitivity of Γ, we only need to show that A is transitive on $V(\Gamma)$ and $E(\Gamma)$. Note that $1_1 \in U_1$ and $(b^m a^n)_1 \in V_1$. By Proposition 4.1, it suffices to construct a $\lambda \in \mathrm{Aut}(G)$ such that

$$\delta_{\lambda,b^m a^n,1} \in I = \{\delta_{\lambda,x,y} \mid \lambda \in \mathrm{Aut}(G), S^{\lambda} = y^{-1}S^{-1}x\},$$

that is $S^{\lambda} = S^{-1}b^m a^n$, because $(1_0, 1_1)^{\delta_{\lambda,b^m a^n,1}} = ((b^m a^n)_1, 1_0)$.

Let $\mu = -(1+p^{\gamma})^m - n(e-1)$ and $\nu = -(e-1)^{-1}\mu^2 - (e-1)^{-1}\mu$. Then $\mu + 1 + n(e-1) \equiv 0 \pmod{p^{\gamma}}$ and hence

$$\begin{aligned}\nu - \mu n &= -(e-1)^{-1}\mu^2 - (e-1)^{-1}\mu - \mu n \\ &\equiv -(e-1)^{-1}\mu[\mu + 1 + n(e-1)] \equiv 0 \pmod{p^{\gamma}}.\end{aligned}$$

By Proposition 2.2, $o(b^m a^{\nu-\mu n}) = p^{\beta}$. Denote by m^{-1} the inverse of m in $\mathbb{Z}_{p^{\beta}}$. Then $(b^m a^{\nu-\mu n})^{m^{-1}} = ba^{\epsilon}$ for some ϵ in $\mathbb{Z}_{p^{\alpha}}$, and it is easy to check that a^{μ} and $(b^m a^{\nu-\mu n})^{m^{-1}}$ have the same relations as do a and b. Define λ as the automorphism of G induced by $a \mapsto a^{\mu}$, $b \mapsto (b^m a^{\nu-\mu n})^{m^{-1}}$. Clearly, $(b^m)^{\lambda} = b^m a^{\nu-\mu n}$.

Note that $S = U \cup V$. First we have

$$\begin{aligned}U^{\lambda} &= \{a^{\eta\mu} \mid \eta \in T\} = \{a^{\eta} \mid \eta \in T\mu\} \text{ and} \\ V^{-1}b^m a^n &= \{(b^m a^{\eta})^{-1}b^m a^n \mid \eta \in T'\} \\ &= \{a^{-\eta+n} \mid \eta \in T'\} = \{a^{\eta} \mid \eta \in -T' + n\}.\end{aligned}$$

Recall that $T' = T[(1+p^\gamma)^m + n(e-1)] + n = -T\mu + n$. Then

$$-T' + n = T\mu - n + n = T\mu,$$

and so $U^\lambda = V^{-1}b^m a^n$.

On the other hand,

$$\begin{aligned} V^\lambda &= \{(b^m a^\eta)^\lambda \mid \eta \in T'\} = \{b^m a^{\nu-\mu n} a^{\eta\mu} \mid \eta \in T'\} \\ &= \{b^m a^\eta \mid \eta \in T'\mu - \mu n + \nu\} \text{ and} \\ U^{-1}b^m a^n &= \{(a^\eta)^{-1} b^m a^n \mid \eta \in T\} = \{b^m a^{-\eta(1+p^\gamma)^m + n} \mid \eta \in T\} \\ &= \{b^m a^\eta \mid \eta \in -T(1+p^\gamma)^m + n\}. \end{aligned}$$

To prove $V^\lambda = U^{-1}b^m a^n$, we only need to show $T'\mu - \mu n + \nu \equiv -T(1+p^\gamma)^m + n$ in $\mathbb{Z}_{p^\alpha}$, which is equivalent to show that $T(1+p^\gamma)^m = T\mu^2 - \nu + n$ because $T' = -T\mu + n$. By Equation (4.2),

$$e^\ell (1+p^\gamma)^m = [(1+p^\gamma)^m - n(1-e)]^2 = \mu^2,$$

and by Lemma 4.5, $T = Te^\ell + (e-1)^{-1}(e^\ell - 1)$. It follows

$$\begin{aligned} T(1+p^\gamma)^m &= Te^\ell(1+p^\gamma)^m + (e-1)^{-1}(e^\ell-1)(1+p^\gamma)^m \\ &= T\mu^2 + (e-1)^{-1}[\mu^2 - (1+p^\gamma)^m]. \end{aligned}$$

Note that

$$\begin{aligned} -\nu + n &= (e-1)^{-1}\mu^2 + (e-1)^{-1}\mu + n \\ &= (e-1)^{-1}[\mu^2 + \mu + n(e-1)] = (e-1)^{-1}[\mu^2 - (1+p^\gamma)^m]. \end{aligned}$$

Then $T(1+p^\gamma)^m = T\mu^2 - \nu + n$, and hence $V^\lambda = U^{-1}b^m a^n$.

Thus, $S^\lambda = U^\lambda \cup V^\lambda = V^{-1}b^m a^n \cup U^{-1}b^m a^n = S^{-1}b^m a^n$, and so Γ is half-arc-transitive.

Let A^* be the subgroup of A fixing the two parts of Γ setwise. Then $A = A^*.\mathbb{Z}_2$. Since $A_{1_0} \cong \mathbb{Z}_k$ and Γ is normal, we have $A^* \cong G \rtimes \mathbb{Z}_k$ and hence $A \cong (G \rtimes \mathbb{Z}_k).\mathbb{Z}_2$. □

Now we prove the necessity of Theorem 1.3.

Lemma 4.8. *For an odd prime p, let Γ be a connected bipartite half-arc-transitive bi-p-metacirculant of valency $2k$ ($k < p$) over G. Then $k \geq 2$, $k \mid (p-1)$, $G \cong G_{\alpha,\beta,\gamma}$ and $\Gamma \cong \Gamma^{\pm}_{m,k,\ell}$, where $m \in \mathbb{Z}^*_{p^{\alpha-\gamma}}$ and $0 \leq \ell < k$ with $\frac{k}{(k,\ell)} \mid \frac{(p-1)}{2}$.*

Proof. Clearly, the two orbits of G are exactly the two parts of Γ. Then we may assume that $\Gamma = \mathrm{BiCay}(G, \emptyset, \emptyset, S)$, where $1 \in S$, $|S| < 2p$ and $G = \langle S \rangle$. Let $A = \mathrm{Aut}(\Gamma)$.

Since Γ is half-arc-transitive, Γ has valency at least 4, that is, $k \geq 2$, and A_{1_0} has exactly two orbits on $S_1 = \{s_1 \mid s \in S\}$, say U_1 and V_1 with $1_1 \in U_1$, where U and V are subsets of G with $1 \in U$. Then $S = U \cup V$, $|U| = |V| = k \geq 2$ and $|S| = 2k$. Since $k < p$, the Orbit-Stabilizer theorem implies that A_{1_0} is a p'-group. By Theorem 1.1, $\hat{G} \trianglelefteq A$, and by Proposition 4.1, $A_{1_0} = F = \{\sigma_{\theta,s} \mid \theta \in \mathrm{Aut}(G), s \in S, S^\theta = s^{-1}S\}$. By Lemma 4.2, A_{1_0} is faithful on S_1, and $F \cong L := \{\sigma \mid \sigma_{\theta,s} \in F\}$.

Suppose that G is non-split. By Lindenberg [36], the automorphism group of G is a p-group. Thus, $p \mid |L|$ and hence $p \mid |A_{1_0}|$, a contradiction.

Thus, G is split, namely $G = G_{\alpha,\beta,\gamma}$, as defined in Equation (2.1). By Proposition 2.1, F is a cyclic subgroup of $\mathbb{Z}_{p-1}$, and hence $F = \langle \sigma_{\theta,s} \rangle$ for some $\theta \in \mathrm{Aut}(G)$ and $s \in S$ with $S^\theta = s^{-1}S$. Then $\sigma_{\theta,s}$ has order k and $\langle \sigma_{\theta,s} \rangle$ is regular on both U_1 and V_1. Furthermore, $A_{1_0} = F = \langle \sigma_{\theta,s} \rangle \cong \mathbb{Z}_k$,

$$U_1 = 1_1^{\langle \sigma_{\theta,s} \rangle} = \{1_1, s_1, (ss^\theta)_1, \ldots, (ss^\theta \cdots s^{\theta^{k-2}})_1\}, \text{ and}$$
$$V_1 = t_1^{\langle \sigma_{\theta,s} \rangle} = \{t_1, (st^\theta)_1, (ss^\theta t^{\theta^2})_1, \ldots, (ss^\theta \cdots s^{\theta^{k-2}} t^{\theta^{k-1}})_1\}$$

with $(ss^\theta \cdots s^{\theta^{k-1}})_1 = 1_1$ for any $t \in V$. It follows

$$U = \{1, s, ss^\theta, \ldots, ss^\theta \cdots s^{\theta^{k-2}}\} \text{ and}$$
$$V = \{t, st^\theta, ss^\theta t^{\theta^2}, \ldots, ss^\theta \cdots s^{\theta^{k-2}} t^{\theta^{k-1}}\}.$$

In particular, $k \mid (p-1)$, θ has order k, and

$$ss^\theta \cdots s^{\theta^{k-1}} = 1. \tag{4.4}$$

By Proposition 2.1, we may assume that θ is the automorphism induced by $a \mapsto a^e$, $b \mapsto b$, where $e \in \mathbb{Z}_{p^\alpha}^*$ has order k.

Let $s = b^i a^j$ and $t = b^m a^n$ with $i, m \in \mathbb{Z}_{p^\beta}$ and $j, n \in \mathbb{Z}_{p^\alpha}$. Since $s^\theta = b^i a^{ej}$, we have $ss^\theta \cdots s^{\theta^{k-1}} = b^{ki} a^\epsilon$ for some $\epsilon \in \mathbb{Z}_{p^\alpha}$. By Equation (4.4), $b^{ki} = 1$, that is, $ki \equiv 0 \pmod{p^\beta}$. Since $k < p$, we have $i \equiv 0 \pmod{p^\beta}$, and hence $s = a^j$. Since

$$ss^\theta \cdots s^{\theta^{i-1}} = a^j a^{je} \cdots a^{je^{i-1}} = a^{j(e-1)^{-1}(e^i-1)},$$

we have

$$U = \{1, a^j, a^j a^{je}, \ldots, a^j a^{je} \cdots a^{je^{k-2}}\} = \{a^{j(e-1)^{-1}(e^i-1)} \mid i \in \mathbb{Z}_k\}.$$

By Lemma 4.3,

$$a^{j(e-1)^{-1}(e^i-1)} b^m = b^m a^{j(e-1)^{-1}(e^i-1)(1+p^\gamma)^m},$$

and since $(b^m a^n)^{\theta^i} = b^m a^{e^i n}$, we have

$$V = \{a^{j(e-1)^{-1}(e^i-1)} (b^m a^n)^{\theta^i} \mid i \in \mathbb{Z}_k\}$$
$$= \{b^m a^{j(e-1)^{-1}(e^i-1)(1+p^\gamma)^m + e^i n} \mid i \in \mathbb{Z}_k\}.$$

By the connectedness of Γ, $G = \langle S \rangle = \langle U \cup V \rangle \le \langle a^j, a^n, b^m \rangle$, forcing $G = \langle a^j, a^n, b^m \rangle$. It follows that $p \nmid m$ and so $m \in \mathbb{Z}_{p^\beta}^*$.

Since Γ is half-arc-transitive, Proposition 4.1 implies that there exists $\delta_{\lambda,x,y} \in I$ such that $(1_0, 1_1)^{\delta_{\lambda,x,y}} = ((b^m a^n)_1, 1_0)$ with $\lambda \in \mathrm{Aut}(G)$ and $S^\lambda = y^{-1} S^{-1} x$. In particular, $(b^m a^n)_1 = 1_0^{\delta_{\lambda,x,y}} = x_1$ and $1_0 = 1_1^{\delta_{\lambda,x,y}} = y_0$. It follows that $x = b^m a^n$, $y = 1$ and $S^\lambda = S^{-1} b^m a^n = U^{-1} b^m a^n \cup V^{-1} b^m a^n$. Furthermore,

$$U^{-1} b^m a^n = \{a^{-j(e-1)^{-1}(e^i-1)} b^m a^n \mid i \in \mathbb{Z}_k\}$$
$$= \{b^m a^{-j(e-1)^{-1}(e^i-1)(1+p^\gamma)^m + n} \mid i \in \mathbb{Z}_k\},$$

and since

$$(b^m a^{j(e-1)^{-1}(e^i-1)(1+p^\gamma)^m+e^i n})^{-1} b^m a^n = a^{-j(e-1)^{-1}(e^i-1)(1+p^\gamma)^m+n(1-e^i)},$$

we have

$$V^{-1}b^m a^n = \{a^{-j(e-1)^{-1}(e^i-1)(1+p^\gamma)^m+n(1-e^i)} \mid i \in \mathbb{Z}_k\}.$$

Suppose $p \mid j$. Since $G = \langle a^j, a^n, b^m \rangle$, we have $p \nmid n$ and $p \nmid m$. By Proposition 2.2, every element in both V and $U^{-1}b^m a^n$ has order $\max\{p^\alpha, p^\beta\}$. Clearly, every element in U has order less than p^α, but the element $a^{-j(1+p^\gamma)^m+n(1-e)} \in V^{-1}b^m a^n$ has order p^α because $p \nmid (1-e)$ by Lemma 4.5. This is impossible as $\lambda \in \mathrm{Aut}(G)$ and $(U \cup V)^\lambda = S^\lambda = S^{-1}b^m a^n = U^{-1}b^m a^n \cup V^{-1}b^m a^n$. Thus, $p \nmid j$.

Now, there is an automorphism of G mapping a^j to a and b to b, and so we may assume $j = 1$ and $s = a$. It follows that

$$U = \{a^\eta \mid \eta \in T\}, \tag{4.5}$$

where $T = \{(e-1)^{-1}(e^i - 1) \mid i \in \mathbb{Z}_k\}$;

$$V = \{b^m a^\eta \mid \eta \in T'\}, \tag{4.6}$$

where $T' = \{(e-1)^{-1}(e^i-1)(1+p^\gamma)^m + e^i n \mid i \in \mathbb{Z}_k\}$.

As

$$\begin{aligned}(e-1)^{-1}&(e^i-1)(1+p^\gamma)^m + e^i n \\ &= [(e-1)^{-1}(e^i-1)][(1+p^\gamma)^m + n(e-1)] + n,\end{aligned}$$

we have

$$T' = T[(1+p^\gamma)^m + n(e-1)] + n. \tag{4.7}$$

Since

$$-(e-1)^{-1}(e^i-1)(1+p^\gamma)^m + n = [(e-1)^{-1}(e^i-1)][-(1+p^\gamma)^m] + n$$

and

$$\begin{aligned}-\,(e-1)^{-1}&(e^i-1)(1+p^\gamma)^m + n(1-e^i) \\ &= [(e-1)^{-1}(e^i-1)][-(1+p^\gamma)^m + n(1-e)],\end{aligned}$$

we have

$$U^{-1}b^m a^n = \{b^m a^\eta \mid \eta \in T_1\}, \tag{4.8}$$

where $T_1 = T[-(1+p^\gamma)^m] + n$;

$$V^{-1}b^m a^n = \{a^\eta \mid \eta \in T_1'\}, \tag{4.9}$$

where $T_1' = T[-(1+p^\gamma)^m + n(1-e)]$.

Noting that $T, T', T_1, T_1' \subseteq \mathbb{Z}_{p^\alpha}$, we have $U, V, U^{-1}b^m a^n, V^{-1}b^m a^n \subseteq G$.

Claim 1. $a^\lambda \in V^{-1}b^m a^n$.

Suppose to the contrary that $a^\lambda \notin V^{-1}b^m a^n$. Since $a^\lambda \in S^\lambda = U^{-1}b^m a^n \cup V^{-1}b^m a^n$, we have $a^\lambda \in U^{-1}b^m a^n$, that is, $a^\lambda = b^m a^\mu$ for $\mu \in T_1$. By Lemma 4.3, $\beta < \alpha$. Recall $k \geq 2$.

Let $k > 2$. Then $a^{1+e} \in U$ and $(a^{1+e})^\lambda = (b^m a^\mu)^{1+e} \in U^{-1}b^m a^n$. Note that $p \nmid m$ and by Lemma 4.5, $p \nmid (1+e)$. Considering the power of b of $(b^m a^\mu)^{1+e}$ and elements in $U^{-1}b^m a^n$, we have $m(1+e) \equiv m \pmod{p^\beta}$ and so $e \equiv 0 \pmod{p^\beta}$, contradicting $e \in \mathbb{Z}_{p^\alpha}^*$.

Let $k = 2$. Then $T = \{0, 1\}$ and $e \equiv -1 \pmod{p^\alpha}$. By Equations (4.5) and (4.6),

$$S = \{1, a, b^m a^n, b^m a^{(1+p^\gamma)^m - n}\},$$

and by Equations (4.8) and (4.9),

$$S^{-1}b^m a^n = \{1, a^{-(1+p^\gamma)^m + 2n}, b^m a^n, b^m a^{-(1+p^\gamma)^m + n}\}.$$

Note that $a^\lambda \in U^{-1}b^m a^n = \{b^m a^n, b^m a^{-(1+p^\gamma)^m + n}\}$.

Case 1. $a^\lambda = b^m a^n$.

As $S^\lambda = S^{-1}b^m a^n$, it is easy to see that

$$\begin{aligned}&((b^m a^n)^\lambda, (b^m a^{(1+p^\gamma)^m - n})^\lambda)\\ &\quad = (a^{-(1+p^\gamma)^m + 2n}, b^m a^{-(1+p^\gamma)^m + n}) \text{ or } (b^m a^{-(1+p^\gamma)^m + n}, a^{-(1+p^\gamma)^m + 2n}).\end{aligned}$$

For the former,

$$\begin{aligned} b^m a^{-(1+p^\gamma)^m + n} &= (b^m a^{(1+p^\gamma)^m - n})^\lambda = [(b^m a^n) a^{(1+p^\gamma)^m - 2n}]^\lambda \\ &= a^{-(1+p^\gamma)^m + 2n} (b^m a^n)^{(1+p^\gamma)^m - 2n}, \end{aligned}$$

implying that $m \equiv m[(1+p^\gamma)^m - 2n] \pmod{p^\beta}$, and since $p \nmid m$, we have $p \mid n$. This is impossible because otherwise $p^\alpha = o(a^\lambda) = o(b^m a^n) < p^\alpha$ ($\beta < \alpha$). For the latter, we can verify that

$$\begin{aligned} a^{-(1+p^\gamma)^m + 2n} &= (b^m a^{(1+p^\gamma)^m - n})^\lambda = [(b^m a^n) a^{(1+p^\gamma)^m - 2n}]^\lambda \\ &= b^m a^{-(1+p^\gamma)^m + n} (b^m a^n)^{(1+p^\gamma)^m - 2n}. \end{aligned}$$

Thus, $0 \equiv m + m[(1+p^\gamma)^m - 2n] \pmod{p^\beta}$, and hence $p \mid (1-n)$, but it is also impossible because otherwise $p^\alpha = o(a^\lambda) = o(b^m a^n) = o((b^m a^n)^\lambda) = o(b^m a^{-(1+p^\gamma)^m + n}) < p^\alpha$.

Case 2. $a^\lambda = b^m a^{-(1+p^\gamma)^m + n}$.

In this case, we have

$$\begin{aligned}&((b^m a^n)^\lambda, (b^m a^{(1+p^\gamma)^m - n})^\lambda)\\ &\qquad = (a^{-(1+p^\gamma)^m + 2n}, b^m a^n) \text{ or } (b^m a^n, a^{-(1+p^\gamma)^m + 2n}).\end{aligned}$$

For the former,

$$\begin{aligned} b^m a^n &= (b^m a^{(1+p^\gamma)^m - n})^\lambda = [(b^m a^n) a^{(1+p^\gamma)^m - 2n}]^\lambda \\ &= a^{-(1+p^\gamma)^m + 2n} (b^m a^{-(1+p^\gamma)^m + n})^{(1+p^\gamma)^m - 2n}, \end{aligned}$$

implying $m \equiv m[(1+p^\gamma)^m - 2n] \pmod{p^\beta}$, and since $p \nmid m$, we have $p \mid n$. By Proposition 2.2,

$$\begin{aligned} o(b^m a^{-(1+p^\gamma)^m+n}) &= o(b^m a^{(1+p^\gamma)^m-n}) \\ &= \max\{o(a^{(1+p^\gamma)^m-n}) = o(a^{-(1+p^\gamma)^m+n}), o(b^m)\}, \end{aligned}$$

and it follows that

$$\begin{aligned} p^\alpha = o(a^\lambda) &= o(b^m a^{-(1+p^\gamma)^m+n}) = o(b^m a^{(1+p^\gamma)^m-n}) \\ &= o((b^m a^{(1+p^\gamma)^m-n})^\lambda) = o(b^m a^n) < p^\alpha, \end{aligned}$$

a contradiction. For the latter,

$$\begin{aligned} a^{-(1+p^\gamma)^m+2n} &= (b^m a^{(1+p^\gamma)^m-n})^\lambda = [(b^m a^n) a^{(1+p^\gamma)^m-2n}]^\lambda \\ &= b^m a^n (b^m a^{-(1+p^\gamma)^m+n})^{(1+p^\gamma)^m-2n}. \end{aligned}$$

Thus, $0 \equiv m + m[(1+p^\gamma)^m - 2n] \pmod{p^\beta}$, and hence $p \mid (1-n)$, but it is also impossible because otherwise $p^\alpha = o(a^\lambda) = o(b^m a^{-(1+p^\gamma)^m+n}) < p^\alpha$. This completes the proof of Claim 1.

By Claim 1, $a^\lambda = a^\mu \in V^{-1} b^m a^n$ for some $\mu \in T_1'$. Since $p^\alpha = o(a^\lambda) = o(a^\mu)$, we have $\mu \in \mathbb{Z}_{p^\alpha}^*$. By Equations (4.5) and (4.9),

$$U^\lambda = \{a^{\eta\mu} \mid \eta \in T\} = \{a^\eta \mid \eta \in T\mu\} \subseteq V^{-1} b^m a^n.$$

Then $U^\lambda = V^{-1} b^m a^n = \{a^\eta \mid \eta \in T_1'\}$, and so $T\mu = T_1'$ in $\mathbb{Z}_{p^\alpha}$. By Equation (4.9), $T\mu = T[-(1+p^\gamma)^m + n(1-e)]$. Since $p \nmid \mu$, we have $T = T[-(1+p^\gamma)^m + n(1-e)]\mu^{-1}$. By Lemma 4.5, $\mu = -(1+p^\gamma)^m + n(1-e)$.

Since $S^\lambda = S^{-1} b^m a^n = U^{-1} b^m a^n \cup V^{-1} b^m a^n$, we have $V^\lambda = U^{-1} b^m a^n$. In particular, $(b^m a^n)^\lambda = b^m a^\nu$ for some $\nu \in T_1$. For $\eta \in T'$, since

$$(b^m a^\eta)^\lambda = [(b^m a^n) a^{\eta-n}]^\lambda = b^m a^\nu a^{\eta\mu-\mu n} = b^m a^{\eta\mu-\mu n+\nu},$$

we have that

$$\begin{aligned} \{b^m a^\eta \mid \eta \in T_1\} &= U^{-1} b^m a^n = V^\lambda = \{(b^m a^\eta)^\lambda \mid \eta \in T'\} \\ &= \{b^m a^{\eta\mu-\mu n+\nu} \mid \eta \in T'\} = \{b^m a^\eta \mid \eta \in T'\mu - \mu n + \nu\}. \end{aligned}$$

By Equations (4.7) and (4.8),

$$\begin{aligned} T[-(1+p^\gamma)^m] + n &= T_1 = T'\mu - \mu n + \nu \\ &= T[(1+p^\gamma)^m + n(e-1)]\mu + \mu n - \mu n + \nu \end{aligned}$$

in $\mathbb{Z}_{p^\alpha}$. Thus, $T[(1+p^\gamma)^m - n(1-e)]^2 (1+p^\gamma)^{-m} - (\nu - n)(1+p^\gamma)^{-m} = T$. By Lemma 4.5, there exists $\ell \in \mathbb{Z}_k$ such that $e^\ell = [(1+p^\gamma)^m - n(1-e)]^2 (1+p^\gamma)^{-m}$, that is, n satisfies Equation (4.2).

Recall that $\alpha - \gamma \leq \beta$ and $m \in \mathbb{Z}_{p^\beta}^*$. Let $m = m_1 + lp^{\alpha-\gamma}$ with $m_1 \in \mathbb{Z}_{p^{\alpha-\gamma}}^*$. Since $(1+p^\gamma)$ has order $p^{\alpha-\gamma}$ in $\mathbb{Z}_{p^\alpha}^*$, we have

$$(1+p^\gamma)^m = (1+p^\gamma)^{m_1+lp^{\alpha-\gamma}} = (1+p^\gamma)^{m_1}.$$

This implies that replacing m by m_1, Equation (4.2) has the same solutions, and

$$T' = \{(e-1)^{-1}(e^i-1)(1+p^\gamma)^{m_1} + e^i n \mid i \in \mathbb{Z}_k\} \subseteq \mathbb{Z}_{p^\alpha}.$$

The automorphism of G induced by $a \mapsto a$ and $b \mapsto b^{m_1 m^{-1}}$, maps U to U, and $V = \{b^m a^\eta \mid \eta \in T'\}$ to $\{b^{m_1} a^\eta \mid \eta \in T'\}$. Thus, we may assume that $m \in \mathbb{Z}^*_{p^{\alpha-\gamma}}$, and therefore, $\Gamma \cong \Gamma^n_{m,k,l}$. □

Proof of Theorem 1.3. This is a consequence of Lemmas 4.7 and 4.8. □

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