The Möbius function of $\text{PSU}(3, 2^{2n})$

Giovanni Zini*

Department of Mathematics and Applications, University of Milano-Bicocca, via Cozzi 55, 20125 Milano, Italy

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Abstract

Let $G$ be the simple group $\text{PSU}(3, 2^{2n})$, $n > 0$. For any subgroup $H$ of $G$, we compute the Möbius function $\mu_L(H, G)$ of $H$ in the subgroup lattice $L$ of $G$, and the Möbius function $\mu_{\tilde{L}}([H], [G])$ of $[H]$ in the poset $\tilde{L}$ of conjugacy classes of subgroups of $G$. For any prime $p$, we provide the Euler characteristic of the order complex of the poset of non-trivial $p$-subgroups of $G$.

Keywords: Unitary groups, Möbius function, subgroup lattice.

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1 Introduction

The Möbius function $\mu(H, G)$ on the subgroups of a finite group $G$ is defined recursively by $\mu(G, G) = 1$ and $\sum_{K \geq H} \mu(K, G) = 0$ if $H < G$. This function was used in 1936 by Hall [12] to enumerate $k$-tuples of elements of $G$ which generate $G$, for a given $k$.

The combinatorial and group-theoretic properties of the Möbius function were investigated by many authors; see the paper [14] by Hawkes, Isaacs, and Özaydin. The Möbius function is defined more generally on a locally finite poset $(P, \leq)$ by the recursive definition $\mu(x, x) = 1$, $\mu(x, y) = 0$ if $x \not\leq y$, and $\sum_{x \leq z \leq y} \mu(z, y) = 0$ if $x \leq y$; for instance, the poset taken into consideration may be the subgroup lattice $L$ of a finite group $G$ ordered by inclusion. Mann [19, 20] studied $\mu(H, G)$ in the broader context of profinite groups $G$ and defined a probabilistic zeta function $P(G, s)$ associated to $G$, related to the probability of generating $G$ with $s$ elements when $G$ is positively finitely generated.

The Möbius function on a poset $\mathcal{P}$ also appears in the context of topological invariants of the order simplicial complex $\Delta(\mathcal{P})$ associated to $\mathcal{P}$, see the works of Brown [2] and
Quillen [25]; if $\mathcal{P}$ is the subgroup lattice of a finite group $G$, then the reduced Euler characteristic of $\Delta(\mathcal{P})$ is equal to $\mu(\{1\}, G)$. This motivates the search for $\mu(\{1\}, G)$ independently of the knowledge of $\mu(H, G)$ for other subgroups $H$ of $G$, see for instance [26, 27] and the references therein; $\mu(\{1\}, G)$ is often called the Möbius number of $G$. Shaprisian provided a formula in [26] for $\mu(\{1\}, \text{Sym}(n))$, and computed $\mu(\{1\}, G)$ in [27] when $G \in \{\text{PGL}(2, q), \text{PSL}(2, q), \text{PGL}(3, q), \text{PSL}(3, q), \text{PGU}(3, q), \text{PSU}(3, q)\}$ with $q$ odd or $G$ is a Suzuki group $Sz(2^{2h+1})$.

Consider the poset $\bar{L}$ of conjugacy classes $[H]$ of subgroups $H$ of a finite group $G$, ordered as follows: $[H] \leq [K]$ if and only if $H$ is contained in some conjugate of $K$ in $G$. After Hawkes, Isaacs, and Özaydin [14], we denote by $\lambda(H, G)$ the Möbius function $\mu([H], [G])$ in $\bar{L}$, while $\mu(H, G)$ is the Möbius function in $L$. Some attempt was done to search relations between the Möbius functions $\mu(H, G)$ and $\lambda(H, G)$; Hawkes, Isaacs, and Özaydin [14] proved that, if $G$ is solvable, then

$$\mu(\{1\}, G') = |G'| \cdot \lambda(\{1\}, G). \quad (1.1)$$

The property $(1.1)$, which we call $(\mu, \lambda)$-property, does not hold in general for non-solvable groups; see [1]. Pahlings [23] proved that, if $G$ is solvable, then

$$\mu(H, G) = [N_{G'}(H) : H \cap G'] \cdot \lambda(H, G) \quad (1.2)$$

for any subgroup $H$ of $G$. The analysis of the generalized $(\mu, \lambda)$-property $(1.2)$, although false in general for non-solvable groups, is of interest since it relates the Möbius functions $\mu(H, G)$ and $\lambda(H, G)$.

A lot of work was done by several authors about probabilistic functions for groups; see for instance [6, 10, 19, 20]. In particular, Mann posed in [19] a conjecture, the validity of which would imply that the sum

$$\sum_{H < G} \frac{\mu(H, G)}{|G : H|^s}$$

over all subgroups $H < G$ of finite index of a positively finitely generated profinite group $G$ is absolutely convergent for $s$ in some right complex half-plane and, for $s \in \mathbb{N}$ large enough, represents the probability of generating $G$ with $s$ elements. Lucchini [18] showed that this problem can be reduced so that Mann’s conjecture is reformulated as follows: there exist two constants $c_1, c_2 \in \mathbb{N}$ such that, for any finite monolithic group $G$ with non-abelian socle,

1. $|\mu(H, G)| \leq |G : H|^{c_1}$ for any $H < G$ such that $G = H \text{soc}(G)$, and

2. the number of subgroups $H < G$ of index $n$ in $G$ such that $H \text{soc}(G) = G$ and $\mu(H, G) \neq 0$ is upper bounded by $n^{c_2}$, for any $n \in \mathbb{N}$.

It seems natural to investigate this conjecture on finite monolithic groups starting by almost simple groups. Mann’s conjecture has been shown to be satisfied by the alternating and symmetric groups [3], as well as by those families of groups $G$ for which $\mu(H, G)$ has been computed for any subgroup $H$: namely, $\text{PSL}(2, q)$ [8, 12], $\text{PGL}(2, q)$ [8], the Suzuki groups $Sz(2^{2h+1})$ [9], and the Ree groups $R(3^{2h+1})$ [24].

In this paper, we take into consideration the three dimensional projective special unitary group $G = \text{PSU}(3, q)$ over the field with $q = 2^{2n}$ elements, for any positive $n$ (note that $\text{PSU}(3, q) = \text{PGU}(3, q)$ as $3 \nmid (q + 1)$). In particular, the following results are obtained.
We compute \(\mu(H, G)\) for any subgroup \(H\) of \(G\), as summarized in Table 1. This shows that the groups \(PSU(3, 2^{2n})\) satisfy Mann’s conjecture.

We compute \(\lambda(H, G)\) for any subgroup \(H\) of \(G\), as summarized in Table 1. This shows that the groups \(PSU(3, 2^{2n})\) satisfy the \((\mu, \lambda)\)-property, but do not satisfy the generalized \((\mu, \lambda)\)-property.

We compute the Euler characteristic \(\chi(\Delta(L_p \setminus \{1\}))\) of the order complex of the poset \(L_p \setminus \{1\}\) of non-trivial \(p\)-subgroups of \(G\), for any prime \(p\), as summarized in Table 2.

For the subgroups listed in Table 1, the isomorphism type determines a unique conjugacy class in \(G\).

### Table 1: Subgroups \(H\) of \(G = PSU(3, q), q = 2^{2n}\), with \(\mu(H) \neq 0\) or \(\lambda(H) \neq 0\).

| Isomorphism type of \(H\) | \(|H|\) | \(N_G(H)\) | \(\mu(H, G)\) | \(\lambda(H, G)\) |
|---------------------------|--------|------------|----------------|----------------|
| \(G\)                     | \(q^3(q^3 + 1)(q^2 - 1)\) | \(H\) | 1 | 1 |
| \((E_q \cdot E_{q^2}) \rtimes C_{q^2 - 1}\) | \(q^3(q^2 - 1)\) | \(H\) | -1 | -1 |
| \(PSL(2, q) \times C_{q+1}\) | \(q(q^2 - 1)(q + 1)\) | \(H\) | -1 | -1 |
| \((C_{q+1} \times C_{q+1}) \rtimes \text{Sym}(3)\) | \(6(q + 1)^2\) | \(H\) | -1 | -1 |
| \(C_{q^2 - q + 1} \rtimes C_3\) | \(3(q^2 - q + 1)\) | \(H\) | -1 | -1 |
| \(E_q \rtimes C_{q^2 - 1}\) | \(q(q^2 - 1)\) | \(H\) | 1 | 1 |
| \((C_{q+1} \times C_{q+1}) \rtimes C_2\) | \(2(q + 1)^2\) | \(H\) | 1 | 1 |
| \(\text{Sym}(3)\) | 6 | \(\text{Sym}(3) \rtimes C_{q+1}\) | \(q + 1\) | 1 |
| \(C_3\) | 3 | \(C_{q^2 - 1} \rtimes C_2\) | \(\frac{2(q^2 - 1)}{3}\) | 1 |
| \(C_2\) | 2 | \((E_q \cdot E_{q^2}) \rtimes C_{q+1}\) | \(-q^3(q+1)\) | -1 |

### Table 2: Euler characteristic of the order complex of the poset of proper \(p\)-subgroups of \(G\).

| Prime \(p\) | \(p \mid |G|\) | \(p = 2\) | \(p \mid (q + 1)\) | \(p \mid (q - 1)\) | \(p \mid (q^2 - q + 1)\) |
|-------------|-------------|------------|----------------|----------------|----------------|
| \(\chi(\Delta(L_p \setminus \{1\}))\) | 0 | \(q^3 + 1\) | \(-q^6 - 2q^5 - q^4 + 2q^3 - 3q^2\) | \(\frac{q^6 + q^3}{2}\) | \(-q^6 + q^5 - q^4 - q^3\) |

The paper is organized as follows. Section 2 contains preliminary results on the Möbius functions \(\mu(H, G)\) and \(\lambda(H, G)\) and the relation between the Möbius function and the Euler characteristic of the order complex; this section contains also preliminary results on the groups \(G = PSU(3, 2^{2n})\), whose elements are described geometrically in their action on the Hermitian curve associated to \(G\). Sections 3 and 4 are devoted to the determination of \(\mu(H, G)\) and \(\lambda(H, G)\), respectively, for any subgroup \(H\) of \(G\). Section 5 provides the Euler characteristic of the order complex of the poset of proper \(p\)-subgroups of \(G\), for any prime \(p\).
2 Preliminary results

Let \((\mathcal{P}, \leq)\) be a finite poset. The Möbius function \(\mu_\mathcal{P}: \mathcal{P} \times \mathcal{P} \to \mathbb{Z}\) is defined recursively as follows:

\[
\mu_\mathcal{P}(x, y) = 0 \quad \text{if} \quad x \not\leq y; \quad \mu_\mathcal{P}(x, x) = 1; \quad \sum_{x \leq z \leq y} \mu_\mathcal{P}(z, y) = 0 \quad \text{if} \quad x < y.
\]

If \(x < y\), then \(\mu_\mathcal{P}(x, y)\) can be equivalently defined by

\[
\sum_{x \leq z \leq y} \mu_\mathcal{P}(x, z) = 0.
\]

To the poset \(\mathcal{P}\) we can associate a simplicial complex \(\Delta(\mathcal{P})\) whose vertices are the elements of \(\mathcal{P}\) and whose \(i\)-dimensional faces are the chains \(a_0 < \cdots < a_i\) of length \(i\) in \(\mathcal{P}\); \(\Delta(\mathcal{P})\) is called the order complex of \(\mathcal{P}\). Provided that \(\mathcal{P}\) has a least element \(0\), the Euler characteristic of the order complex of \(\mathcal{P}\) is computed as follows (see [28, Proposition 3.8.6]):

\[
\chi(\Delta(\mathcal{P})) = -\sum_{x \in \mathcal{P}} \mu_\mathcal{P}(0, x).
\]

Given a finite group \(G\), we will consider the following two Möbius functions associated to \(G\).

(i) The Möbius function on the subgroup lattice \(L\) of \(G\), ordered by inclusion. We will denote \(\mu_L(H, G)\) simply by \(\mu(H)\).

(ii) The Möbius function on the poset \(\bar{L}\) of conjugacy classes \([H]\) of subgroups \(H\) of \(G\), ordered as follows: \([H] \leq [K]\) if and only if \(H\) is contained in the conjugate \(gKg^{-1}\) for some \(g \in G\). We will denote \(\mu_{\bar{L}}([H], [G])\) simply by \(\lambda(H)\).

Two facts will be used to compute \(\mu(H)\). The first easy fact is that, if \(H\) and \(K\) are conjugate in \(G\), then \(\mu(H) = \mu(K)\). The second fact is due to Hall [12, Theorem 2.3], and is stated in the following lemma.

**Lemma 2.1.** If \(H < G\) satisfies \(\mu(H) \neq 0\), then \(H\) is the intersection of maximal subgroups of \(G\).

For any prime \(p\), let \(L_p\) be the subposet of \(L\) given by all \(p\)-subgroups of \(G\), so that

\[
\chi(\Delta(L_p \setminus \{1\})) = -\sum_{H \in L_p \setminus \{1\}} \mu_L(\{1\}, H).
\]

By a result of Brown [2], \(\chi(\Delta(L_p \setminus \{1\}))\) is congruent to 1 modulo the order \(|G|_p\) of a Sylow \(p\)-subgroup of \(G\). In order to compute explicitly \(\chi(\Delta(L_p \setminus \{1\}))\) we will use the following result of Hall [12, Equation (2.7)]:

**Lemma 2.2.** Let \(H\) be a \(p\)-group of order \(p^r\). If \(H\) is not elementary abelian, then \(\mu_{L_p}(\{1\}, H) = 0\). If \(H\) is elementary abelian, then \(\mu_{L_p}(\{1\}, H) = (-1)^r p^r\).

We describe now the group \(G\) which will be considered in the following sections. Let \(n\) be a positive integer, \(q = 2^a\); \(\mathbb{F}_q\) be the finite field with \(q\) element, and \(\overline{\mathbb{F}}_q\) be the algebraic...
closure of $\mathbb{F}_q$. Let $\mathcal{U}$ be a non-degenerate unitary polarity of the plane $\text{PG}(2, q^2)$ over $\mathbb{F}_{q^2}$, and $\mathcal{H}_q \subset \text{PG}(2, \mathbb{F}_q)$ be the Hermitian curve defined by $\mathcal{U}$. The following homogeneous equations define models for $\mathcal{H}_q$ which are projectively equivalent over $\mathbb{F}_{q^2}$:

\[ X^{q+1} + Y^{q+1} + Z^{q+1} = 0; \quad (2.2) \]
\[ X^q Z + X Z^q - Y^{q+1} = 0. \quad (2.3) \]

The models (2.2) and (2.3) are called the Fermat and the Norm-Trace model of $\mathcal{H}_q$, respectively. The set of $\mathbb{F}_{q^2}$-rational points of $\mathcal{H}_q$ is denoted by $\mathcal{H}_q(\mathbb{F}_{q^2})$, and consists of the $q^3 + 1$ isotropic points of $\mathcal{U}$. The full automorphism group $\text{Aut}(\mathcal{H}_q)$ of $\mathcal{H}_q$ is defined over $\mathbb{F}_{q^2}$, and coincides with the unitary subgroup $\text{PGU}(3, q)$ of $\text{PGL}(3, q^2)$ stabilizing $\mathcal{H}_q(\mathbb{F}_{q^2})$, of order $|\text{PGU}(3, q)| = q^3(q^3 + 1)(q^2 - 1)$.

The combinatorial properties of $\mathcal{H}_q(\mathbb{F}_{q^2})$ can be found in [16]. In particular, any line $\ell$ of $\text{PG}(2, q^2)$ has either 1 or $q + 1$ common points with $\mathcal{H}_q(\mathbb{F}_{q^2})$, that is, $\ell$ is either a tangent line or a chord of $\mathcal{H}_q(\mathbb{F}_{q^2})$; in the former case $\ell$ contains its pole with respect to $\mathcal{U}$, in the latter case $\ell$ doesn’t. Also, $\text{PGU}(3, q)$ acts 2-transitively on $\mathcal{H}_q(\mathbb{F}_{q^2})$ and transitively on $\text{PG}(2, q^2) \setminus \mathcal{H}_q$; $\text{PGU}(3, q)$ acts transitively also on the non-degenerate self-polar triangles $T = \{P_1, P_2, P_3\} \subset \text{PG}(2, q^2) \setminus \mathcal{H}_q$ with respect to $\mathcal{U}$. Recall that, if $\sigma \in \text{PGU}(3, q)$ stabilizes a point $P \in \text{PG}(2, q^2)$, then $\sigma$ stabilizes also the polar line of $P$ with respect to $\mathcal{U}$, and vice versa.

The curve $\mathcal{H}_q$ is non-singular and $\mathbb{F}_{q^2}$-maximal of genus $g = \frac{q(q - 1)}{2}$, that is, the size of $\mathcal{H}_q(\mathbb{F}_{q^2})$ attains the Hasse-Weil upper bound $q^2 + 1 + 2qg$. This implies that $\mathcal{H}_q$ is $\mathbb{F}_{q^4}$-minimal and $\mathbb{F}_{q^6}$-maximal, so that $\mathcal{H}_q(\mathbb{F}_{q^4}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2}) = \emptyset$ and $|\mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})| = q^6 - q^5 - q^4 - q^3$. Let $\Phi_q$ be the Frobenius map $(X, Y, Z) \mapsto (X^q, Y^q, Z^q)$ over $\text{PG}(2, \mathbb{F}_{q^2})$; then the $\mathbb{F}_{q^6} \setminus \mathbb{F}_{q^2}$-rational points of $\mathcal{H}_q$ split into $\frac{q^6 + q^5 - q^4 - q^3}{3}$ non-degenerate triangles $\{P, \Phi_q(P), \Phi_q^2(P)\}$. The group $\text{PGU}(3, q)$ is transitive on such triangles.

Since $3 \nmid (q + 1)$, we have $\text{PGU}(3, q) = \text{PSU}(3, q)$; henceforth, we denote by $G$ the simple group $\text{PSU}(3, q)$. The following classification of subgroups of $G$ goes back to Hartley [13]; here we use that $\log_2(q)$ has no odd divisors different from 1. The notation $S_2$ stands for a Sylow 2-subgroup of $G$, which is a non-split extension $E_q, E_{q^2}$ of its elementary abelian center of order $q$ by an elementary abelian group of order $q^2$.

**Theorem 2.3.** Let $n > 0$, $q = 2^{2n}$, and $G = \text{PSU}(3, q)$. Up the conjugation, the maximal subgroups of $G$ are the following.

(i) The stabilizer $M_1(P) \cong S_2 \rtimes C_{q^2-1}$ of a point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$, of order $q^3(q^2 - 1)$.

(ii) The stabilizer $M_2(P) \cong \text{PSL}(2, q) \rtimes C_{q+1}$ of a point $P \in \text{PG}(2, q^2) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$, of order $q(q^2 - 1)(q + 1)$.

(iii) The stabilizer $M_3(T) \cong (C_{q+1} \rtimes C_{q+1}) \rtimes \text{Sym}(3)$ of a non-degenerate self-polar triangle $T = \{P, Q, R\} \subset \text{PG}(2, q^2) \setminus \mathcal{H}_q$ with respect to $\mathcal{U}$, of order $6(q + 1)^2$.

(iv) The stabilizer $M_4(T) \cong C_{q^2-q+1} \rtimes C_3$ of a triangle $T = \{P, \Phi_q(P), \Phi_q^2(P)\} \subset \mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$, of order $3(q^2 - q + 1)$.

For a detailed description of the maximal subgroups of $G$, both from an algebraic and a geometric point of view, we refer to [11, 21, 22].
In our investigation it is useful to know the geometry of the elements of \( \text{PGU}(3, q) \) on \( \text{PG}(2, \mathbb{F}_q) \), and in particular on \( \mathcal{H}_q(\mathbb{F}_{q^2}) \). This can be obtained as a corollary of Theorem 2.3, and is stated in Lemma 2.2 with the usual terminology of collineations of projective planes; see [16]. In particular, a linear collineation \( \sigma \) of \( \text{PG}(2, \mathbb{F}_q) \) is a \((P, \ell)\)-perspectivity, if \( \sigma \) preserves each line through the point \( P \) (the center of \( \sigma \)), and fixes each point on the line \( \ell \) (the axis of \( \sigma \)). A \((P, \ell)\)-perspectivity is either an elation or a homology according to \( P \in \ell \) or \( P \notin \ell \). Lemma 2.4 was obtained in [21] in a more general form (i.e., for any prime power \( q \)).

**Lemma 2.4.** For a nontrivial element \( \sigma \in G = \text{PSU}(3, q) \), \( q = 2^{2n} \), one of the following cases holds.

(A) \( \text{ord}(\sigma) \mid (q + 1) \) and \( \sigma \) is a homology, with center \( P \in \text{PG}(2, q^2) \setminus \mathcal{H}_q \) and axis \( \ell_P \) which is a chord of \( \mathcal{H}_q(\mathbb{F}_{q^2}) \); \((P, \ell_P)\) is a pole-polar pair with respect to \( \mathcal{U} \).

(B) \( 2 \nmid \text{ord}(\sigma) \) and \( \sigma \) fixes the vertices \( P_1, P_2, P_3 \) of a non-degenerate triangle \( T \subset \text{PG}(2, q^2) \).

\( (B1) \) \( \text{ord}(\sigma) \mid (q + 1) \), \( P_1, P_2, P_3 \in \text{PG}(2, q^2) \setminus \mathcal{H}_q \), and the triangle \( T \) is self-polar with respect to \( \mathcal{U} \).

\( (B2) \) \( \text{ord}(\sigma) \mid (q^2 - 1) \) and \( \text{ord}(\sigma) \nmid (q + 1) \); \( P_1 \in \text{PG}(2, q^2) \setminus \mathcal{H}_q \) and \( P_2, P_3 \in \mathcal{H}_q(\mathbb{F}_{q^2}) \).

\( (B3) \) \( \text{ord}(\sigma) \mid (q^2 - q + 1) \) and \( P_1, P_2, P_3 \in \mathcal{H}_q(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2}) \).

(C) \( \text{ord}(\sigma) = 2 \); \( \sigma \) is an elation with center \( P \in \mathcal{H}_q(\mathbb{F}_{q^2}) \) and axis \( \ell_P \) which is tangent to \( \mathcal{H}_q \) at \( P \), such that \((P, \ell_P)\) is a pole-polar pair with respect to \( \mathcal{U} \).

(D) \( \text{ord}(\sigma) = 4 \); \( \sigma \) fixes a point \( P \in \mathcal{H}_q(\mathbb{F}_{q^2}) \) and a line \( \ell_P \) which is tangent to \( \mathcal{H}_q \) at \( P \), such that \((P, \ell_P)\) is a pole-polar pair with respect to \( \mathcal{U} \).

(E) \( \text{ord}(\sigma) = 2d \) where \( d \) is a nontrivial divisor of \( q + 1 \); \( \sigma \) fixes two points \( P \in \mathcal{H}_q(\mathbb{F}_{q^2}) \) and \( Q \in \text{PG}(2, q^2) \setminus \mathcal{H}_q \), the polar line \( PQ \) of \( P \), and the polar line of \( Q \) which passes through \( P \).

For a detailed description of the elements and subgroups of \( G \), both from an algebraic and a geometric point of view, we refer to [11, 21, 22], on which our geometric arguments are based.

Throughout the paper, a nontrivial element of \( G \) is said to be of type (A), (B), (B1), (B2), (B3), (C), (D), or (E), as given in Lemma 2.4. Also, the polar line to \( \mathcal{H}_q \) at \( P \in \text{PG}(2, q^2) \) is denoted by \( \ell_P \). Note that, under our assumptions, any element of order 3 in \( G \) is of type (B2). We will denote a cyclic group of order \( d \) by \( C_d \) and an elementary abelian group of order \( d \) by \( E_q \). The center \( Z(S_2) \) of \( S_2 \) is elementary abelian of order \( q \), and any element in \( S_2 \setminus Z(S_2) \) has order 4; see [11, Section 3].

### 3 Determination of \( \mu(H) \) for any subgroup \( H \) of \( G \)

Let \( n > 0 \), \( q = 2^{2n} \), \( G = \text{PSU}(3, q) \). This section is devoted to the proof of the following theorem.
Theorem 3.1. Let $H$ be a proper subgroup of $G$. Then $H$ is the intersection of maximal subgroups of $G$ if and only if $H$ is one of the following groups:

$$
\begin{align*}
S_2 \rtimes C_{q^2-1}, & \quad \text{PSL}(2,q) \rtimes C_{q+1}, & \quad C_{q^2-q+1} \rtimes C_3, \\
(C_{q+1} \rtimes C_{q+1}) \rtimes \text{Sym}(3), & \quad E_q \rtimes C_{q^2-1}, & \quad (C_{q+1} \rtimes C_{q+1}) \rtimes C_2, \\
C_{q+1} \rtimes C_{q+1}, & \quad C_{q^2-1}, & \quad C_{2(q+1)}, \\
C_{q+1} = Z(M_2(P)) \text{ for some } P, & \quad E_q, & \quad \text{Sym}(3), \\
C_3, & \quad C_2, & \quad \{1\}.
\end{align*}
$$

(3.1)

Given a type of groups in Equation (3.1), there is just one conjugacy class of subgroups of $G$ of that isomorphism type.

The normalizer $N_G(H)$ of $H$ in $G$ for the groups $H$ in Equation (3.1) are, respectively:

$$
\begin{align*}
H, & \quad H, & \quad H, \\
H, & \quad H, & \quad H, \\
H \rtimes \text{Sym}(3), & \quad H \rtimes C_2, & \quad E_q \rtimes C_{q+1}, \\
\text{PSL}(2,q) \rtimes H, & \quad S_2 \rtimes C_{q^2-1}, & \quad H \rtimes C_{q+1}, \\
C_{q^2-1} \rtimes C_2, & \quad S_2 \rtimes C_{q+1}, & \quad G.
\end{align*}
$$

(3.2)

The values $\mu(H)$ for the groups $H$ in Equation (3.1) are, respectively:

$$
\begin{align*}
-1, & \quad -1, & \quad -1, \\
-1, & \quad 1, & \quad 1, \\
0, & \quad 0, & \quad 0, \\
0, & \quad 0, & \quad q+1, \\
\frac{2(q^2-1)}{3}, & \quad -\frac{q^3(q+1)}{2}, & \quad 0.
\end{align*}
$$

(3.3)

The proof of Theorem 3.1 is divided into several propositions.

Proposition 3.2. The group $G$ contains exactly one conjugacy class for any group in Equation (3.1).

Proof. Case 1: The first four groups in Equation (3.1), i.e.,

$$S_2 \rtimes C_{q^2-1}, \quad \text{PSL}(2,q) \rtimes C_{q+1}, \quad C_{q^2-q+1} \rtimes C_3, \quad \text{and } (C_{q+1} \rtimes C_{q+1}) \rtimes \text{Sym}(3),$$

are the maximal subgroups of $G$, for which there is just one conjugacy class by Theorem 2.3.

Case 2: Let $\alpha_1, \alpha_2 \in G$ have order 3, so that they are of type (B2) and $\alpha_i$ fixes two distinct points $P_i, Q_i \in \mathcal{H}_q(\mathbb{F}_{q^2})$. The group $G$ is 2-transitive on $\mathcal{H}_q(\mathbb{F}_{q^2})$, and the pointwise stabilizer of $\{P_i, Q_i\}$ is cyclic of order $q^2-1$. Hence, $\langle \alpha_1 \rangle$ and $\langle \alpha_2 \rangle$ are conjugated in $G$.

Case 3: Let $\alpha_1, \alpha_2 \in G$ have order 2, so that they are of type (C) and $\alpha_i$ fixes exactly one point $P_i$ on $\mathcal{H}_q(\mathbb{F}_{q^2})$. Up to conjugation $P_1 = P_2$, as $G$ is transitive on $\mathcal{H}_q(\mathbb{F}_{q^2})$. The involutions fixing $P_1$ in $G$, together with the identity, form an elementary abelian group $E_q$, which is normalized by a cyclic group $C_{q-1}$; no nontrivial element of $C_{q-1}$ commutes
with any nontrivial element of $E_q$ (see [11, Section 4]). Hence, $\alpha_1$ and $\alpha_2$ are conjugated under an element of $C_{q^{-1}}$.

**Case 4:** Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in G$ satisfy $o(\alpha_i) = 3$, $o(\beta_i) = 2$, and $H_i := \langle \alpha_i, \beta_i \rangle \cong \text{Sym}(3)$. As shown in the previous point, we can assume $\alpha_1 = \alpha_2$ up to conjugation. Let $P, Q \in H_q(\mathbb{F}_{q^2})$ and $R \in \text{PG}(2, q^2) \setminus H_q$ be the fixed points of $\alpha_1$. Since $\beta_i \alpha_1 \beta_i^{-1} = \alpha_1^{-1}$, we have that $\beta_i$ fixes $R$ and interchanges $P$ and $Q$; $\beta$ is then uniquely determined from the $\mathbb{F}_{q^2}$-rational point of $PQ$ fixed by $\beta$ (namely, the intersection between $PQ$ and the axis of $\beta$). Since the pointwise stabilizer $C_{q^2-1}$ of $\{P, Q\}$ acts transitively on $PQ(\mathbb{F}_{q^2}) \setminus H_q$, $\beta_1$ and $\beta_2$ are conjugated, and the same holds for $H_1$ and $H_2$.

**Case 5:** Any two groups isomorphic to $C_{q^2-1}$ are conjugated in $G$, because they are generated by elements of type $(B2)$ and $G$ is 2-transitive on $H_q(\mathbb{F}_{q^2})$.

**Case 6:** Any two groups isomorphic to $E_q$ are conjugated in $G$, because any such group fixes exactly one point $P \in H_q(\mathbb{F}_{q^2})$, $G$ is transitive on $H_q(\mathbb{F}_{q^2})$, and the stabilizer $G_P = M_1(P)$ contains just one subgroup $E_q$.

**Case 7:** Any two groups $H_1, H_2 \cong E_q \rtimes C_{q^2-1}$ are conjugated in $G$. In fact, their Sylow 2-subgroups $E_q$ coincide up to conjugation, as shown in the previous point. The normalizer $N_G(E_q)$ fixes the fixed point $P \in H_q(\mathbb{F}_{q^2})$ of $E_q$, and hence $N_G(E_q) = M_1(P) = S_2 \rtimes C_{q^2-1}$. The complements $C_{q^2-1}$ are conjugated by Schur-Zassenhaus Theorem; hence, $H_1$ and $H_2$ are conjugated.

**Case 8:** Any two groups isomorphic to $C_{2(q+1)}$ are conjugated in $G$, because they are generated by elements of type (E) and two elements $\alpha_1, \alpha_2$ of type (E) of the same order are conjugated in $G$. In fact, $\alpha_i$ is uniquely determined by its fixed points $P_i \in H_q(\mathbb{F}_{q^2})$ and $Q_i \in \ell_{P_i}(\mathbb{F}_{q^2}) \setminus H_q$; here, $\ell_{P_i}$ is the polar line of $P_i$. Up to conjugation $P_1 = P_2$, from the transitivity of $G$ on $H_q(\mathbb{F}_{q^2})$. Also, $S_2$ has order $q^3$ and acts on the $q^2$ points of $\ell_{P_1}(\mathbb{F}_{q^2}) \setminus H_q$ with kernel $E_q$, hence transitively. We can then assume $Q_1 = Q_2$.

**Case 9:** Let $Z_{P_i}$ be the center of $M_2(P_i)$, $i = 1, 2$. As shown in [5, Section 4], $Z_{P_i} \cong C_{q+1}$ and $Z_{P_i}$ is made by the homologies with center $P_i$, together with the identity. Since $G$ is transitive on $\text{PG}(2, q^2) \setminus H_q$, we have up to conjugation that $M_2(P_1) = M_2(P_2)$ and $Z_{P_1} = Z_{P_2}$.

**Case 10:** Any two groups $H_1, H_2 \cong C_{q+1} \times C_{q+1}$ are conjugated in $G$. In fact, $H_i$ is the pointwise stabilizer of a self-polar triangle $T_i = \{P_i, Q_i, R_i\} \subset \text{PG}(2, q^2) \setminus H_q$ (see [5, Section 3]), and the stabilizers of $T_1$ and $T_2$ are conjugated by Theorem 2.3.

**Case 11:** Any two groups $H_1, H_2 \cong (C_{q+1} \times C_{q+1}) \rtimes C_2$ are conjugated in $G$. In fact, their subgroups $C_{q+1} \times C_{q+1}$ coincide up to conjugation as shown above, and fix pointwise a self-polar triangle $T = \{P, Q, R\} \subset \text{PG}(2, q^2) \setminus H_q$. Let $\beta_i \in H_i$ have order 2, $i = 1, 2$. Then $\beta_i$ fixes one vertex of $T$ and interchanges the other two vertexes. Up to conjugation in $M_3(T)$ we have $\beta_1(P) = \beta_2(P) = P$. Then $H_1 = H_2$, as they coincide with the stabilizer of $P$ in $M_3(T)$.

**Proposition 3.3.** The normalizers $N_G(H)$ of the groups $H$ in Equation (3.1) are given in Equation (3.2).

**Proof.** **Case 1:** Clearly $N_G(H) = H$ for any $H$ from the first four groups of Equation (3.1) as $H$ is maximal in $G$. 


Case 2: Let \( H = E_q \rtimes C_{q^2-1} \). Then \( H \leq M_2(P) \), where \( P \) is the unique fixed point of \( C_{q^2-1} \) in \( \text{PG}(2, q^2) \setminus \mathcal{H}_q \). The group \( H \) has a unique cyclic subgroup \( C_{q+1} \) of order \( q+1 \); namely, \( C_{q+1} \) is the center of \( M_2(P) \) and is made by the homologies with center \( P \); since \( q \) is even, \( H \) is a split extension \( C_{q+1} \times (E_q \rtimes C_{q-1}) \). Hence, \( N_G(H) \leq N_G(C_{q+1}) = M_2(P) \). The group \( H/C_{q+1} \cong E_q \rtimes C_{q-1} \) is maximal and hence self-normalizing in \( M_2(P)/C_{q+1} \cong \text{PSL}(2, q) \); thus, \( N_G(E_q \rtimes C_{q-1}) = H \) and \( N_G(H) = H \).

Case 3: Let \( H = C_{q+1} \times C_{q+1} \). Then \( N_G(H) \leq M_3(T) \), where \( T \) is the self-polar triangle fixed pointwise by \( H \). Since \( H \) is the kernel of \( M_3(T) \) in its action on \( T \), we have \( N_G(H) = M_3(T) \) and \( |N_G(H)| = 6|H| \).

Case 4: Let \( H = (C_{q+1} \times C_{q+1}) \rtimes C_2 \). Then \( C_{q+1} \times C_{q+1} \) is normal in \( N_G(H) \), being the unique subgroup of index 2 in \( H \). Hence \( N_G(H) \leq M_3(T) \), where \( T \) is the self-polar triangle fixed pointwise by \( H \). Also, \( N_G(H) \) fixes the vertex \( P \) of \( T \) fixed by \( H \), so that \( N_G(H) \neq M_3(T) \). This implies \( N_G(H) = H \).

Case 5: Let \( H = C_{q^2-1} \). Then \( H \) is generated by an element \( \alpha \) of type (B2) with fixed points \( P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2}) \) and \( R \in \text{PG}(2, q^2) \setminus \mathcal{H}_q \). Let \( \beta \) be an involution satisfying \( \beta(R) = R, \beta(P) = Q, \) and \( \beta(Q) = P \); then \( \beta \in N_G(H) \), because \( H \) coincides with the pointwise stabilizer of \( \{P, Q\} \) in \( G \). An explicit description is the following: given \( \mathcal{H}_q \) with equation (2.3), we can assume up to conjugation that \( \alpha = \text{diag}(a^{q+1}, a, 1) \) where \( a \) is a generator if \( \mathbb{F}_{q^2}^* \) (see [11]); then take

\[
\beta = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

Since \( N_G(H) \) acts on \( \{P, Q\} \) and \( \beta \in N_G(H) \), the pointwise stabilizer \( H \) of \( \{P, Q\} \) has index 2 in \( N_G(H) \). This implies \( N_G(H) = C_{q^2-1} \rtimes C_2 \) and \( |N_G(H)| = 2|H| \).

Case 6: Let \( H = C_{2(q+1)} \), so that \( H \) is generated by an element \( \alpha \) of type (E) fixing exactly two points \( P \in \mathcal{H}_q(\mathbb{F}_{q^2}) \) and \( Q \in \ell_P(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q \). Then \( N_G(H) \) fixes \( P \) and \( Q \). The subgroup \( E_q \) of \( M_1(P) \) commuting with \( H \) elementwise, while any 2-element in \( M_1(P) \setminus E_q \) has order 4 and does not fix \( Q \); hence, the Sylow 2-subgroup of \( N_G(H) \) is \( E_q \). Also, \( N_G(H) = E_q \rtimes C_d \), where \( C_d \) is a subgroup of \( C_{q^2-1} \) containing the subgroup \( C_{q+1} \) of \( H \). Let \( C_2 \) be the subgroup of \( H \) of order 2; the quotient group \( (C_2 \rtimes C_d)/C_{q+1} \cong C_2 \rtimes C_4 \) acts faithfully as a subgroup of \( \text{PGL}(2, q) \) on the \( q+1 \) points of \( \ell_Q \cap \mathcal{H}_q \). By the classification of subgroups of \( \text{PGL}(2, q) \) ([7]; see [17, Hauptsatz 8.27]), this implies \( d = 1 \); that is, \( N_G(H) = E_q \rtimes C_{q+1} \) and \( |N_G(H)| = \frac{q(q^2-1)}{2} |H| \).

Case 7: Let \( H = C_{q+1} = Z(M_2(P)) \). Since \( H \) is the center of \( M_2(P), M_2(P) \leq N_G(H) \). Conversely, \( H \) is made by homologies with center \( P \), and hence \( N_G(H) \) fixes \( P \). Thus, \( N_G(H) = M_2(P) \) and \( |N_G(H)| = q(q^2-1)|H| \).

Case 8: Let \( H = E_q \). Since \( E_q \) has a unique fixed point \( P \) on \( \mathcal{H}_q(\mathbb{F}_{q^2}) \) and \( E_q = Z(M_1(P)) \), we have \( N_G(H) \leq M_1(P) \) and \( M_1(P) \leq N_G(H) \), so that \( N_G(H) = M_1(P) \) and \( |N_G(H)| = q^2(q^2-1)|H| \).

Case 9: Let \( H = \text{Sym}(3) = \langle \alpha, \beta \rangle \), with \( o(\alpha) = 3 \) and \( o(\beta) = 2 \). Let \( P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2}) \) and \( R \in \text{PG}(2, q^2) \setminus \mathcal{H}_q \) be the fixed points of \( \alpha, \beta \) fixes \( R \), interchanges \( P \) and \( Q \), and fixes another point \( A_\beta \) on \( \ell_R \cap \mathcal{H}_q \). The group \( N_G(H) \) acts on \( \{P, Q\} \) and on \( \{A_\beta, A_\alpha \beta, A_{\alpha^2 \beta}\} \).
The pointwise stabilizer $C_{q^2-1}$ has a subgroup $C_{q+1}$ which is the center of $M_2(P)$ and fixes $PQ$ pointwise, while any element in $C_{q^2-1} \setminus C_{q+1}$ acts semiregularly on $PQ \setminus \{P, Q\}$; hence, $C_{q^2-1} \cap N_G(H) = C_{3(q+1)}$. If an element $\gamma \in N_G(H)$ fixes $\{P, Q\}$ pointwise, then $\gamma$ fixes a point in $\{A_\beta, A_\alpha\beta, A_\alpha^2\beta\}$, and hence $\gamma \in \{\beta, \alpha\beta, \alpha^2\beta\}$. Therefore, $N_G(H) = C_{3(q+1)} \rtimes C_2 = H \times C_{q+1}$ and $|N_G(H)| = (q+1)|H|$. 

**Case 10:** Let $H = C_3$ and $\alpha$ be a generator of $H$, with fixed points $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$. The normalizer $N_G(H)$ fixes $R$ and acts on $\{P, Q\}$. There exists an involution $\beta \in G$ normalizing $H$ and interchanging $P$ and $Q$ (see Equation (3.4)). Then the pointwise stabilizer of $\{P, Q\}$ has index $2$ in $N_G(H)$. Also, the pointwise stabilizer of $\{P, Q\}$ in $G$ is cyclic of order $q^2 - 1$. Then $N_G(H) = C_{q^2-1} \rtimes C_2$ and $|N_G(H)| = \frac{2(q^2-1)}{3}|H|$.

**Lemma 3.4.** Let $\alpha \in G$ be an involution, and hence an elation, with center $P$ and axis $\ell_P$. Then there exist exactly $q^3/2$ self-polar triangles $T_{i,j} = \{P_i, Q_{i,j}, R_{i,j}\}$, $i = 1, \ldots, q^2$, $j = 1, \ldots, \frac{q^2}{2}$, such that $\alpha$ stabilizes $T_{i,j}$. Also, $P_i \in \ell_P$ and $P \in Q_{i,j} R_{i,j}$ for any $i$ and $j$.

**Proof.** The number of involutions in $G$ is $(q^3+1)(q-1)$, since for any of the $q^3+1$ $\mathbb{F}_{q^2}$-rational points $P$ of $\mathcal{H}_q$ the involutions fixing $P$ form a group $E_q$. The number of self-polar triangles $T \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ is $[G : M_3(T)] = \frac{(q^3+1)q^3(q^2-1)}{6(q+1)^2}$. For any self-polar triangle $T = \{A_1, A_2, A_3\} \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$, the number of involutions in $G$ stabilizing $T$ is $3(q+1)$. In fact, for any of the 3 vertices of $T$ there are exactly $q+1$ involutions $\alpha_1, \ldots, \alpha_{q+1}$ fixing that vertex, say $A_1$, and interchanging $A_2$ and $A_3$: $\alpha_i$ is uniquely determined by its center $A_2 A_3 \cap \mathcal{H}_q$. Then, by double counting the size of

$$\{(\beta, T) \mid \beta \in G, o(\beta) = 2, T \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q \text{ is a self-polar triangle,} \beta \text{ stabilizes } T\},$$

$\alpha$ stabilizes exactly $\frac{q^2}{2}$ self-polar triangles $T$. For any such $T$, one vertex $P_i$ of $T$ lies on the axis of $\alpha$, because $\alpha$ is an elation, and the other two vertices $\{Q_{i,j}, R_{i,j}\}$ of $T$ lie on the polar line $\ell_{P_i}$ of $P_i$. Since $M_1(P)$ is transitive on the $q^2$ points $P_1, \ldots, P_{q^2}$ of $\ell_P(\mathbb{F}_{q^2}) \setminus \{P\}$, any point $P_i$ is contained in the same number $\frac{q^2}{2}$ of self-polar triangles $T_{i,j}$ stabilized by $\alpha$. 

**Lemma 3.5.** Let $\alpha \in G$ have order 3. Then there are exactly $\frac{q^2-1}{3}$ self-polar triangles

$$T_i \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q, \quad i = 1, \ldots, \frac{q^2-1}{3},$$

which are stabilized by $\alpha$. Also, there are exactly $\frac{2(q^2-1)}{3}$ triangles

$$\hat{T}_j = \{P_j, \Phi_{q^2}(P_j), \Phi_{q^2}^2(P_j)\} \subset \mathcal{H}_q(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2}), \quad j = 1, \ldots, \frac{2(q^2-1)}{3},$$

which are stabilized by $\alpha$. 


Proof. By Proposition 3.2, any two subgroups of $G$ of order 3 are conjugated in $G$. Also, any element of order 3 is conjugated to its inverse by an involution of $G$. Hence, any two element of order 3 are conjugated in $G$.

Now the claim follows by double counting the size of

$$\{(\beta, T) \mid \beta \in G, o(\beta) = 3, T \subset PG(2, q^2) \setminus H_q \text{ is a self-polar triangle}, \beta \text{ stabilizes } T\},$$

and

$$\{(\beta, \tilde{T}) \mid \beta \in G, o(\beta) = 3, \tilde{T} = \{P, \Phi_{q^2}(P), \Phi_{q^2}(P)\} \text{ with } P \in H_q(\mathbb{F}_{q^2}) \setminus H_q(\mathbb{F}_{q^2}), \beta \text{ stabilizes } \tilde{T}\},$$

using the following facts. The number of elements of order 3 in $G$ is $(q^3+1) \cdot 2$. The number of self-polar triangles $T \subset PG(2, q^2) \setminus H_q$ is $[G : M_3(T)]$. The number of elements of order 3 stabilizing a fixed self-polar triangle $T$ is $2(q+1)^2$, because any element acting as a 3-cycle on the vertices of $T$ has order 3 (see [5, Section 3]). The number of triangles $\tilde{T} = \{P, \Phi_{q^2}(P), \Phi_{q^2}(P)\} \subset H_q(\mathbb{F}_{q^2}) \setminus H_q(\mathbb{F}_{q^2})$ is $[G : M_4(\tilde{T})]$. The number of elements of order 3 stabilizing a fixed triangle $\tilde{T}$ is $2(q^2 - q + 1)$, because any element in $M_4(\tilde{T}) \setminus C_{q^2-1}$ has order 3 (see [4, Section 4]).

Lemma 3.6. Let $H < G$ be isomorphic to $\text{Sym}(3)$, $H = \langle \alpha \rangle \rtimes \langle \beta \rangle$. Then there are exactly $q + 1$ self-polar triangles

$$T_i = \{P_i, Q_i, R_i\} \subset PG(2, q^2) \setminus H_q, \quad i = 1, \ldots, q + 1,$$

which are stabilized by $H$. Up to relabeling the vertexes, we have that $P_1, \ldots, P_{q+1}$ lie on the axis of the elation $\beta$, $Q_1, \ldots, Q_{q+1}$ lie on the axis of the elation $\alpha \beta$, and $R_1, \ldots, R_{q+1}$ lie on the axis of the elation $\alpha^2 \beta$.

Proof. By Proposition 3.2, any two subgroups $K_1, K_2 < G$ with $K_i \cong \text{Sym}(3)$ are conjugated, and $|N_G(K_i)| = 6(q + 1)$; hence, the number of subgroups of $G$ isomorphic to $\text{Sym}(3)$ is $[G : N_G(K_i)] = (q^3+1)q^2(q-1)$. The number of self-polar triangles $T$ is $[G : M_3(T)] = (q^2-q+1)q^2(q-1)$. Then the claim on the number of self-polar triangles follows by double counting the size of

$$\{(K, T) \mid K < G, K \cong \text{Sym}(3), T \subset PG(2, q^2) \setminus H_q \text{ is a self-polar triangle}, K \text{ stabilizes } T\},$$

once we show that, for any self-polar triangle $T = \{A, B, C\}$, there are in $G$ exactly $(q + 1)^2$ subgroups isomorphic to $\text{Sym}(3)$ which stabilize $T$.

Let $K < M_3(T)$, $K \cong \text{Sym}(3)$, $K = \langle \alpha, \beta \rangle$ with $o(\alpha) = 3$, $o(\beta) = 2$. Let $P, Q, R$ be the fixed points of $\alpha$, with $P \in PG(2, q^2) \setminus H_q$, $Q, R \in H_q(\mathbb{F}_{q^2})$. By Proposition 3.3, $N_G(K) = K \times C_{q+1}$ where $C_{q+1}$ is made by homologies with center $P$; this implies $N_G(K) \cap M_3(T) = K$. Hence, there are at least $[M_3(T) : \text{Sym}(3)] = (q+1)^2$ distinct groups $\text{Sym}(3)$ stabilizing $T$, namely the conjugates of $K$ through elements of $M_3(T)$. On the other side, $M_3(T)$ contains exactly $(q + 1)^2$ subgroups $K$ of order 3, with fixed points $P \in PG(2, q^2) \setminus H_q$, $Q, R \in H_q(\mathbb{F}_{q^2})$. Any involution $\beta$ of $M_3(T)$ normalizing
$K$ is uniquely determined by the vertex of $T$ that $\beta$ fixes, because $\beta(P) = P$, $\beta(Q) = R$, and $\beta(R) = Q$. Thus, $K$ is contained in exactly one subgroup of $M_3(T)$ isomorphic to $\text{Sym}(3)$. Therefore the number of subgroups isomorphic to $\text{Sym}(3)$ which stabilize $T$ is $(q + 1)^2$.

Finally, the configuration of the vertexes of $T_1, \ldots, T_{q+1}$ on the axes of the involutions of $H$ follows from Lemma 2.4 and the fact that every involution fixes a different vertex of $T_1$.

\[ \square \]

**Proposition 3.7.** Any group $H$ in Equation (3.1) is the intersection of maximal subgroups of $G$.

**Proof.** Case 1: The first four groups of Equation (3.1) are exactly the maximal subgroups of $G$.

Case 2: Let $H = E_q \times C_{q^2-1}$. Let $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ be the unique point of $\mathcal{H}_q$ fixed by $E_q$; $E_q$ fixes $\ell_P$ pointwise. Also, the fixed points of $C_{q^2-1}$ are $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$, where $R \in \ell_P$ and $PQ = \ell_R$. Then $H \leq M_1(P) \cap M_2(R)$. Conversely, from $M_1(P) \cap M_2(R) \leq M_1(P)$ follows $M_1(P) \cap M_2(R) = K \times C_d$ with $K \leq S_2$ and $C_d \leq C_{q^2-1}$. From $M_1(P) \cap M_2(R) \leq M_2(R)$ follows that $K$ does not contain any element of type (D), so that $K \leq E_q$. Thus, $M_1(P) \cap M_2(R) \leq H$, and $H = M_1(P) \cap M_2(R)$.

Case 3: Let $H = (C_{q+1} \times C_{q^2+1}) \times C_2$. Let $T = \{ P, Q, R \} \subset \text{PG}(2, q^2) \setminus \mathcal{H}_q$ be the self-polar triangle fixed pointwise by $C_{q+1} \times C_{q^2+1}$, and let $P$ be the vertex of $T$ fixed by $C_2$. Then $H \leq M_3(T) \cap M_2(P)$. Conversely, since $M_3(T) \cap M_2(P)$ fixes $P$ and acts on $\{Q, R\}$, the pointwise stabilizer $C_{q+1} \times C_{q^2+1}$ of $T$ has index at most 2 in $M_3(T) \cap M_2(P)$, so that $M_3(T) \cap M_2(P) \leq H$. Thus, $H = M_3(T) \cap M_2(P)$.

Case 4: Let $H = C_{q+1} \times C_{q+1}$. Let $T = \{ P, Q, R \} \subset \text{PG}(2, q^2) \setminus \mathcal{H}_q$ be the self-polar triangle fixed pointwise by $C_{q+1} \times C_{q+1}$. Since $H$ is the whole pointwise stabilizer of $T$ in $G$, we have $H = M_2(P) \cap M_2(Q) \cap M_2(R)$.

Case 5: Let $H = C_{q^2-1}$ and let $\alpha$ be a generator of $H$, with fixed points $P, Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $R \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$. The pointwise stabilizer of $\{P, Q\}$ in $G$ is exactly $H$; thus, $H = M_1(P) \cap M_2(Q)$.

Case 6: Let $H = C_{2(q+1)}$ and let $\alpha$ be a generator of $H$, of type (E), with fixed points $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q \in \ell_P(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q$. By Lemma 3.4 there are $\frac{q}{2}$ self-polar triangles stabilized by the involution $\alpha^{q+1}$ having one vertex in $Q$ and two vertexes on $\ell_Q$; let $T = \{ Q, R_1, R_2 \}$ be one of these triangles. Then $H \leq M_1(P) \cap M_2(Q) \cap M_3(T)$. Conversely, let $\sigma \in (M_1(P) \cap M_2(Q) \cap M_3(T)) \setminus \{1\}$. If $\sigma$ fixes $\{R_1, R_2\}$ pointwise, then from $\sigma \in M_1(P)$ follows that $\sigma$ is in the kernel $C_{q+1} \leq H$ of the action of $M_2(Q)$ on $\ell_Q$. The quotient $(M_1(P) \cap M_2(Q) \cap M_3(T))/C_{q+1}$ acts on $\ell_Q$ as a subgroup of $\text{PSL}(2, q)$ fixing $P$ and interchanging $R_1$ and $R_2$. From [17, Hauptsatz 8.27] follows $(M_1(P) \cap M_2(Q) \cap M_3(T))/C_{q+1} \cong C_2$, and hence $H = M_1(P) \cap M_2(Q) \cap M_3(T)$.

Case 7: Let $H = C_{q+1} = Z(M_2(P))$. Then $H$ is made by the homologies of $G$ with center $P$, together with the identity. Thus, $H = M_1(P_1) \cap M_1(P_2) \cap M_1(P_3)$, where $P_1, P_2, P_3$ are distinct point in $\ell_P \cap \mathcal{H}_q$.\[ \square \]
Case 8: Let $H = E_q$ and let $P$ be the unique point of $H_q(F_q^2)$ fixed by any element in $H$. Then $H = M_2(P_1) \cap M_2(P_2) \cap M_2(P_3)$, where $P_1, P_2, P_3$ are distinct points in $\ell_P(F_q^2) \setminus \{P\}$.

Case 9: Let $H = C_2$, $\alpha$ be a generator of $H$ with fixed point $P \in H_q(F_q^2)$, and $P_1, P_2, P_3 \in \ell_P(F_q^2) \setminus \{P\}$. Let $T = \{P_1, Q_{1,1}, R_{1,1}\}$ be a self-polar triangle stabilized by $\alpha$. Then $H \leq M_2(P_1) \cap M_2(P_2) \cap M_2(P_3) \cap M_3(T)$. Since the elation $\alpha$ is uniquely determined by the image of one point not on its axis $\ell_P$, $H \leq M_3(T)$ implies $H = M_2(P_1) \cap M_2(P_2) \cap M_2(P_3) \cap M_3(T)$.

Case 10: Let $H = C_3$. By Lemma 3.5, $H$ stabilizes $\frac{2(q^2-1)}{3}$ triangles $\tilde{T} \subset H_q(F_q^2)$; let $\tilde{T}_1$ and $\tilde{T}_2$ be two of them. Then $H \leq M_4(\tilde{T}_1) \cap M_4(\tilde{T}_2)$. If $H < M_4(\tilde{T}_1) \cap M_4(\tilde{T}_2)$, then there exist a nontrivial $\sigma \in G$ stabilizing pointwise both $\tilde{T}_1$ and $\tilde{T}_2$, a contradiction to Lemma 2.4. Thus, $H = M_4(\tilde{T}_1) \cap M_4(\tilde{T}_2)$.

Case 11: Let $H = \text{Sym}(3)$. By Lemma 3.6, $H$ stabilizes $q + 1$ self-polar triangles $T_1, \ldots, T_{q+1}$, so that $H \leq M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$. Suppose by contradiction that $H \neq M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$. Then $M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$ contains a nontrivial element $\sigma$ fixing every triangle $T_i$ pointwise. Since the triangles $T_i$’s do not have vertexes in common, this is a contradiction to Lemma 2.4. Thus, $H = M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$.

Case 12: Let $H = \{1\}$. Since $G$ is simple, $H$ is the Frattini subgroup of $G$.

Proposition 3.8. If $H < G$ is the intersection of maximal subgroups, then $H$ is one of the groups in Equation (3.1).

Proof. We proceed as follows: we take every subgroup $K < G$ in Equation (3.1), starting from the maximal subgroups $M_i$ of $G$; we consider the intersections $H = K \cap M_i$ of $K$ with the maximal subgroups of $G$; here, we assume that $K \nleq M_i$. We show that $H$ is again one of the groups in Equation (3.1).

Case 1: Let $K = S_2 \times C_{q^2-1} = M_1(P)$ for some $P \in H_q(F_q^2)$.

Let $H = K \cap M_1(Q)$, $Q \neq P$. Then $H$ is the pointwise stabilizer of $\{P, Q\} \subset H_q(F_q^2)$, which is cyclic of order $q^2 - 1$, i.e. $H = C_{q^2-1}$.

Let $H = K \cap M_2(Q)$. Suppose $Q \in \ell_P$. Then $H = E_{q^2} \times C_{q^2-1}$, where $E_{q^2}$ is made by the elations with axis $PQ$ and $C_{q^2-1}$ is generated by an element of type (B2) with fixed points $Q, P$, and another point $R \in \ell_Q$. Now suppose $Q \notin \ell_P$. Then $H$ stabilizes $\ell_Q$ and hence also the point $R = \ell_P \cap \ell_Q$. Then $H$ stabilizes $QR$ and hence also the pole $A$ of $QR$; by reciprocity, $A \in PQ$. Thus, $H$ fixes three collinear point $A, P, Q$, and hence every point on $AP$. Then $H = C_{q^2-1} = Z(M_2(R))$.

Let $H = K \cap M_3(T)$, $T = \{A, B, C\}$, with $P$ on a side of $T$, say $P \in AB$. Then $H$ fixes $C$ and acts on $\{A, B\}$. Thus, $H$ is generated by an element of type (E) with fixed points $P, C$ and fixed lines $PC, AB$; hence, $H = C_{2(q^2+1)}$.

Let $H = K \cap M_3(T)$, $T = \{A, B, C\}$, with $P$ out of the sides of $T$. By reciprocity, no vertex of $T$ lies on $\ell_P$. This implies that no elation acts on $T$, so that $2 \nmid |H|$; this also implies that no homology in $M_3(T)$ fixes $P$, so that $H$ has no nontrivial elements fixing $T$ pointwise. Thus $H \leq C_3$.

Let $H = K \cap M_4(T)$. By Lagrange’s theorem, $H \leq C_3$.

Case 2: Let $K = \text{PSL}(2, q) \times C_{q+1} = M_2(P)$ for some $P \in \text{PG}(2, q^2) \setminus H_q.$
Let $H = K \cap M_2(Q)$, $Q \neq P$, and $R$ be the pole of $PQ$. If $R \in PQ$, then $H$ is the pointwise stabilizer of $PQ$ and is made by the elations with center $R$; thus, $H = E_q$. If $R \notin PQ$, then $H$ is the pointwise stabilizer of $T = \{P, Q, R\}$; thus, $H = C_{q+1} \times C_{q+1}$.

Let $H = K \cap M_3(T)$ with $T = \{A, B, C\}$. If $P$ is a vertex of $T$, then $H = (C_{q+1} \times C_{q+1}) \times C_2$. If $P$ is on a side of $T$ but is not a vertex, say $P \in AB$, then $H$ fixes the pole $D \in AB$ of $C$. Then $H$ fixes pointwise $T' = \{P, C, D\}$ and acts on $\{A, B\}$. This implies that $H$ fixes $AB$ pointwise and $H = C_{q+1} = Z(M_2(C))$. If $P$ is out of the sides of $T$, then no nontrivial element of $H$ fixes $T$ pointwise; thus, $H \leq \text{Sym}(3)$.

Let $H = K \cap M_4(T)$. By Lagrange’s theorem, $H \leq C_3$.

**Case 3:** Let $K = (C_{q+1} \times C_{q+1}) \times \text{Sym}(3) = M_3(T)$ for some self-polar triangle $T = \{A, B, C\}$.

Let $H = K \cap M_3(T')$ with $T' = \{A', B', C'\} \neq T$. If $T$ and $T'$ have one vertex $A = A'$ in common, then $H = C_{2(q+1)}$ is generated by an element of type (E) fixing $A$ and a point $D \in BC = B'C'$. If $A' \in AC \setminus \{A, C\}$, then $H$ stabilizes $B'C'$, because $B'C'$ is the only line containing 4 points of $\{A, B, C', A', B', C'\}$. Then $H$ fixes $A', A$, and $C$; hence also $B$. Since $H$ acts on $\{B', C'\}$, $H$ cannot be made by nontrivial homologies of center $B$; thus, $H = \{1\}$.

Let $H = K \cap M_4(T)$. By Lagrange’s theorem, $H \leq C_3$.

**Case 4:** Let $K = C_{q+2-q+1} \times C_3 = M_4(T)$ for some $T \in \mathcal{H}_q(\mathbb{F}_q)$. Let $H = K \cap M_4(T')$ with $T' \neq T$. Since 3 does not divide the order of the pointwise stabilized $C_{q+2-q+1}$ of $T$, $H$ contains no nontrivial elements fixing $T$ or $T'$ pointwise. Thus, $H \leq C_3$.

**Case 5:** Let $K = E_q \times C_{q+2}$ and $P \in \mathcal{H}_q(\mathbb{F}_q)$, $Q \in \ell_P \setminus \{P\}$ be the fixed points of $K$.

Let $H = K \cap M_3(R)$ with $R \neq P$. If $R \in \ell_Q$, then $H = C_{q+1}$. If $R \notin \ell_Q$, then $H$ fixes the pole $S$ of $PR$; by reciprocity $S \in PQ$, so that $H$ fixes $PQ$ pointwise and also $R \notin PQ$. Thus, $H = \{1\}$.

Let $H = K \cap M_3(R)$ with $R \notin Q$. If $R \in \ell_P$, then $H$ is the pointwise stabilizer $E_q$ of $PQ$. If $R \notin \ell_P$, then $H$ fixes pointwise the self-polar triangle $\{Q, R, S\}$ where $S$ is the pole of $QR$. Hence, either $H = C_{q+1} = Z(M_2(Q))$ or $H = \{1\}$ according to $P \in RS$ or $P \notin RS$, respectively.

Let $H = K \cap M_3(T)$ with $T = \{A, B, C\}$. If $P$ is on a side of $T$, say $P \in BC$, then either $H = \{1\}$ or $H = C_{q+1} = Z(M_2(A))$. If $P$ is out of the sides of $T$, then no nontrivial element of $H$ can fix $T$ pointwise; thus, $H \leq \text{Sym}(3)$.

Let $H = K \cap M_4(T)$. By Lagrange’s theorem, $H \leq C_3$.

**Case 6:** Let $K = (C_{q+1} \times C_{q+1}) \times C_2 = M_3(T) \cap M_2(A)$, where $T = \{A, B, C\}$.

Let $H = K \cap M_3(P)$. If $P \in BC$, then $H = C_{2(q+1)}$ is generated by an element of type (E). If $P \notin BC$, then $H = \{1\}$.

Let $H = K \cap M_2(P)$, $P \neq A$. If $P \in \{B, C\}$, then $H$ is the pointwise stabilizer $C_{q+1} \times C_{q+1}$ of $T$. If $P \in AB \setminus \{A, B\}$ or $P \in AC \setminus \{A, C\}$, then $H = C_{q+1} = Z(M_2(C))$ or $H = C_{q+1} = Z(M_2(B))$, respectively. If $P \in BC \setminus \{B, C\}$, then $H$ fixes $A$, $P$, the pole of $AP$, and acts on $\{B, C\}$; thus, $H = C_{q+1} = Z(M_2(A))$. If $P$ is not on the sides of $T$, then no nontrivial element of $H$ can fix $T$ pointwise; thus, $H \leq C_2$.

Let $H = K \cap M_3(T')$ with $T' = \{A', B', C'\} \neq T$. Since $3 \mid |H|$, $H$ fixes a vertex of $T'$, say $A'$. If $A' = A$, then $H = C_{2(q+1)}$. If $A' \in \{B, C\}$, then $H$ fixes $T$ pointwise and acts on $\{B', C'\}$; thus, $H = C_{q+1} = Z(M_2(A'))$. If $A' \in (AB \cup AC) \setminus \{A, B, C\}$, then $H$ fixes $AB$ or $AC$ pointwise and acts on $\{B', C'\}$; thus, $H = \{1\}$. If $A' \in BC$, then $H$
Proposition 3.9. The values $\mu(H)$ for the groups in Equation (3.1) are given in Equation (3.3).

Proof. Let $H$ be one of the groups in Equation (3.1). By Lemma 2.1 and Proposition 3.8, $\mu(H)$ only depends on the subgroups $K$ of $G$ such that $H < K$ and $K$ is in Equation (3.1).

Case 1: If $H$ is one of the first four groups in Equation (3.1), then $H$ is maximal in $G$, and hence $\mu(H) = -1$.

Case 2: Let $H = E_q \rtimes C_{q^2-1}$. Let $P \in H_q(\mathbb{F}_{q^2})$ and $Q \in \text{PG}(2, q^2) \setminus H_q$ be the fixed points of $H$. Then $H = M_1(P) \cap M_2(Q)$ and $H$ is not contained in any other maximal
subgroup of $G$. Thus, $\mu(H) = -\{\mu(G) + \mu(M_1(P)) + \mu(M_2(Q))\} = 1$.

**Case 3:** Let $H = (C_{q+1} \times C_{q+1}) \rtimes C_2$. Let $T = \{P, Q, R\}$ be the self-polar triangle stabilized by $H$, with $H(P) = P$. No point different from $P$ is fixed by $H$. Also, if a triangle $T' = \{P', Q'\} \neq T$ is fixed by $H$, then $P$ is a vertex of $T'$, say $P = P'$, and $\{Q', R'\} \subset QR$; but $C_{q+1} \times C_{q+1}$ has orbits of length $q + 1 > |\{Q', R'\}|$, so that $H$ cannot fix $T'$. Then $H = M_2(P) \cap M_3(T)$ and $H$ is not contained in any other maximal subgroup of $G$. Thus, $\mu(H) = 1$.

**Case 4:** Let $H = C_{q+1} \times C_{q+1}$ and $T = \{P, Q, R\}$ be the self-polar triangle fixed pointwise by $H$. The vertices of $T$ are the unique fixed points of the elements of type (B1) in $H$. Also, any triangle $T' \neq T$ fixed by an element of type (A) in $H$ has two vertices on a side $\ell$ of $T'$; but $H$ has orbits of length $q + 1 > 2$ on $\ell$, so that $H$ does not fix $T'$. Then $H = M_3(T) \cap M_2(P) \cap M_2(Q) \cap M_2(R)$ and $H$ is not contained in any other maximal subgroup of $G$.

If $K$ is one of the groups $M_3(T) \cap M_2(P), M_3(T) \cap M_2(Q), M_3(T) \cap M_2(R)$, then $K$ contains $H$ properly, and $\mu(K) = 1$ as shown in the previous point. The intersection of three groups between $M_3(T), M_2(P), M_2(Q),$ and $M_2(R)$ is equal to $H$. Thus, by direct computation, $\mu(H) = 0$.

**Case 5:** Let $H = C_{q-1}$ with fixed points $P \in PG(2,q^2) \setminus Q, R \in H_q(\mathbb{F}_{q^2})$. Then $H = M_1(Q) \cap M_1(R) = M_1(Q) \cap M_1(R) \cap M_2(P)$. We already know $\mu(M_1(Q) \cap M_2(P)) = \mu(M_1(R) \cap M_2(P)) = 1$. Moreover, $C_{q-1}$ has no fixed triangles, by Lagrange’s theorem, and no other fixed points. Thus, by direct computation, $\mu(H) = 0$.

**Case 6:** Let $H = C_{q+1} = \langle \alpha \rangle$; $\alpha$ is of type (E), fixes the points $P \in H_q(\mathbb{F}_{q^2})$ and $Q \in PG(2,q^2) \setminus H_q$, and fixes the lines $\ell_P$ and $\ell_Q$. Since $\alpha^2$ is a homology with center $Q$, the orbits on $\ell_Q$ of $H$ coincide with the orbits on $\ell_Q$ of the elation $\alpha^{q+1}$. By Lemma 3.4, the self-polar triangles $T_i$ stabilized by $H$ have a vertex in $Q$ and two vertices on $\ell_Q$; there are exactly $q+1$ such triangles $T_1, \ldots, T_{q+1}$. No other triangle and no other point different from $P$ and $Q$ is fixed by $H$, so that $H = M_1(P) \cap M_2(Q) \cap M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$ and $H$ is not contained in any other maximal subgroup of $G$.

If $K$ is the intersection of $M_2(Q)$ with one of the groups $M_1(P), M_3(T_1), \ldots, M_3(T_{q+1})$, then $K = E_q \rtimes C_{q-1}$ or $K = (C_{q+1} \times C_{q+1}) \rtimes C_2$; hence, $K$ contains $H$ properly and $\mu(K) = 1$ as shown above. The intersection of $K$ with a third maximal subgroup of $G$ containing $H$ coincides with $H$. Finally, the intersection of any two groups in $\{M_1(P), M_3(T_1), \ldots, M_3(T_{q+1})\}$ coincides with $H$. Thus, by direct computation, $\mu(H) = 0$.

**Case 7:** Let $H = C_{q+1} = Z(M_2(P))$. Denote $\ell_P \cap \mathcal{H}_q = \{P_1, \ldots, P_{q+1}\}$ and $\ell_{(\mathbb{F}_{q^2})} \setminus \mathcal{H}_q = \{Q_1, \ldots, Q_{q^2-q}\}$ such that, for $i = 1, \ldots, q^2-q$, $T_i = \{P, Q_i, Q_i+q^2-q\}$ are the self-polar triangles with a vertex in $P$. Then

$$H = \bigcap_{i=1}^{q+1} M_1(P_i) \cap M_2(P) \cap \bigcap_{i=1}^{q^2-q} M_2(Q_i) \cap \bigcap_{i=1}^{(q^2-q)/2} M_3(T_i)$$

and $H$ is not contained in any other maximal subgroup of $G$. By direct inspection, the intersections $K$ of some (at least two) maximal subgroups of $G$ such that $H < K < G$ are exactly the following.
(i) $K = M_1(P_i) \cap M_1(P_j)$ for some $i \neq j$; in this case, $K = C_{q^2-1}$ and $\mu(K) = 0$.

(ii) $K = M_1(P_i) \cap M_2(P)$ with $i \in \{1, \ldots, q+1\}$; in this case, $K = E_q \rtimes C_{q^2-1}$ and $\mu(K) = 1$. These $q + 1$ groups are pairwise distinct.

(iii) $K = M_1(P_i) \cap M_3(T_j)$ for some $i, j$; in this case, $K = C_{2(q+1)}$ and $\mu(K) = 0$.

(iv) $K = M_2(P) \cap M_2(Q_i)$ for some $i$; in this case, $K = C_{q+1} \times C_{q+1}$ and $\mu(K) = 0$.

(v) $K = M_2(P) \cap M_3(T_i)$ with $i \in \{1, \ldots, \frac{q^2-q}{2}\}$; in this case, $K = (C_{q+1} \times C_{q+1}) \rtimes C_2$ and $\mu(K) = 1$. These $\frac{q^2-q}{2}$ groups are pairwise distinct.

(vi) $K = M_2(Q_i) \cap M_3(T_i)$ or $K = M_2(Q_{i+\frac{q^2-q}{2}}) \cap M_3(T_i)$, with $i \in \{1, \ldots, \frac{q^2-q}{2}\}$; in this case, $K = (C_{q+1} \times C_{q+1}) \rtimes C_2$ and $\mu(K) = 0$. These $q^2 - q$ groups are pairwise distinct.

To sum up, the only subgroups $K$ with $H < K < G$ and $\mu(K) \neq 0$ are the maximal subgroups, $q + 1$ distinct groups of type $E_q \rtimes C_{q^2-1}$, and $\frac{3(q^2-q)}{2}$ distinct groups of type $(C_{q+1} \times C_{q+1}) \rtimes C_2$. Thus, $\mu(H) = 0$.

**Case 8:** Let $H = E_q$. Let $P$ be the point of $H_q(\mathbb{F}_{q^2})$ fixed by $H$; $H$ fixes $\ell_P$ pointwise. We have $H = M_1(P) \cap M_2(Q_1) \cap \cdots \cap M_2(Q_{q^2})$, where $Q_1, \ldots, Q_{q^2}$ are the $\mathbb{F}_{q^2}$-rational points of $\ell_P \setminus \{P\}$; $H$ is not contained in any other maximal subgroup of $G$. The intersections $K$ of at least two maximal subgroups of $G$ such that $H < K < G$ are exactly the $q^2$ groups $M_1(P) \cap M_2(Q_i) = E_q \rtimes C_{q^2-1}$, with $\mu(K) = 1$. Thus, by direct computation, $\mu(H) = 0$.

**Case 9:** Let $H = \text{Sym}(3) = \langle \alpha, \beta \rangle$ with $o(\alpha) = 3$ and $o(\beta) = 2$. Let $P \in \text{PG}(2, q^2) \setminus H_q$ and $Q, R \in H_q$ be the fixed points of $\alpha$, and $A \in QR$ be the fixed point of $\beta$ on $H_q$, so that $\beta$ fixes $\ell_A = AP$. By Lemma 3.6 and its proof, $H = M_2(P) \cap M_3(T_1) \cap \cdots \cap M_3(T_{q+1})$, where $T_i$ has one vertex on $\ell_A \setminus \{P, A\}$ and the other two vertices are collinear with $A$; $H$ is not contained in any other maximal subgroup of $G$.

For any $i, j \in \{1, \ldots, q+1\}$ with $i \neq j$, no vertex of $T_j$ is on a side of $T_i$; hence, no nontrivial element of $M_3(T_i) \cap M_3(T_j)$ fixes $T_i$ pointwise. This implies $M_3(T_i) \cap M_3(T_j) = H$. Analogously, no nontrivial element in $M_3(T_i) \cap M_2(P)$ fixes $T_i$ pointwise, and this implies $M_3(T_i) \cap M_2(P) = H$. Thus, by direct computation, $\mu(H) = q + 1$.

**Case 10:** Let $H = C_3 = \langle \alpha \rangle$ with fixed points $P \in \text{PG}(2, q^2) \setminus H_q$ and $Q, R \in H_q$. By Lemma 3.5,

$$H = M_1(Q) \cap M_1(R) \cap M_2(P) \cap \bigcap_{i=1}^{(q^2-1)/3} M_3(T_i) \cap \bigcap_{i=1}^{2(q^2-1)/3} M_4(T_i)$$

and $H$ is not contained in any other maximal subgroup of $G$. By direct inspection, the intersections $K$ of at least two maximal subgroups of $G$ such that $H < K < G$ are exactly the following.

(i) $K = M_1(Q) \cap M_2(P)$ or $K = M_1(R) \cap M_2(P)$; in this case, $K = E_q \rtimes C_{q^2-1}$ and $\mu(K) = 1$.

(ii) $K = M_1(Q) \cap M_1(R)$; in this case, $K = C_{q^2-1}$ and $\mu(K) = 0$. 
(iii) There are exactly \( \frac{q-1}{3} \) groups \( K \) containing \( H \) with \( K \cong \text{Sym}(3) \), and hence \( \mu(K) = q + 1 \). In fact, any involution \( \beta \in G \) satisfying \( \langle H, \beta \rangle \cong \text{Sym}(3) \) interchanges \( Q \) and \( R \) and fixes a point of \( (QR \cap H_q) \setminus \{P, Q\} \); conversely, any of the \( q-1 \) points \( A_1, \ldots, A_{q-1} \) of \( (QR \cap H_q) \setminus \{P, Q\} \) determines uniquely the involution \( \beta_i \in G \) such that \( \beta(A_i), \beta_i(Q) = R, \beta_i(R) = Q \), and hence \( \langle H, \beta_i \rangle \cong \text{Sym}(3) \). The involutions \( \beta_i, \alpha \beta_i, \) and \( \alpha^2 \beta_i \), together with \( H \), generate the same group; thus, there are exactly \( \frac{q-1}{3} \) groups \( \text{Sym}(3) \) containing \( H \).

Thus, by direct computation, \( \mu(H) = \frac{2(q^2-1)}{3} \).

**Case 11:** Let \( H = C_2 = \langle \alpha \rangle \), where \( \alpha \) has center \( P \). Let \( \ell_P(\mathbb{F}_{q^2}) \setminus \{P\} = \{P_1, \ldots, P_{q^2}\} \).

By Lemma 3.4,

\[
H = M_1(P) \cap \bigcap_{i=1}^{q^2} M_2(P_i) \cap \bigcap_{i=1}^{q^2} M_3(T_{i,j}),
\]

where the triangles \( T_{i,j} \) are described in Lemma 3.4; \( H \) is not contained in any other maximal subgroup of \( G \). By direct inspection, the intersections \( K \) of at least two maximal subgroups of \( G \) such that \( H < K < G \) are exactly the following.

(i) \( K = M_1(P) \cap M_2(P_i) \) for \( i = 1, \ldots, q^2 \); in this case, \( K = E_q \rtimes C_{q^2-1} \) and \( \mu(K) = 0 \).

(ii) \( K = M_2(P_i) \cap M_2(P_j) \) with \( i \neq j \); in this case, \( K = E_q \) and \( \mu(K) = 0 \).

(iii) \( K = M_1(P) \cap M_3(T_{i,j}) \); in this case, \( K = E_q \times C_{2(q+1)} \) and \( \mu(K) = 0 \).

(iv) \( K = M_2(Q_i) \cap M_3(T_{i,j}) \) with \( i \in \{1, \ldots, q^2\} \) and \( j \in \{1, \ldots, q/2\} \); these \( 2q^3 \) distinct groups are of type \( (C_{q+1} \times C_{q+1}) \rtimes C_2 \), so that \( \mu(K) = 1 \).

(v) There are exactly \( N = \frac{q^3}{2} \) groups \( K \) containing \( H \) such that \( K \cong \text{Sym}(3) \), and hence \( \mu(K) = q + 1 \). This follows by double counting the size of

\[ I = \{(H, K) \mid H, K < G, \ H \cong C_2, \ K \cong \text{Sym}(3), \ H < K\}. \]

Arguing as in the proof of Lemma 3.4, \( |I| = (q^3+1)(q-1)N \); arguing as in the proof of Lemma 3.6, \( |I| = \frac{q^3(q^3+1)(q-1)}{6} \cdot 3 \). Hence, \( N = \frac{q^3}{2} \).

Thus, by direct computation, \( \mu(H) = -\frac{q^3(q+1)}{2} \).

**Case 12:** Let \( H = \{1\} \). Then \( \mu(H) = -\sum_{1 \leq K \leq G} \mu(K, G) \). By the values \( \mu(K) \) computed in the previous cases, Propositions 3.2, and Proposition 3.3, only the following groups \( K \) have to be considered:

(i) 1 group \( G \);

(ii) \( q^3 + 1 \) groups \( S_2 \rtimes C_{q^2-1} \);

(iii) \( q^2(q^2 - q + 1) \) groups \( \text{PSL}(2, q) \rtimes C_{q+1} \);

(iv) \( \frac{q^3(q-1)(q^2-q+1)}{6} \) groups \( (C_{q+1} \times C_{q+1}) \rtimes \text{Sym}(3) \);

(v) \( \frac{q^3(q+1)^2(q-1)}{3} \) groups \( C_{q^2-q+1} \rtimes C_3 \);

(vi) \( (q^3 + 1)q^2 \) groups \( E_q \rtimes C_{q^2-1} \).
(vii) \(\frac{q^3(q-1)(q^2-q+1)}{2}\) groups \((C_{q+1} \times C_{q+1}) \rtimes C_2\);
(viii) \(\frac{q^3(q^3+1)(q-1)}{6}\) groups \(\text{Sym}(3)\);
(ix) \(\frac{q^3(q^3+1)}{2}\) groups \(C_3\);
(x) \((q^3 + 1)(q - 1)\) groups \(C_2\).

Thus, by direct computation, \(\mu(H) = 0\).

4 Determination of \(\lambda(H)\) for any subgroup \(H\) of \(G\)

Let \(n > 0, q = 2^{2n}, G = \text{PSU}(3, q)\). This section is devoted to the proof of the following theorem.

**Theorem 4.1.** Let \(H\) be a proper subgroup of \(G\). Then \(\lambda(H) \neq 0\) if and only \(H\) is one of the following groups:

\[
\begin{align*}
&E_q \rtimes C_{q^2-1}, & (C_{q+1} \times C_{q+1}) \rtimes C_2, & \text{Sym}(3), \\
&C_3, & S_2 \rtimes C_{q^2-1}, & \text{PSL}(2, q) \times C_{q+1}, \quad (4.1) \\
&(C_{q+1} \times C_{q+1}) \rtimes \text{Sym}(3), & C_{q^2-q+1} \rtimes C_3, & C_2.
\end{align*}
\]

For any isomorphism type in Equation (4.1) there is just one conjugacy class of subgroups of \(G\).

If \(H\) is in the first row of Equation (4.1), then \(\lambda(H) = -1\); if \(H\) is in the second row of Equation (4.1), then \(\lambda(H) = 1\).

**Proof.** By Proposition 3.2, for any isomorphism type in Equation (4.1) there is just one conjugacy class of subgroups of \(G\) of that type. Hence, we can use the notation \([M_1]\), \([M_2]\), \([M_3]\) and \([M_4]\) for the conjugacy classes of \(M_1(P), M_2(P), M_3(T)\) and \(M_4(T)\), respectively. If \(H = G\), then \(\lambda(H) = 1\); if \(H\) is one of the groups in the second row of Equation (4.1) and \(H \neq C_2\), then \(\lambda(H) = -1\) as \(H\) is maximal in \(G\).

**Case 1:** Firstly, we assume that \(H\) is not a subgroup of \(\text{Sym}(3)\), and that \(H\) is not a group of homologies, i.e. \(H \nsubseteq C_{q+1} = Z(M_2(Q))\) for any point \(Q\).

(i) Let \(H < M_4(T)\) for some \(T\). From \(H \neq C_3\) follows that some nontrivial element in \(H\) fixes \(T\) pointwise; hence, \(H\) is not contained in any maximal subgroup of \(G\) other than \(M_4(T)\). Thus, inductively, \(\lambda(H) = -\{\lambda(G) + \lambda(M_4(T))\} = 0\).

(ii) Let \(H < M_1(P)\) for some \(P\); we assume in addition that \(\gcd(|H|, q-1) > 1\). Here, the assumption \(H \nsubseteq \text{Sym}(3)\) reads \(H \notin \{\{1\}, C_2, C_3\}\). If \(H\) contains an element of order 4, then \(H\) is not contained in any maximal subgroup of \(G\) other than \(M_1(P)\). Thus, inductively, \(\lambda(H) = 0\).

We can then assume that the 2-elements of \(H\) are involutions, so that \(H = E_{2^r} \rtimes C_d\) with \(0 \leq r \leq 2^n\) and \(d \mid (q^2 - 1)\) (see [15, Theorem 11.49]). This implies that \(H \leq M_1(P) \cap M_2(Q)\) for some \(Q \in \mathcal{P}\); the eventual nontrivial elements in \(H\) whose order divides \(q + 1\) are homologies with center \(Q\). Then we have \([H] \leq [M_1], [H] \leq [M_2]\); by Lagrange’s theorem, \([H] \nleq [M_4]\). From the assumptions \(\gcd(|H|, q-1) > 1\) and \(H \nsubseteq \text{Sym}(3)\) follows \([H] \nleq [M_3]\).

If \(H = E_q \rtimes C_{q^2-1}\), then no proper subgroup of \(M_1(P)\) or \(M_2(Q)\) contains \(H\) properly; thus, \(\lambda(H) = 1\). If \(H \neq E_q \rtimes C_{q^2-1}\), then \(H < E_q \rtimes C_{q^2-1} = M_1(P) \cap\)
Let $H < M_2(Q)$ for some $Q$, and assume also $H \not\leq M_1(P)$ for any $P$. As $H \not\leq C_3$, we have $[H] \not\leq [M_4]$. The group $\tilde{H} := H/(H \cap Z(M_2(Q)))$ acts as a subgroup of $\text{PSL}(2,q)$ on $\ell_Q \cap H_q$; we assume in this point that $H$ is one of the following groups (see [17, Hauptsatz 8.27]): $\text{PSL}(2,2^{2h})$ with $0 < h \leq n$; a dihedral group of order $2d$ where $d$ is a divisor of $q - 1$ greater than 3; $\text{Alt}(5)$. Then, by Lagrange’s theorem, $[H] \not\leq [M_3]$. Thus, inductively, $G$ and $M_2(Q)$ are the only groups $K$ with $H < K$ and $\lambda(K) \neq 0$, so that $\lambda(H) = 0$.

Note that, since we are under the assumptions $H \not\leq M_1(P)$ for any $P$, $H \not\leq \text{Sym}(3)$, and $H \not\leq C_{q+1} = Z(M_2(Q))$, we have that the only subgroups $\tilde{H}$ of $\text{PSL}(2,q)$ for which $\lambda(H)$ still has not been computed are the cyclic or dihedral groups of order $d$ or $2d$ (respectively), where $d$ is a nontrivial divisor of $q + 1$.

Let $H < M_3(T)$ for some $T$, and assume also $H \not\leq M_1(P)$ for any $P$. As $H \not\leq C_3$, we have $[H] \not\leq [M_4]$. Here, the assumption $H \not\leq \text{Sym}(3)$ means that some nontrivial element of $H$ fixes $T$ pointwise. Hence, the assumption $H \not\leq C_{q+1} = Z(M_2(Q))$ for any vertex $Q$ of $T$, together with $H \not\leq M_1(P)$, implies that $H$ contains some element of type $(B1)$. Write $H = L \rtimes K$, with $L \leq \text{Sym}(3)$ and $L < C_{q+1} \times C_{q+1}$.

If $K = C_3$ or $K = \text{Sym}(3)$, then $[H] \not\leq [M_2]$; thus, inductively, $G$ and $M_3(T)$ are the only groups $K$ with $H < K$ and $\lambda(K) \neq 0$, so that $\lambda(H) = 0$.

If $K = C_2$ and $L = C_{q+1} \times C_{q+1}$, then $H \leq M_2(Q)$ for some vertex $Q$ of $T$. Since $\tilde{H} := H/(H \cap Z(M_2(Q)))$ is dihedral of order $2(q + 1)$, [17, Hauptsatz 8.27] implies the non-existence of groups $K$ with $H < K < M_2(Q)$ (except for $q = 4$ and $K = \text{Alt}(5)$; in this case, $\lambda(K) = 0$ by the previous point). Thus, $\lambda(H) = -\{\lambda(G) + \lambda(M_2(Q)) + \lambda(M_3(T))\} = 1$.

If $K = C_2$ and $L < C_{q+1} \times C_{q+1}$, then again $H \leq M_2(Q)$ with $Q$ vertex of $T$.

The group $\tilde{H}$ is dihedral of order $2d$, where $d \mid (q + 1)$; $d > 1$ because $L$ contains elements of type $(B1)$. By the previous point and [17, Hauptsatz 8.27], the only groups $K$ with $H < K < M_2(Q)$ are such that $\tilde{K}$ is dihedral of order dividing $q + 1$. Thus, inductively, $\lambda(K) = 0$.

If $K = \{1\}$, then $H \leq M_2(Q)$ for any vertex $Q$ of $T$. The group $\tilde{H} < \text{PSL}(2,q)$ on the line $\ell_Q \cap H_q$ is cyclic of order $d \mid (q + 1)$; $d > 1$ because $H$ has elements of type $(B1)$. By [17, Hauptsatz 8.27], the groups $K$ with $H < K < M_2(Q)$ are such that either $\tilde{K}$ is cyclic of order dividing $q + 1$, or we have already proved that $\lambda(K) = 0$. Thus, inductively, $\lambda(K) = 0$.

Let $H < M_2(Q)$ for some $Q$. Let $\tilde{H} \neq \{1\}$ be the induced subgroup of $\text{PSL}(2,q)$ acting on $\ell_Q \cap H_q$. If $\tilde{H}$ is cyclic or dihedral of order $d$ or $2d$ (respectively) with $d \mid (q + 1)$, then $\tilde{H} \leq M_3(T)$ for some $T$. Hence, $\lambda(H) = 0$, as already computed in the previous point in the case $K = \{1\}$ if $\tilde{H}$ is cyclic, or in the case $K = C_2$ if $H$ is dihedral.

Under the assumptions that $H \not\leq \text{Sym}(3)$ and $H$ is not a group of homologies, the only remaining case is $H < M_1(P)$ for some $P$ with $\gcd(|H|, q - 1) = 1$. In this case $H = E_{2r} \times C_d$, where $C_d$ is cyclic of order $d \mid (q + 1)$ and made by homologies, whose axis passes through $P$ and whose center $Q$ lies on $\ell_P$. We have $r > 0$, because $H \not\leq Z(M_2(Q))$. 


If $r = 1$, then $H$ is cyclic of order $2d$ generated by an element of type (E). By Lemma 3.4, $H \leq M_3(T)$, where $T$ has a vertex in $Q$ and two vertexes on $\ell_Q$. Hence, $[H] \leq [M_1], [H] \leq [M_2], [H] \leq [M_3]$, and $[H] \not\leq [M_4]$. Let $K$ be such that $H < K \leq G$ and $K$ is not of the same type of $H$, i.e. $K$ is not cyclic of order $2d'$ with $d' \mid (q + 1)$. As shown in the previous points, $\lambda(K) \neq 0$ if and only if $[K] \in \{[G], [M_1], [M_2], [M_3], [E_q \rtimes C_{q^2 - 1}], [(C_{q+1} \times C_{q+1}) \rtimes C_2]\}$. Thus, inductively, $\lambda(H) = 0$.

**Case 2:** Let $H = C_{q+1} = Z(M_2(Q))$ for some $Q$ and $K$ be a subgroup of $G$ properly containing $H$. As shown above, $\lambda(K) \neq 0$ if and only if

$$[K] \in \{[G], [M_1], [M_2], [M_3], [E_q \rtimes C_{q^2 - 1}], [(C_{q+1} \times C_{q+1}) \rtimes C_2]\}.$$ Thus $\lambda(Z(M_2(Q))) = 0$ and, inductively, $\lambda(H) = 0$.

**Case 3:** Let $H = \text{Sym}(3) = \langle \alpha \rangle \rtimes \langle \beta \rangle$ with $o(\alpha) = 3$ and $o(\beta) = 2$. Let $P \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$ and $Q, R \in \mathcal{H}_q(\mathbb{F}_{q^2})$ be the fixed point of $\alpha$, so that $\beta$ fixes $P$ and interchanges $Q$ and $R$. This implies $[H] \leq [M_2]$. By Lemma 3.6, $[H] \leq [M_3]$. From the computations above and Lagrange’s theorem, no class $[K]$ with $K \leq G$ other than $[G], [M_2]$ and $[M_3]$ satisfies $[H] \leq [K]$ and $\lambda(H) \neq 0$. Thus, $\lambda(H) = 1$.

**Case 4:** Let $H = C_3$. By Lagrange’s theorem and Proposition 3.2, $H < K \leq G$ and $\lambda(K) \neq 0$ if and only if

$$[K] \in \{[G], [M_1], [M_2], [M_3], [M_4], [E_q \rtimes C_{q^2 - 1}], [\text{Sym}(3)]\}.$$ Thus, $\lambda(H) = 1$.

**Case 5:** Let $H = C_2$. By Lagrange’s theorem and Proposition 3.2, $H < K \leq G$ and $\lambda(K) \neq 0$ if and only if

$$[K] \in \{[G], [M_1], [M_2], [M_3], [E_q \rtimes C_{q^2 - 1}], [(C_{q+1} \times C_{q+1}) \rtimes C_2], [\text{Sym}(3)]\}.$$ Thus, $\lambda(H) = -1$.

**Case 6:** Let $H = \{1\}$. Collecting all the classes $[K]$ with $\lambda(K) \neq 0$, we have by direct computation $\lambda(H) = 0$.

5 Determination of $\chi(\Delta(L_p \setminus \{1\}))$ for any prime $p$

Let $n > 0$, $q = 2^n$, $G = \text{PSU}(3, q)$. If $p$ is a prime number, we denote by $L_p$ the poset of $p$-subgroups of $G$ ordered by inclusion, by $L_p \setminus \{1\}$ its subposet of proper $p$-subgroups of $G$, and by $\Delta(L_p \setminus \{1\})$ the order complex of $L_p \setminus \{1\}$. In this section we determine the Euler characteristic $\chi(\Delta(L_p \setminus \{1\}))$ of $\Delta(L_p \setminus \{1\})$ for any prime $p$, using Equation (2.1) and Lemma 2.2. The results are stated in Theorem 5.1 and in Table 2.

**Theorem 5.1.** For any prime number $p$ one of the following cases holds:

(i) $p \nmid |G|$ and $\chi(\Delta(L_p \setminus \{1\})) = 0$;

(ii) $p = 2$ and $\chi(\Delta(L_2 \setminus \{1\})) = q^3 + 1$;

(iii) $p \mid (q + 1)$ and $\chi(\Delta(L_p \setminus \{1\})) = -q^{p-2}q^8 - 2q^4 + 2q^3 - 3q^2$;
(iv) $p \mid (q - 1)$ and $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^6 + q^3}{2}$;

(v) $p \mid (q^2 - q + 1)$ and $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^6 + q^5 - q^4 - q^3}{3}$.

**Proof.** Since $|G| = q^3(q+1)^2(q-1)(q^2 - q + 1)$, $q$ is even, and $3 \mid (q - 1)$, the cases $p \mid |G|, \ p = 2, p \mid (q + 1), p \mid (q - 1),$ and $p \mid (q^2 - q + 1)$ are exhaustive and pairwise incompatible. We denote by $S_p$ a Sylow $p$-subgroup of $G$.

**Case 1:** Let $p \mid |G|$. Then $\Delta(L_p \setminus \{1\}) = \emptyset$, and hence $\chi(\Delta(L_p \setminus \{1\})) = \chi(\emptyset) = 0$.

**Case 2:** Let $p = 2$. The group $G$ has $q^3 + 1$ Sylow 2-subgroups, and any two of them intersect trivially; see [15, Theorem 11.133]. Any nontrivial element $\sigma$ of $S_2$ fixes exactly one point $P$ on $H_q(\mathbb{F}_q^2)$ which is the same for any $\sigma \in S_2$; $S_2$ is uniquely determined among the Sylow 2-subgroups of $G$ by $P$. Hence, Equation (2.1) reads

$$\chi(\Delta(L_2 \setminus \{1\})) = -(q^3 + 1) \sum_{H \in L_2 \setminus \{1\}, \ H(P) = P} \mu_{L_2}(\{1\}, H),$$

where $P$ is a given point of $H_q(\mathbb{F}_q^2)$. By Lemma 2.2, we only consider those 2-groups in $M_1(P)$ which are elementary abelian. Then we consider all nontrivial subgroups $H$ of an elementary abelian 2-group $E_q$ of order $q$. For any such group $H = E_{2^r}$ of order $2^r$, with $1 \leq r \leq 2^n$, we have $\mu_{L_2}(\{1\}, H) = (-1)^r \cdot 2^{\binom{r}{2}}$ by Lemma 2.2. Thus,

$$\chi(\Delta(L_2 \setminus \{1\})) = -(q^3 + 1) \sum_{r=1}^{2^n} (-1)^r 2^{\binom{r}{2}} \binom{2^n}{r} \binom{2^n}{2^r}$$

where the Gaussian coefficient $\binom{2^n}{2^r}$ counts the subgroups of $E_q$ of order $2^r$. Using the property

$$\binom{2^n}{r} = \binom{2^n}{r - 1} + 2^r \binom{2^{n-1}}{r}$$

we obtain

$$\sum_{r=1}^{2^n} (-1)^r 2^{\binom{r}{2}} \binom{2^n}{r} = \sum_{r=1}^{2^n} (-1)^r 2^{\binom{r}{2}} \binom{2^n}{r - 1} + \sum_{r=1}^{2^n} (-1)^r 2^{\binom{r}{2}} \binom{2^n}{r}$$

$$= \sum_{r=0}^{2^n-1} (-1)^{r+1} 2^{\binom{r+1}{2}} \binom{2^n}{r - 1} + \sum_{r=1}^{2^n} (-1)^{r+1} 2^{\binom{r+1}{2}} \binom{2^n}{r}$$

$$= (-1)^0 2^{\binom{1}{2}} \binom{2^n - 1}{0} + (-1)^{2^n} 2^{\binom{2^n+1}{2}} \binom{2^n - 1}{2^n} = -1.$$

Thus, $\chi(\Delta(L_2 \setminus \{1\})) = q^3 + 1$.

**Case 3:** Let $p \mid (q + 1)$. Then $S_p \leq C_{q+1} \times C_{q+1}$, and hence $S_p \cong C_{p^s} \times C_{p^s}$, where $p^s \mid (q + 1)$ and $p^{s+1} \nmid (q + 1)$. Let $H$ be a subgroup of $S_p$. By Lemma 2.2, $\mu_{L_p}(\{1\}, H) \neq 0$ only if $H$ is elementary abelian of order $p$ or $p^2$; in this case, $\mu_{L_p}(\{1\}, C_p) = -1$ and $\mu_{L_p}(\{1\}, C_p \times C_p) = r$. Now we count the number of elementary abelian subgroups of order $p$ or $p^2$ in $G$. 


(i) A subgroup $E_{p^2}$ of $G$ of type $C_p \times C_p$ is uniquely determined by the maximal subgroup $M_3(T)$ such that $E_{p^2}$ is the Sylow $p$-subgroup of $M_3(T)$. Hence, $G$ contains exactly $[G : N_G(M_3(T))] = \frac{q^3(q^2 - q + 1)(q-1)(p - 2)}{6}$ elementary abelian subgroups of order $p^2$.

(ii) A subgroup $C_p$ made by homologies is uniquely determined by its center $P \in \mathrm{PG}(2,q^2) \setminus \mathcal{H}_q$ of homology, because the group of homologies with center $P$ is cyclic. Hence, $G$ contains exactly $|\mathrm{PG}(2,q^2) \setminus \mathcal{H}_q| = q^2(q^2 - q + 1)$ cyclic subgroups of order $p$ made by homologies.

(iii) A subgroup $C_p$ which is not made by homologies is made by elements of type (B1), and fixes pointwise a unique self-polar triangle $T$. The Sylow $p$-subgroup $C_p \times C_p$ of $M_3(T)$ contains exactly 3 subgroups $C_p$ made by homologies, namely the groups of homologies with center one of the vertexes of $T$. Since $C_p \times C_p$ contains $p + 1$ subgroups $C_p$ altogether, $C_p \times C_p$ contains exactly $p - 2$ subgroups $C_p$ not made by homologies. Thus, the number of subgroups $C_p$ of $G$ not made by homologies is $(p - 2) \cdot |G : N_G(M_3(T))| = \frac{q^3(q^2 - q + 1)(q-1)(p - 2)}{6}$.

Thus, by direct computation,

$$
\chi(\Delta(L_p \setminus \{1\})) = -\left\{ \frac{q^3(q^2 - q + 1)(q-1)(p - 2)}{6} \cdot r + \left[ q^2(q^2 - q + 1) + \frac{q^3(q^2 - q + 1)(q-1)(p - 2)}{6} \right] \cdot (-1) \right\}
= -\frac{q^6 - 2q^5 - q^4 + 2q^3 - 3q^2}{3}.
$$

**Case 4:** Let $p \mid (q - 1)$. By Lemma 2.4, $S_p$ is a subgroup of the cyclic group $C_{q^2-1}$ fixing two points $P, Q$ on $\mathcal{H}_q(\mathbb{F}_{q^2})$; then a proper $p$-subgroup $H$ of $G$ satisfies $\mu_{L_p}(\{1\}) \neq 0$ if and only if $H$ has order $p$; in this case, $\mu_{L_p}(\{1\}, H) = -1$. Also, by Lemma 2.4, any two Sylow $p$-subgroups of $G$ have trivial intersection. Then the number of subgroups $C_p$ of $G$ is equal to the number of couples of points in $\mathcal{H}_q(\mathbb{F}_{q^2})$; equivalently, this number is equal to $|G : N_G(C_{q^2})|$, where $|N_G(C_{q^2-1})| = 2(q^2 - 1)$ by Proposition 3.3. Thus, $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^6 + q^3}{2}$.

**Case 5:** Let $p \mid (q^2 - q + 1)$. Then $S_p \leq C_{q^2-q+1}$, and hence a proper $p$-subgroup $H$ of $G$ satisfies $\mu_{L_p}(\{1\}, H) \neq 0$ if and only if $H$ has order $p$; in this case, $\mu_{L_p}(\{1\}, H) = -1$. The number of subgroups $C_p$ of $G$ is equal to the number of subgroups $C_{q^2-q+1}$ of $G$. Thus, $\chi(\Delta(L_p \setminus \{1\})) = -\frac{q^3(q^2 + 1)(q-1)}{3} = -\frac{q^6 + q^3 - q^4 - q^3}{3}$.

**References**


