

Sums of r -Lah numbers and r -Lah polynomials

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Abstract

The total number of partitions of a finite set into nonempty ordered subsets such that r distinguished elements belong to distinct ordered blocks can be described as sums of r -Lah numbers. In this paper we study this possible variant of Bell-like numbers, as well as the related r -Lah polynomials.

Keywords: Summed r -Lah numbers, r -Lah polynomials.

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1 Introduction

Bell numbers play a crucial role in enumerative combinatorics. The n th Bell number B_n counts the number of partitions of an n -element set, or in other words, it is the sum of Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ ($k = 0, \dots, n$). In connection with these numbers, it is possible to introduce the n th Bell polynomial

$$B_n(x) = \sum_{j=0}^n \left\{ \begin{smallmatrix} n \\ j \end{smallmatrix} \right\} x^j,$$

whose value at 1 is simply $B_n(1) = B_n$. (These polynomials should not be confused with partial Bell polynomials which are multivariate polynomials.)

Using r -Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r$, defined by L. Carlitz [5], A. Z. Broder [4], and later rediscovered by R. Merris [12], I. Mező [13, 14] introduced and investigated the corresponding r -Bell numbers $B_{n,r}$ as the number of partitions of a set with $n + r$

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elements such that r distinguished elements belong to distinct blocks, and the r -Bell polynomials as

$$B_{n,r}(x) = \sum_{j=0}^n \left\{ \begin{matrix} n \\ j \end{matrix} \right\}_r x^j.$$

(We have to mention that there is some confusion in notation of r -Stirling numbers in the literature, therefore we need to underline that for various reasons, we prefer to denote by $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r$ the number of partitions of an $(n + r)$ -element set into $k + r$ nonempty subsets such that r distinguished elements belong to distinct blocks.) The r -Bell numbers were studied from a graph theoretical point of view by Zs. Kereskényi-Balogh and G. Nyul [9]. We shall discuss these numbers and polynomials in detail in Section 2.

Lah numbers $\lfloor \begin{matrix} n \\ k \end{matrix} \rfloor$, named after I. Lah [10, 11], are close relatives of Stirling numbers. Sometimes they are called Stirling numbers of the third kind. G. Nyul and G. Rác [19] defined and extensively studied the r -generalization of Lah numbers. The r -Lah number $\lfloor \begin{matrix} n \\ k \end{matrix} \rfloor_r$ is the number of partitions of a set with $n + r$ elements into $k + r$ nonempty ordered subsets such that r distinguished elements have to be in distinct ordered blocks. We notice that some identities for r -Lah numbers were derived by H. Belbachir, A. Belkhir [1] and H. Belbachir, E. Bousbaa [2], and they appear as the results of substitutions into partial r -Bell polynomials by M. Mihoubi and M. Rahmani [17]. The r -Lah numbers are special cases of r -Whitney-Lah numbers defined by G.-S. Cheon and J.-H. Jung [6] (see also [8]), and recently M. Shattuck [21] introduced a further generalization of these numbers.

Similarly to Bell numbers, one could be interested in summation

$$L_n = \sum_{j=0}^n \lfloor \begin{matrix} n \\ j \end{matrix} \rfloor$$

of Lah numbers. Although these numbers slightly appear in the literature [7, 18, 20, 22], they have not been studied systematically yet. This will be done in our paper at a more general level, namely we shall prove several properties of sums $L_{n,r}$ of r -Lah numbers and r -Lah polynomials $L_{n,r}(x)$, for instance, we express summed r -Lah numbers by sums of $(r - s)$ -Lah numbers, we derive Spivey and Dobiński type identities, second-order linear recurrence relations, exponential generating functions. Finally, we show that r -Lah polynomials have only real roots. We prefer purely combinatorial arguments in the proofs where it is possible. As we shall see, some of these results could be viewed as the summed or polynomial counterparts of certain theorems from [19]. They are also included in this paper, because we aim to give a self-contained presentation of these numbers and polynomials.

2 r -Bell numbers and r -Bell polynomials

Above, we have defined r -Bell numbers and r -Bell polynomials. In the following table we collect their properties, especially those ones which correspond to our theorems about summed r -Lah numbers and r -Lah polynomials. We indicate the references for the known identities (star symbol means that a certain paper contains the formula only for r -Bell numbers, not for polynomials), but it also contains some new results. For example, to the best of our knowledge, the Spivey type identity never appeared previously in this full generality. All of these properties can be proved along the lines of our proofs in the next section. We notice that these proofs are based on a completely new idea even for several known identities of the table. We should draw attention to that our purely combinatorial

argument will fail to work in the most general case (Theorem 3.3) for r -Lah polynomials, but even so, it works for r -Bell numbers and polynomials.

Table 1: Properties of r -Bell numbers and r -Bell polynomials.

$B_{n,0}(x) = B_n(x)$ [14], $xB_{n,1}(x) = B_{n+1}(x)$
$B_{n,r}(x) = \sum_{j=0}^n \binom{n}{j} B_{j,r-s}(x) s^{n-j}$ [16]
$B_{n,r}(x) = \sum_{j=0}^n \binom{n}{j} B_{j,r-1}(x)$ [14]
$B_{n,r}(x) = \sum_{j=0}^n \binom{n}{j} B_j(x) r^{n-j}$ [5]*, [14]
$B_{m+n,r}(x) = \sum_{i=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ i \end{matrix} \right\}_r \binom{n}{j} B_{j,r-s}(x) (i+s)^{n-j} x^i$
$B_{m+n,r}(x) = \sum_{i=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ i \end{matrix} \right\}_r \binom{n}{j} B_{j,r}(x) i^{n-j} x^i$ [16]
$B_{m+n,r}(x) = \sum_{i=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ i \end{matrix} \right\}_r \binom{n}{j} B_{j,r-1}(x) (i+1)^{n-j} x^i$
$B_{m+n,r}(x) = \sum_{i=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ i \end{matrix} \right\}_r \binom{n}{j} B_j(x) (i+r)^{n-j} x^i$ [15]*, [16]
$B_{n,r}(x) = \frac{1}{\exp(x)} \sum_{j=0}^{\infty} \frac{(j+r)^n}{j!} x^j$ [14]
$\sum_{n=0}^{\infty} \frac{B_{n,r}(x)}{n!} y^n = \exp(x(\exp(y) - 1) + ry)$ [5]*, [14]
The roots of $B_{n,r}(x)$ are simple, real and negative ($r \geq 1$). [13]

3 Summed r -Lah numbers and r -Lah polynomials

We begin this section with the exact definitions of summed r -Lah numbers and r -Lah polynomials, which can be viewed as relatives of r -Bell numbers and polynomials (in the sense that r -Lah numbers are relatives of r -Stirling numbers of the second kind).

For non-negative integers n, r , not both 0, denote by $L_{n,r}$ the number of partitions of a set with $n + r$ elements into nonempty ordered subsets such that r distinguished elements belong to distinct ordered blocks. Moreover, let $L_{0,0} = 1$. We can call $L_{n,r}$ the n th summed r -Lah number, because the formula

$$L_{n,r} = \sum_{j=0}^n \left[\begin{matrix} n \\ j \end{matrix} \right]_r$$

immediately follows from the definitions. This suggests us to define the polynomial analogues of these numbers. If $n, r \geq 0$, then the n th r -Lah polynomial is

$$L_{n,r}(x) = \sum_{j=0}^n \left[\begin{matrix} n \\ j \end{matrix} \right]_r x^j.$$

If we have no distinguished elements, then the summands in the first formula and the coefficients of the polynomial are the ordinary Lah numbers. In this case, we simply call them the n th summed Lah number and Lah polynomial, and denote them by L_n and $L_n(x)$.

Obviously, $L_{n,r}(x)$ is a monic polynomial of degree n with non-negative integer coefficients. Since $L_{n,r}(1) = L_{n,r}$, it is enough to state our theorems for r -Lah polynomials throughout this paper, the corresponding properties for summed r -Lah numbers follows simply by the substitution $x = 1$.

It will be useful to associate a combinatorial interpretation to r -Lah polynomials, as well. If $n, r \geq 0$, not both 0, and $c \geq 1$, then $L_{n,r}(c)$ counts the number of partitions of a set with $n + r$ elements into nonempty ordered subsets and colourings of the blocks with c colours such that r distinguished elements belong to distinct uncoloured ordered blocks. For brevity, in the rest of the paper we shall call these objects c -coloured r -Lah partitions of an $(n + r)$ -element set into ordered blocks.

If $r = 0$ or $r = 1$, then we have no restriction for the partition into ordered blocks, hence $L_{n,0}(x) = L_n(x)$ and $xL_{n,1}(x) = L_{n+1}(x)$ ($n \geq 0$).

In our first theorem, we express r -Lah polynomials in terms of $(r - s)$ -Lah polynomials. It is the polynomial counterpart and could be derived directly from [19, Theorem 3.4], but we carry out the necessary modification of the combinatorial proof.

Theorem 3.1. *If $n, r, s \geq 0$ and $s \leq r$, then*

$$L_{n,r}(x) = \sum_{j=0}^n \binom{n}{j} L_{j,r-s}(x) (2s)^{\overline{n-j}}.$$

Proof. We may assume that n, r are not both 0, and let c be a positive integer. Then, $L_{n,r}(c)$ is the number of c -coloured r -Lah partitions of an $(n + r)$ -element set into ordered blocks. These can be enumerated in another way:

Let j be the number of those non-distinguished elements which belong to other ordered blocks than the first s distinguished elements ($j = 0, \dots, n$). We can choose them in $\binom{n}{j}$ ways, thereafter we have $L_{j,r-s}(c)$ possibilities for their c -coloured $(r - s)$ -Lah partitions into ordered blocks together with the last $r - s$ distinguished elements. Finally, we can put the remaining $n - j$ non-distinguished elements into the ordered blocks of the first s distinguished elements in $(2s)^{\overline{n-j}}$ ways. It means that, for a fixed j , the number of possibilities is $\binom{n}{j} L_{j,r-s}(c) (2s)^{\overline{n-j}}$. \square

Remark 3.2. For the most important choices $s = 1$ and $s = r$, the identity becomes

$$L_{n,r}(x) = \sum_{j=0}^n \binom{n}{j} L_{j,r-1}(x) (n - j + 1)!,$$

$$L_{n,r}(x) = \sum_{j=0}^n \binom{n}{j} L_j(x) (2r)^{\overline{n-j}}.$$

Now, we prove a general Spivey type formula for r -Lah polynomials. It is named after M. Z. Spivey [23], who discovered his remarkable formula for Bell numbers just over a decade ago.

Theorem 3.3. *If $m, n, r, s \geq 0$ and $s \leq r$, then*

$$L_{m+n,r}(x) = \sum_{i=0}^m \sum_{j=0}^n \left[\begin{matrix} m \\ i \end{matrix} \right]_r \binom{n}{j} L_{j,r-s}(x) (m+i+2s)^{\overline{n-j}} x^i.$$

Proof. By [19, Theorem 3.2], we get

$$(x+2r)^{\overline{m+n}} = \sum_{k=0}^{m+n} \left[\begin{matrix} m+n \\ k \end{matrix} \right]_r x^k.$$

On the other hand, using again [19, Theorem 3.2] and the binomial theorem for rising factorials, we also have

$$\begin{aligned} (x+2r)^{\overline{m+n}} &= (x+2r)^{\overline{m}}(x+2r+m)^{\overline{n}} \\ &= \sum_{i=0}^m \left[\begin{matrix} m \\ i \end{matrix} \right]_r x^i (x-i+2r-2s+m+i+2s)^{\overline{n}} \\ &= \sum_{i=0}^m \left[\begin{matrix} m \\ i \end{matrix} \right]_r x^i \sum_{j=0}^n \binom{n}{j} (x-i+2r-2s)^{\overline{j}} (m+i+2s)^{\overline{n-j}} \\ &= \sum_{i=0}^m \sum_{j=0}^n \left[\begin{matrix} m \\ i \end{matrix} \right]_r x^i \binom{n}{j} (m+i+2s)^{\overline{n-j}} \sum_{k=0}^j \left[\begin{matrix} j \\ k \end{matrix} \right]_{r-s} (x-i)^k \\ &= \sum_{i=0}^m \sum_{j=0}^n \sum_{k=0}^j \left[\begin{matrix} m \\ i \end{matrix} \right]_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \left[\begin{matrix} j \\ k \end{matrix} \right]_{r-s} x^{i+k} \\ &= \sum_{i=0}^m \sum_{j=0}^n \sum_{k=i}^{i+j} \left[\begin{matrix} m \\ i \end{matrix} \right]_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \left[\begin{matrix} j \\ k-i \end{matrix} \right]_{r-s} x^k \\ &= \sum_{k=0}^{m+n} \sum_{i=0}^{\min\{m,k\}} \sum_{j=\max\{0,k-i\}}^n \left[\begin{matrix} m \\ i \end{matrix} \right]_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \left[\begin{matrix} j \\ k-i \end{matrix} \right]_{r-s} x^k. \end{aligned}$$

Comparing the coefficients of x^k in the above two expressions gives

$$\left[\begin{matrix} m+n \\ k \end{matrix} \right]_r = \sum_{i=0}^{\min\{m,k\}} \sum_{j=\max\{0,k-i\}}^n \left[\begin{matrix} m \\ i \end{matrix} \right]_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \left[\begin{matrix} j \\ k-i \end{matrix} \right]_{r-s},$$

which identity is interesting on its own.

If we multiply both sides by x^k and sum for k ($k = 0, \dots, m+n$), we obtain

$$\begin{aligned} L_{m+n,r}(x) &= \sum_{k=0}^{m+n} \left[\begin{matrix} m+n \\ k \end{matrix} \right]_r x^k \\ &= \sum_{k=0}^{m+n} \sum_{i=0}^{\min\{m,k\}} \sum_{j=\max\{0,k-i\}}^n \left[\begin{matrix} m \\ i \end{matrix} \right]_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \left[\begin{matrix} j \\ k-i \end{matrix} \right]_{r-s} x^k \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=0}^m \sum_{j=0}^n \sum_{k=i}^{i+j} \begin{bmatrix} m \\ i \end{bmatrix}_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \begin{bmatrix} j \\ k-i \end{bmatrix}_{r-s} x^k \\
 &= \sum_{i=0}^m \sum_{j=0}^n \sum_{k=0}^j \begin{bmatrix} m \\ i \end{bmatrix}_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \begin{bmatrix} j \\ k \end{bmatrix}_{r-s} x^{i+k} \\
 &= \sum_{i=0}^m \sum_{j=0}^n \begin{bmatrix} m \\ i \end{bmatrix}_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} x^i L_{j,r-s}(x). \quad \square
 \end{aligned}$$

Remark 3.4. First, we note that this formula gives back Theorem 3.1 and the definition of r -Lah polynomials for $m = 0$ and $n = 0$, respectively.

While, in the special cases of $s = 0$, $s = 1$ and $s = r$, we have

$$\begin{aligned}
 L_{m+n,r}(x) &= \sum_{i=0}^m \sum_{j=0}^n \begin{bmatrix} m \\ i \end{bmatrix}_r \binom{n}{j} L_{j,r}(x) (m+i)^{\overline{n-j}} x^i, \\
 L_{m+n,r}(x) &= \sum_{i=0}^m \sum_{j=0}^n \begin{bmatrix} m \\ i \end{bmatrix}_r \binom{n}{j} L_{j,r-1}(x) (m+i+2)^{\overline{n-j}} x^i, \\
 L_{m+n,r}(x) &= \sum_{i=0}^m \sum_{j=0}^n \begin{bmatrix} m \\ i \end{bmatrix}_r \binom{n}{j} L_j(x) (m+i+2r)^{\overline{n-j}} x^i.
 \end{aligned}$$

For the last identity, we give a combinatorial proof, as well. The reason is that the extension of Spivey’s idea works for r -Lah polynomials only if $s = r$. However, as we mentioned previously, a similar argument proves the Spivey type formula listed in the table of Section 2 for r -Bell polynomials in full generality. It would be interesting to find a purely combinatorial proof of the general identity as stated in Theorem 3.3.

Proof. We may assume that m, n, r are not all 0, and let c be a positive integer. Then, $L_{m+n,r}(c)$ gives the number of c -coloured r -Lah partitions of an $(m+n+r)$ -element set into ordered blocks. We find an alternative way to count them:

First, we consider a c -coloured r -Lah partition of the distinguished elements and the first m non-distinguished elements into $i+r$ ordered blocks ($i = 0, \dots, m$). We have $\begin{bmatrix} m \\ i \end{bmatrix}_r c^i$ such partitions. Denote by j the number of those non-distinguished elements among the last n ones which do not belong to these $i+r$ ordered blocks ($j = 0, \dots, n$). They can be chosen in $\binom{n}{j}$ ways, and there are $L_j(c)$ possibilities to partition them into coloured ordered blocks with c colours. As our last step, we place the remaining $n-j$ non-distinguished elements into the $i+r$ original ordered blocks, which can be done in $(m+i+2r)^{\overline{n-j}}$ ways. Summarizing, the number of possibilities is

$$\begin{bmatrix} m \\ i \end{bmatrix}_r \binom{n}{j} L_j(c) (m+i+2r)^{\overline{n-j}} c^i$$

for a fixed pair of i, j . □

The r -Lah polynomials satisfy the following second-order linear recurrence relation. In the special case of sums of ordinary Lah numbers (i.e., for $r = 0$), it appears in [18, 20, 22] in different contexts.

Theorem 3.5. *If $n \geq 1$ and $r \geq 0$, then*

$$L_{n+1,r}(x) = (x + 2n + 2r)L_{n,r}(x) - n(n + 2r - 1)L_{n-1,r}(x).$$

Proof. Let c be a positive integer. Then, $L_{n+1,r}(c)$ counts the number of c -coloured r -Lah partitions of an $(n + r + 1)$ -element set into ordered blocks. The rest of the proof gives another enumeration of them:

We have $L_{n,r}(c)$ c -coloured r -Lah partitions of our set excluding the last non-distinguished element into ordered blocks. If this last element constitutes a singleton, then we only need to colour its one-element ordered block with c colours. Otherwise, we can place the excluded element before or after any other elements, i.e., to $2n + 2r$ places. It means that there would be $(c + 2n + 2r)L_{n,r}(c)$ possibilities.

But, of course, we counted twice those cases when our last element is put between two elements. This could happen in two different ways. If the j th non-distinguished element stands directly before the originally excluded element ($j = 1, \dots, n$), then there are $L_{n-1,r}(c)$ c -coloured r -Lah partitions of our set without these two elements into ordered blocks, and this pair of elements can be put back to $n + r - 1$ places (they cannot be at the end of an ordered block). If a distinguished element stands directly before and the j th non-distinguished element stands directly after the originally excluded element ($j = 1, \dots, n$), then we have $L_{n-1,r}(c)$ c -coloured r -Lah partitions of our set without the latter two elements into ordered blocks, and they can be put back to r places (directly after one of the distinguished elements). Therefore, the number of the possibilities to be subtracted is $(n(n + r - 1) + nr)L_{n-1,r}(c)$, altogether. \square

We can derive a Dobiński type formula for r -Lah polynomials, named after the well-known Dobiński formula for Bell numbers.

Theorem 3.6. *If $n, r \geq 0$, then*

$$L_{n,r}(x) = \frac{1}{\exp(x)} \sum_{j=0}^{\infty} \frac{(j + 2r)^{\overline{n}}}{j!} x^j.$$

Proof. I. First, we prove it for polynomials. Through this proof, let $\lfloor n \rfloor_r = 0$ if $i > n$. Applying [19, Theorem 3.2], we have

$$(j + 2r)^{\overline{n}} = \sum_{i=0}^n \lfloor n \rfloor_r^i j^i = \sum_{i=0}^{\infty} \lfloor n \rfloor_r^i j^i = \sum_{i=0}^j \lfloor n \rfloor_r^i \frac{j!}{(j-i)!}.$$

Dividing both sides by $j!$ gives

$$\frac{(j + 2r)^{\overline{n}}}{j!} = \sum_{i=0}^j \lfloor n \rfloor_r^i \frac{1}{(j-i)!},$$

which means that $\left(\frac{(j+2r)^{\overline{n}}}{j!}\right)_{j=0}^{\infty}$ is the convolution of the sequences $\left(\lfloor n \rfloor_r^i\right)_{j=0}^{\infty}$ and $\left(\frac{1}{j!}\right)_{j=0}^{\infty}$. Therefore, its generating function is

$$\sum_{j=0}^{\infty} \frac{(j + 2r)^{\overline{n}}}{j!} x^j = L_{n,r}(x) \exp(x).$$

II. Now, we can give another proof for summed r -Lah numbers using probability theory. Let λ be a positive real number and ξ a Poisson random variable with parameter λ . Then, again by [19, Theorem 3.2], we get

$$\begin{aligned} E(\xi + 2r)^{\bar{n}} &= \sum_{j=0}^{\infty} (j + 2r)^{\bar{n}} \frac{\lambda^j}{j!} e^{-\lambda} = e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_r j^i \\ &= e^{-\lambda} \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_r \sum_{j=0}^{\infty} \frac{j^i}{j!} \lambda^j = e^{-\lambda} \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_r \sum_{j=i}^{\infty} \frac{\lambda^j}{(j-i)!} \\ &= e^{-\lambda} \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_r \lambda^i \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_r \lambda^i = L_{n,r}(\lambda). \end{aligned}$$

Especially, for $\lambda = 1$, we have

$$L_{n,r} = L_{n,r}(1) = E(\xi + 2r)^{\bar{n}} = \sum_{j=0}^{\infty} (j + 2r)^{\bar{n}} \frac{1}{j!} e^{-1}. \quad \square$$

The next theorem gives the exponential generating function of the sequence of r -Lah polynomials. We note that a special case, the exponential generating function of $(L_n)_{n=0}^{\infty}$ can be found in [7, 18, 22].

Theorem 3.7. For $r \geq 0$, the exponential generating function of $(L_{n,r}(x))_{n=0}^{\infty}$ is

$$\sum_{n=0}^{\infty} \frac{L_{n,r}(x)}{n!} y^n = \exp\left(\frac{xy}{1-y}\right) \frac{1}{(1-y)^{2r}}.$$

Proof. I. We use [19, Theorem 3.10] to get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{L_{n,r}(x)}{n!} y^n &= \sum_{n=0}^{\infty} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_r x^j \frac{1}{n!} y^n = \sum_{j=0}^{\infty} x^j \sum_{n=j}^{\infty} \begin{bmatrix} n \\ j \end{bmatrix}_r \frac{1}{n!} y^n \\ &= \sum_{j=0}^{\infty} x^j \frac{1}{j!} \left(\frac{y}{1-y}\right)^j \frac{1}{(1-y)^{2r}} = \frac{1}{(1-y)^{2r}} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{xy}{1-y}\right)^j \\ &= \exp\left(\frac{xy}{1-y}\right) \frac{1}{(1-y)^{2r}}. \end{aligned}$$

II. We can prove the theorem in another way for summed r -Lah numbers. Denote by $\ell_r(y)$ the exponential generating function to be find.

From the first special case of Theorem 3.1, it follows that $(L_{n+1})_{n=0}^{\infty} = (L_{n,1})_{n=0}^{\infty}$ is the binomial convolution of the sequences $(L_n)_{n=0}^{\infty}$ and $((n+1)!)_{n=0}^{\infty}$, hence their exponential generating functions give the differential equation

$$\ell'_0(y) = \ell_0(y) \frac{1}{(1-y)^2}.$$

For $n \geq 0$, it shows that $[y^j]\ell_0(y)$ ($j = 0, \dots, n$) uniquely determine $[y^{n+1}]\ell_0(y)$, whence our differential equation with the initial condition $[y^0]\ell_0(y) = \frac{L_0}{0!} = 1$ is uniquely solvable among formal power series, and this solution is $\ell_0(y) = \exp\left(\frac{y}{1-y}\right)$.

The second special case of Theorem 3.1 says that $(L_{n,r})_{n=0}^{\infty}$ is the binomial convolution of the sequences $(L_n)_{n=0}^{\infty}$ and $((2r)^{\overline{n}})_{n=0}^{\infty}$, therefore its exponential generating function is

$$\ell_r(y) = \ell_0(y) \sum_{n=0}^{\infty} \frac{(2r)^{\overline{n}}}{n!} y^n = \exp\left(\frac{y}{1-y}\right) \frac{1}{(1-y)^{2r}}. \quad \square$$

In the following theorem, we show the real-rootedness of r -Lah polynomials, where the proof will contain a further recurrence for them.

Theorem 3.8. *If $n \geq 1$, then the roots of $L_n(x)$ are simple, real, one of them is 0 and the others are negative. If $n, r \geq 1$, then the roots of $L_{n,r}(x)$ are simple, real and negative. Furthermore, for any $r \geq 0$, $(L_{n,r}(x))_{n=0}^{\infty}$ is an interlacing sequence of polynomials.*

Proof. We perform the proof by induction on n only for $r \geq 1$. We can easily check the assertion for $n = 1, 2$, and assume that it holds for some n .

Using [19, Theorem 3.1] and the special values of r -Lah numbers, we get

$$\begin{aligned} L_{n+1,r}(x) &= \sum_{k=0}^{n+1} \left[\begin{matrix} n+1 \\ k \end{matrix} \right]_r x^k = \left[\begin{matrix} n+1 \\ 0 \end{matrix} \right]_r + \sum_{k=1}^n \left[\begin{matrix} n+1 \\ k \end{matrix} \right]_r x^k + \left[\begin{matrix} n+1 \\ n+1 \end{matrix} \right]_r x^{n+1} \\ &= (2r)^{\overline{n+1}} + \sum_{k=1}^n \left(\left[\begin{matrix} n \\ k-1 \end{matrix} \right]_r + (n+k+2r) \left[\begin{matrix} n \\ k \end{matrix} \right]_r \right) x^k + x^{n+1} \\ &= \sum_{k=0}^{n-1} \left[\begin{matrix} n \\ k \end{matrix} \right]_r x^{k+1} + x^{n+1} + (n+2r) \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_r x^k + (2r)^{\overline{n+1}} + \sum_{k=1}^n k \left[\begin{matrix} n \\ k \end{matrix} \right]_r x^k \\ &= x \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_r x^k + (n+2r) \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_r x^k + x \sum_{k=1}^n k \left[\begin{matrix} n \\ k \end{matrix} \right]_r x^{k-1} \\ &= x L_{n,r}(x) + (n+2r) L_{n,r}(x) + x L'_{n,r}(x). \end{aligned}$$

Then, multiplying this equation by $e^x x^{n+2r-1}$ gives

$$e^x x^{n+2r-1} L_{n+1,r}(x) = (e^x x^{n+2r} L_{n,r}(x))'.$$

The induction hypothesis tells us that $L_{n,r}(x)$ has n simple real roots which are negative, hence $e^x x^{n+2r} L_{n,r}(x)$ has exactly $n+1$ zeros, one of them is 0, and the others are negative. Moreover, $\lim_{x \rightarrow -\infty} e^x x^{n+2r} L_{n,r}(x) = 0$. Then it follows from Rolle's mean value theorem that $(e^x x^{n+2r} L_{n,r}(x))' = e^x x^{n+2r-1} L_{n+1,r}(x)$ has at least $n+1$ negative zeros, therefore $L_{n+1,r}(x)$ has $n+1$ distinct negative roots.

The proof also shows the interlacing property. \square

This result together with a theorem of Newton (see, e.g., [24]) immediately implies the following consequence, which was proved in [19, Theorem 3.8] by different means.

Corollary 3.9. *If $n \geq 1$ and $r \geq 0$, then the sequence $\left(\left[\begin{matrix} n \\ j \end{matrix} \right]_r \right)_{j=0}^n$ is strictly log-concave and unimodal.*

The theorem also allows us to give a good approximation of the quotient of two consecutive summed r -Lah numbers.

Corollary 3.10. *If $n \geq 1$ and $r \geq 0$, then*

$$\left| \frac{L_{n+1,r}}{L_{n,r}} - (n+r+1) - \left\lfloor \sqrt{n+r^2+1} \right\rfloor \right| < 1.$$

Proof. From the recurrence derived in the proof of Theorem 3.8, we get

$$L'_{n,r}(1) = L_{n+1,r} - (n+2r+1)L_{n,r}.$$

Then the assertion follows from Theorem 3.8, a theorem of Darroch (see, e.g., [3]) and [19, Theorem 3.9]. □

Finally, we prove that the r -Stirling transform of the first kind of the sequence of s -Bell polynomials is the sequence of $\frac{r+s}{2}$ -Lah polynomials if r and s have the same parity.

Theorem 3.11. *If $n, r, s \geq 0$ and $r+s$ is even, then*

$$L_{n, \frac{r+s}{2}}(x) = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_r B_{j,s}(x).$$

Proof. By [19, Theorem 3.11], we have

$$\begin{aligned} L_{n, \frac{r+s}{2}}(x) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{\frac{r+s}{2}} x^k = \sum_{k=0}^n \sum_{j=k}^n \begin{bmatrix} n \\ j \end{bmatrix}_r \begin{Bmatrix} j \\ k \end{Bmatrix}_s x^k \\ &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_r \sum_{k=0}^j \begin{Bmatrix} j \\ k \end{Bmatrix}_s x^k = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_r B_{j,s}(x). \end{aligned} \quad \square$$

Remark 3.12. If $r = s$, then the identity simply becomes

$$L_{n,r}(x) = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_r B_{j,r}(x).$$

In this case, we can provide a combinatorial proof.

Proof. We may again assume that n, r are not both 0, and let c be a positive integer. A c -coloured r -Lah partition of an $(n+r)$ -element set into ordered blocks can be constructed as follows: First, we decompose the elements into $j+r$ disjoint cycles such that the r distinguished elements belong to distinct cycles ($j = 0, \dots, n$). These latter cycles will be referred to as distinguished cycles. After that, we partition all the cycles such that distinguished cycles are in distinct blocks, and we colour the blocks containing no distinguished cycle with c colours. Finally, we multiply the cycles in each block to obtain the ordered blocks of the original $(n+r)$ -element set. Therefore, for a fixed j , the number of c -coloured r -Lah partitions is $\begin{bmatrix} n \\ j \end{bmatrix}_r B_{j,r}(c)$. □

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