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Symplectic semifield spreads of $\mathrm{PG}(5, q^t)$, q even

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Abstract

Let $q > 2 \cdot 3^{4t}$ be even. We prove that the only symplectic semifield spread of $\mathrm{PG}(5, q^t)$, whose associate semifield has center containing $\mathbb{F}_q$, is the Desarguesian spread. Equivalently, a commutative semifield of order q^{3t}, with middle nucleus containing $\mathbb{F}_{q^t}$ and center containing $\mathbb{F}_q$, is a field. We do that by proving that the only possible $\mathbb{F}_q$-linear set of rank $3t$ in $\mathrm{PG}(5, q^t)$ disjoint from the secant variety of the Veronese surface is a plane of $\mathrm{PG}(5, q^t)$.

Keywords: Semifields, spreads, symplectic polarity, linear sets, Veronese variety.

Math. Subj. Class.: 05B25, 51E15, 51E23, 14M12

1 Introduction

Let $\mathrm{PG}(r-1, q)$ be the projective space of dimension $r-1$ over the finite field $\mathbb{F}_q$ of order q. An $(n-1)$-*spread* $\mathcal{S}$ of $\mathrm{PG}(2n-1, q)$, which we will call simply *spread* from now on, is a partition of the point-set in $(n-1)$-dimensional subspaces. With any spread $\mathcal{S}$ it is associated a translation plane $A(\mathcal{S})$ of order q^n via the André-Bruck-Bose construction (see e.g. [7, Section 5.1]). Translation planes associated with different spreads of $\mathrm{PG}(2n-1, q)$ are isomorphic if and only if there is a collineation of $\mathrm{PG}(2n-1, q)$ mapping one spread to the other (see [1] or [16, Chapter 1]). A spread $\mathcal{S}$ is said to be *Desarguesian* if $A(\mathcal{S})$ is isomorphic to $\mathrm{AG}(2, q^n)$ and hence a plane coordinatized by the field of order q^n. The spread $\mathcal{S}$ is said to be a *semifield spread* if $A(\mathcal{S})$ is a plane of Lenz-Barlotti class V and this is equivalent to saying that $A(\mathcal{S})$ is coordinatized by a semifield.

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A *finite semifield* $\mathbb{S} = (\mathbb{S}, +, \star)$ is a finite algebra satisfying all the axioms for a skew-field except (possibly) associativity of multiplication. The subsets

$$\begin{aligned} \mathbb{N}_l &= \{a \in \mathbb{S} : (a \star b) \star c = a \star (b \star c),\ \forall b, c \in \mathbb{S}\}, \\ \mathbb{N}_m &= \{b \in \mathbb{S} : (a \star b) \star c = a \star (b \star c),\ \forall a, c \in \mathbb{S}\}, \\ \mathbb{N}_r &= \{c \in \mathbb{S} : (a \star b) \star c = a \star (b \star c),\ \forall a, b \in \mathbb{S}\} \text{ and} \\ \mathcal{K} &= \{a \in \mathbb{N}_l \cap \mathbb{N}_m \cap \mathbb{N}_r : a \star b = b \star a,\ \forall b \in \mathbb{S}\} \end{aligned}$$

are fields and are known, respectively, as the *left nucleus*, the *middle nucleus*, the *right nucleus* and the *center* of the semifield. A finite semifield is a vector space over its nuclei and its center.

If $A(\mathcal{S})$ is coordinatized by the semifield $\mathbb{S}$, then $\mathbb{S}$ has order q^n and its *left nucleus* contains $\mathbb{F}_q$.

Semifields are studied up to an equivalence relation called *isotopy*, which corresponds to the study of semifield planes up to isomorphisms (for more details on semifields see, e.g., [7]).

The spread $\mathcal{S}$ is said to be *symplectic* if the elements of $\mathcal{S}$ are totally isotropic with respect to a *symplectic polarity* of $\mathrm{PG}(2n-1, q)$. If $A(\mathcal{S})$ is coordinatized by the semifield $\mathbb{S}$, then $\mathbb{S}$ is called symplectic semifield and if its *center* contains $\mathbb{F}_s \leq \mathbb{F}_q$, then from $\mathbb{S}$ we get by the cubical array (see [13]) a semifield isotopic to a commutative semifield with *middle nucleus* containing $\mathbb{F}_q$ and *center* containing $\mathbb{F}_s$ ([11]).

Let q be even. For $n = 2$, there is the following remarkable theorem due to Cohen and Ganley.

Theorem 1.1 ([6]). *A commutative semifield of order q^2 with middle nucleus containing $\mathbb{F}_q$ is a field.*

For $n > 2$, the only known commutative semifields, that are not a field, are the Kantor-Williams symplectic pre-semifields of order q^n and $n > 1$ odd ([12]) and their commutative Knuth derivatives ([11]). Symplectic semifield spreads in characteristic 2 with odd dimension over $\mathbb{F}_2$ give arise to $\mathbb{Z}_4$-linear codes and extremal line sets in Euclidean spaces ([4]).

Most of the above mentioned results are obtained with an algebraic approach, whereas ours is mainly geometric. For small n, the study of semifield spreads has shown to be a good way to classify semifields.

Let $M(n, \mathbb{F}_q)$ be the set of all $n \times n$ matrices over $\mathbb{F}_q$. Without loss of generality, we may always assume that $S(\infty) := \{(\mathbf{0}, \mathbf{y}) : \mathbf{y} \in \mathbb{F}_q^n\}$ and $S(0) := \{(\mathbf{x}, \mathbf{0}) : \mathbf{x} \in \mathbb{F}_q^n\}$ belong to $\mathcal{S}$, hence we may write $\mathcal{S} = \{S(A) : A \in \mathbb{C}\} \cup S(\infty)$, with $S(A) := \{(\mathbf{x}, \mathbf{x}A) : \mathbf{x} \in \mathbb{F}_q^n\}$, with $\mathbb{C} \subset M(n, \mathbb{F}_q)$ such that $|\mathbb{C}| = q^n$ and $\mathbb{C}$ contains the zero matrix. The set $\mathbb{C}$ is called the *spread set* associated with $\mathcal{S}$. In order to have a semifield spread, the non-zero elements of $\mathbb{C}$ must be invertible and $\mathbb{C}$ must be a subgroup of the additive group of $M(n, \mathbb{F}_q)$ ([7, Section 5.1]), hence $\mathbb{C}$ is a vector space over some subfield of $\mathbb{F}_q$. If we choose the symplectic polarity induced by the alternating bilinear form $\beta((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2\mathbf{y}_2)) = \mathbf{x}_1\mathbf{y}_2^T - \mathbf{y}_1\mathbf{x}_2^T$, $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{F}_q^n$, then the subspace $S(A) \in \mathcal{S}$ is totally isotropic if and only if A is symmetric. The symmetric matrices form an $\frac{n(n+1)}{2}$-dimensional subspace of $M(n, \mathbb{F}_q)$ that then induces a $\mathrm{PG}\left(\frac{n(n+1)}{2} - 1, q\right)$. The rank-1 symmetric matrices form the Veronese variety $\mathcal{V}$ of degree 2 of $\mathrm{PG}\left(\frac{n(n+1)}{2} - 1, q\right)$ (this is the so called determinantal representation of the Veronese variety of degree 2, see [8, Example 2.6]). Hence the singular symmetric matrices form the $(n-2)$-th secant variety, say $\mathcal{V}_{n-2}$, of the Veronese variety. If $\mathbb{C}$ is an $\mathbb{F}_s$-vector space, $q = s^t$, then $\dim_{\mathbb{F}_s} \mathbb{C} = nt$ and it defines a subset L of $\mathrm{PG}\left(\frac{n(n+1)}{2} - 1, q\right)$ called $\mathbb{F}_s$-*linear set* of rank nt (for a complete overview on linear sets see [18]). So to a symplectic semifield spread of $\mathrm{PG}(2n-1, q)$ there corresponds an $\mathbb{F}_s$-linear set L, $q = s^t$, of $\mathrm{PG}\left(\frac{n(n+1)}{2} - 1, q\right)$ of rank tn such that $L \cap \mathcal{V}_{n-2} = \emptyset$ (see also [15]). We recall the associated semifield has left nucleus containing $\mathbb{F}_q$ and if $\mathbb{F}_s$ is the maximum subfield with respect to L is linear, then the center of the semifield is isomorphic to $\mathbb{F}_s$. So the isotopic commutative semifield we get has middle nucleus containing $\mathbb{F}_q$ and center isomorphic to $\mathbb{F}_s$.

In this article, we are focused on the case $n = 3$, i.e., on symplectic semifield spreads of $\mathrm{PG}(5, q)$, when q is even. In such a case, only two non-sporadic examples are known: the Desarguesian spread and one of its cousin (see [10]), so they are both obtained by slicing the so called Desarguesian spread of $\mathcal{Q}^+(7, q)$. In the former case, the associated translation plane is the Desarguesian plane, hence it is coordinatized by the finite field of order q^3 and the relevant linear set is actually linear on $\mathbb{F}_q$. In the latter case, the semifield spread is associated to a spread set $\mathbb{C}$ that is an $\mathbb{F}_2$-linear set L of $\mathrm{PG}(5, q)$, where $\mathbb{F}_2$ is the maximum subfield of $\mathbb{F}_q$ for which L is linear, and the associate semifield has order q^3 and center $\mathbb{F}_2$.

In [5], it is proven that the only symplectic semifield spread of $\mathrm{PG}(5, q^2)$, $q > 2^{14}$, whose associate semifield has center containing $\mathbb{F}_q$, is the Desarguesian spread, meaning that a commutative semifield of order q^6, with middle nucleus containing $\mathbb{F}_{q^2}$ and center containing $\mathbb{F}_q$ is a field, provided q is not too small. That was done by studying the intersection of the five non-equivalent $\mathbb{F}_q$-linear sets of $\mathrm{PG}(5, q^2)$ with the secant variety $\mathcal{V}_1$ of the Veronese variety and the only one that can have empty intersection with $\mathcal{V}_1$ is a plane. A classification of the $\mathbb{F}_q$-linear sets of $\mathrm{PG}(5, q^t)$ of rank $3t$ is not feasible, as the number of non-equivalent ones quickly grows with t. In fact, the present paper, we had a slightly different approach which allowed us to generalize the result of [5] in $\mathrm{PG}(5, q^t)$ for any t: by field reduction, a $\mathrm{PG}(5, q^t)$ can be seen as $\mathrm{PG}(6t-1, q)$, a linear set of rank $3t$ as a subspace $\cong \mathrm{PG}(3t-1, q)$ and $\mathcal{V}_1$ an algebraic variety, say $\mathcal{V}_1^t$, of codimension t in $\mathrm{PG}(6t-1, q)$. Hence, we have studied when a subspace of dimension $3t-1$ can have empty intersection with $\mathcal{V}_1^t$ (over $\mathbb{F}_q$), regardless the geometric feature of the linear set in $\mathrm{PG}(5, q^t)$.

2 Preliminary results

2.1 $\mathbb{F}_q$-linear sets and the $\mathbb{F}_q$-linear representation of $\mathrm{PG}(r-1, q^t)$

The set $L \subset \mathrm{PG}(V, \mathbb{F}_{q^t}) = \mathrm{PG}(r-1, q^t)$, with V an r-dimensional vector space over $\mathbb{F}_{q^t}$, is said to be an $\mathbb{F}_q$-*linear* set of rank m if it is defined by the non-zero vectors of an $\mathbb{F}_q$-vector subspace U of V of dimension m, i.e.

$$L = L_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^t}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}.$$

If $r = m$ and $\langle L_U \rangle = \mathrm{PG}(r-1, q^t)$, then $L_U \cong \mathrm{PG}(r-1, q)$. In this case, L_U is said to be a *subgeometry* (of order q) of $\mathrm{PG}(r-1, q^t)$. Throughout this paper, we shall extensively use the following result: a subset Σ of $\mathrm{PG}(r-1, q^t)$ is a subgeometry of order q if and only if there exists an $\mathbb{F}_q$-linear collineation σ of $\mathrm{PG}(r-1, q^t)$ of order t such that $\Sigma = \operatorname{Fix}\sigma$, where $\operatorname{Fix}\sigma$ is the set of points fixed by σ. This is a straightforward consequence of the fact that there is just one conjugacy class of $\mathbb{F}_q$-linear collineations of order t in $\mathrm{P\Gamma L}(r, q^t)$, namely that of

$$\varsigma\colon (x_0, x_1, \dots, x_{r-1}) \mapsto (x_0^q, x_1^q, \dots, x_{r-1}^q).$$

In particular, all subgeometries $\cong \mathrm{PG}(r-1, q)$ of $\mathrm{PG}(r-1, q^t)$ are projectively equivalent to the subgeometry induced by $\{(x_0, x_1, \dots, x_{r-1}) : x_i \in \mathbb{F}_q\}$. A subspace Π of $\mathrm{PG}(r-1, q^t)$ defines a subspace of $\operatorname{Fix}\sigma \cong \mathrm{PG}(r-1, q)$ of the same dimension if and only if $\Pi = \Pi^\sigma$ (see [14, Lemma 1]). It will be more convenient for us to explicitly state the following equivalent result.

Notation. Let $\mathbb{F}$ be any field containing $\mathbb{F}_q$. Throughout the paper we will denote by $\Pi(\mathbb{F})$ the unique subspace of $\mathrm{PG}(r-1, \mathbb{F})$ containing Π.

Lemma 2.1. *If we consider* $\mathrm{PG}(r-1, q)$ *embedded as a subgeometry of* $\mathrm{PG}(r-1, q^t)$ *and* Π *is a subspace of* $\mathrm{PG}(r-1, q)$ *of dimension* $s-1$*, then the subspace* $\Pi(\mathbb{F}_{q^t})$ *of* $\mathrm{PG}(r-1, q^t)$ *containing* Π *has dimension* $s-1$ *as well.*

Analogously, if $\mathcal{W}$ is an algebraic variety of $\mathrm{PG}(r-1, q^t)$, then $\mathcal{W} \cap \operatorname{Fix}\sigma \subset \mathcal{W} \cap \mathcal{W}^\sigma \cap \dots \cap \mathcal{W}^{\sigma^{t-1}}$ and hence $\mathcal{W} \cap \operatorname{Fix}\sigma$ has the same dimension and degree of $\mathcal{W}$ if and only if $\mathcal{W} = \mathcal{W}^\sigma$.

Remark 2.2. An algebraic variety $\mathcal{W}$ is said to be a variety of $\mathrm{PG}(r-1, q)$ if it consists of the set of zeros of polynomials $f_1, f_2, \dots, f_k \in \mathbb{F}_q[x_0, x_1, \dots, x_{r-1}]$, and we will write $\mathcal{W} = V(f_1, f_2, \dots, f_k)$. By *dimension* and *degree* of $\mathcal{W}$ we will mean the dimension and degree of the variety when considered as variety of $\mathrm{PG}(r-1, \overline{\mathbb{F}_q})$, with $\overline{\mathbb{F}_q}$ the algebraic closure of $\mathbb{F}_q$.

In the remaining part of this section, we will describe the setting we adopt to study the $\mathbb{F}_q$-linear sets of $\mathrm{PG}(V, \mathbb{F}_{q^t}) = \mathrm{PG}(r-1, q^t)$.

When we regard V as an $\mathbb{F}_q$-vector space, $\dim_{\mathbb{F}_q} V = rt$ and hence $\mathrm{PG}(V, q) = \mathrm{PG}(rt-1, q)$. Furthermore, a point $\langle v \rangle_{\mathbb{F}_{q^t}} \in \mathrm{PG}(r-1, q^t)$ corresponds to the $(t-1)$-dimensional subspace of $\mathrm{PG}(rt-1, q)$ given by $\{\lambda v : \lambda \in \mathbb{F}_{q^t}\}$. This is the so-called $\mathbb{F}_q$*-linear representation* of $\langle v \rangle_{\mathbb{F}_{q^t}}$ and the set $\mathcal{S}$, consisting of the $(t-1)$-subspaces of $\mathrm{PG}(rt-1, q)$ that are the linear representation of the points of $\mathrm{PG}(r-1, q^t)$, is a partition of the point set of $\mathrm{PG}(rt-1, q)$. Such a partition $\mathcal{S}$ is called *Desarguesian spread* of $\mathrm{PG}(rt-1, q)$. In this setting, a linear set L_U is the subset of the Desarguesian spread $\mathcal{S}$ with non-empty intersection with the projective subspace Π_U of $\mathrm{PG}(rt-1, q)$ induced by U.

We shall adopt the following cyclic representation of $\mathrm{PG}(rt-1, q)$ in $\mathrm{PG}(rt-1, q^t)$. Let $\mathrm{PG}(rt-1, q^t) = \mathrm{PG}(V', q^t)$, with V' the standard rt-dimensional vector space over $\mathbb{F}_{q^t}$ and let e_i the i-th element of the canonical base of V'. Consider the semi-linear collineation σ with accompanying automorphism $x \mapsto x^q$ and such that $e_i \mapsto e_{i+r}$, where the subscript are taken mod rt. Then σ is an $\mathbb{F}_q$-linear collineation of order t and $\operatorname{Fix}\sigma = \{(\mathbf{x}, \mathbf{x}^q, \dots, \mathbf{x}^{q^{t-1}}) : \mathbf{x} = (x_0, x_1, \dots, x_{r-1}), x_i \in \mathbb{F}_{q^t}, \mathbf{x} \neq \mathbf{0}\}$ is isomorphic to $\mathrm{PG}(rt-1, q)$. The elements of $\mathcal{S}$ are the subspaces $\Pi_P := \langle P, P^\sigma, \dots, P^{\sigma^{t-1}} \rangle \cap \operatorname{Fix}\sigma$, with $P \in \Pi_0 \cong \mathrm{PG}(r-1, q^t)$ and Π_0 defined by $x_i = 0 \ \forall i > r-1$ (see [14]). Let Π_i be $\Pi_0^{\sigma^i}$. In the following, we shall identify a point P of $\Pi_0 = \mathrm{PG}(r-1, q^t)$ with the spread element Π_P. We observe that P is just the projection of Π_P from $\langle \Pi_1, \Pi_2, \ldots, \Pi_{t-1}\rangle$ on Π_0. If L_U is a linear set of rank m, then it is induced by an $(m-1)$-dimensional subspace $\Pi_U \subset \mathrm{PG}(rt-1,q) = \mathrm{Fix}\,\sigma$ and it can be viewed both as the subset of Π_0 that is the projection of Π_U from $\langle \Pi_1, \Pi_2, \ldots, \Pi_{t-1}\rangle$ on Π_0 as well as the subset of $\mathcal{S}$ consisting of the elements Π_P such that $\Pi_P \cap \Pi_U \neq \emptyset$. We stress out that we have defined the subspaces Π_U and Π_P as subspaces of $\mathrm{Fix}\,\sigma = \mathrm{PG}(rt-1,q)$. Let $\mathbb{F}$ be any field containing $\mathbb{F}_{q^t}$, then the projection of $\Pi_U(\mathbb{F})$ on Π_0 from $\langle \Pi_1, \Pi_2, \ldots, \Pi_{t-1}\rangle$ is $\langle L_U\rangle_{\mathbb{F}}$.

Let $\mathcal{H}$ be a hypersurface of $\mathrm{PG}(r-1,q^t)$ and let $f \in \mathbb{F}_{q^t}[x_0, x_1, \ldots, x_{r-1}]$ a polynomial defining $\mathcal{H}$, i.e., $\mathcal{H} = V(f)$. In the linear representation of $\mathrm{PG}(r-1,q^t) = \Pi_0$, the points of $\mathcal{H}$ correspond to the spread elements Π_P such that $P \in \mathcal{H}$, hence it is the intersection of the variety $V(f, f^\sigma, \ldots, f^{\sigma^{t-1}})$ of $\mathrm{PG}(rt-1,q^t)$ with $\mathrm{Fix}\,\sigma$, where, by abuse of notation, we extend the action of σ also to polynomials. We observe that the variety $V(f, f^\sigma, \ldots, f^{\sigma^{t-1}})$ is the *join* of the varieties $\mathcal{H}, \mathcal{H}^\sigma, \ldots, \mathcal{H}^{\sigma^{t-1}}$ (see [8, Chapter 8]) and hence it has dimension $t(\dim \mathcal{H} + 1) - 1 = t(r-1) - 1 = tr - t - 1$ and degree $\deg(\mathcal{H})^t$. We observe that $V(f, f^\sigma, \ldots, f^{\sigma^{t-1}})$ it is defined by t polynomials and $\dim V(f, f^\sigma, \ldots, f^{\sigma^{t-1}}) = tr - t - 1 = \dim \mathrm{PG}(rt-1,q^t) - t$, hence $V(f, f^\sigma, \ldots, f^{\sigma^{t-1}})$ is a *complete intersection* (see [8, Example 11.8]). We will denote the join of the varieties $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k$ by $\mathrm{Join}(\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k)$.

Let $T_P(\mathcal{W})$ be the tangent space to the algebraic variety $\mathcal{W}$ at the point $P \in \mathcal{W}$.

Proposition 2.3 (Terracini's Lemma [20])**.** *Let* $\mathcal{W} = \mathrm{Join}(\mathcal{Y}_1, \mathcal{Y}_2)$ *and let* $P = \langle P_1, P_2\rangle \in \mathcal{W}$ *with* $P_i \in \mathcal{Y}_i$*. Then* $\langle T_{P_1}(\mathcal{Y}_1), T_{P_2}(\mathcal{Y}_2)\rangle \subseteq T_P(\mathcal{W})$*.*

The variety $V(f, f^\sigma, \ldots, f^{\sigma^{t-1}})$ is the join of the varieties $\mathcal{H}^{\sigma^i}$, $i = 0, 1, \ldots, t-1$. We recall that $\mathcal{H}^{\sigma^i}$ is a hypersurface of Π_i, hence $T_{P_i}(\mathcal{H}^{\sigma^i})$ is a hypersurface of Π_i for a *non-singular point* $P_i \in \mathcal{H}^{\sigma^i}$. By $\Pi_i \cap \langle \Pi_j, j \neq i\rangle = \emptyset$, we get

$$\dim\langle T_{P_0}(\mathcal{H}), T_{P_1}(\mathcal{H}^\sigma), \ldots, T_{P_{t-1}}(\mathcal{H}^{\sigma^{t-1}})\rangle = rt - 1 - t$$

for non-singular points $P_0, P_1, \ldots, P_{t-1}$. Since for a non-singular point $P \in V(f, f^\sigma, \ldots, f^{\sigma^{t-1}})$, $\dim T_P(V(f, f^\sigma, \ldots, f^{\sigma^{t-1}})) = rt - 1 - t$, we have

$$\langle T_{P_0}(\mathcal{H}), T_{P_1}(\mathcal{H}^\sigma), \ldots, T_{P_{t-1}}(\mathcal{H}^{\sigma^{t-1}})\rangle = T_P(V(f, f^\sigma, \ldots, f^{\sigma^{t-1}}))$$

for a non-singular $P \in V(f, f^\sigma, \ldots, f^{\sigma^{t-1}})$.

Let $\mathrm{Sing}(\mathcal{W})$ be the set of the singular points of a variety $\mathcal{W}$; we recall that $\mathrm{Sing}(\mathcal{W})$ is a subvariety of $\mathcal{W}$. From the discussion above, it is clear that

$$\mathrm{Sing}(V(f, f^\sigma, \ldots, f^{\sigma^{t-1}})) = \bigcup_{i=0}^{t-1} S_i,$$

with $S_i = \mathrm{Join}(\mathrm{Sing}(\mathcal{H}^{\sigma^i}), \mathcal{H}^{\sigma^j}, j \neq i)$.

2.2 The Veronese surface and its secant variety

In this section we denote by $\mathbb{P}^{n-1}$ the $(n-1)$-dimensional projective space over a generic field $\mathbb{F}$.

The Veronese map of degree 2

$$v_2\colon (x_0, x_1, x_2) \in \mathbb{P}^2 \longmapsto (\dots, \mathbf{x}^l, \dots) \in \mathbb{P}^5$$

is such that $\mathbf{x}^l$ ranges over all monomials of degree 2 in x_0, x_1, x_2. The image $\mathcal{V} := v_2(\mathbb{P}^2)$ is the *quadric Veronese surface*, a variety of dimension 2 and degree 4. A section $H \cap \mathcal{V}$, where H is a hyperplane of $\mathbb{P}^5$, consists of the points of $v_2(\mathcal{C})$, where $\mathcal{C}$ is a conic of $\mathbb{P}^2$.

If we use the so-called determinantal representation of $\mathcal{V}$ (see [8, Example 2.6]), then we can take $\mathbb{P}^5$ as induced by the subspace of $M(3, \mathbb{F})$ consisting of symmetric matrices and $v_2(x_0, x_1, x_2) = A$ such that $a_{ij} = x_i x_j$, i.e., $\mathcal{V}$ consists of the rank 1 matrices of $M(3, \mathbb{F})$.

Hence, the secant variety of $\mathcal{V}$, say $\mathcal{V}_1$, consists of the symmetric matrices of rank at most 2, i.e., $\mathcal{V}_1$ consists of the singular symmetric 3×3 matrices. So $\mathcal{V}_1$ is a hypersurface of $\mathbb{P}^5$ of degree 3. It is well known that the singular points of $\mathcal{V}_1$ are the points of $\mathcal{V}$.

The automorphism group $\hat{G}$ of $\mathcal{V}$ is the lifting of $G = \mathrm{PGL}(3, \mathbb{F})$ acting in the obvious way: $v_2(p)^{\hat{g}} = v_2(p^g)\ \ \forall g \in \mathrm{PGL}(3, \mathbb{F})$. The group $\hat{G}$ obviously fixes $\mathcal{V}_1$.

The maximal subspaces contained in $\mathcal{V}_1$ are planes and they are of three types: the span of $v_2(\ell)$, with ℓ a line of $\mathbb{P}^2$, the tangent planes $T_P(\mathcal{V})$ for $P \in \mathcal{V}$, and, when the characteristic of $\mathbb{F}$ is even, the *nucleus plane* π_N.

Let the characteristic of $\mathbb{F}$ be even. The plane π_N of $\mathbb{P}^5$ consists of the symmetric matrices with zero diagonal, hence π_N is contained in $\mathcal{V}_1$. By the Jacobi's formula, $\frac{\partial}{\partial a_{ij}} \det A = \mathrm{tr}(\mathrm{adj}(A) \frac{\partial A}{\partial a_{ij}})$, where $\mathrm{tr}(M)$ is the trace of a matrix M and $\mathrm{adj}(M)$ is the adjoint matrix of M. Let E_{ij} be the 3×3 matrix with 1 in the ij-position and 0 elsewhere, so we have $\frac{\partial}{\partial a_{ij}} \det A = \mathrm{tr}(\mathrm{adj}(A) \frac{\partial A}{\partial a_{ij}}) = \mathrm{tr}(\mathrm{adj}(A)(E_{ij} + E_{ji})) = 0\ \ \forall i \neq j$. It follows that a hyperplane is tangent to $\mathcal{V}_1$ if and only if it contains π_N. Also, each point of π_N is the nucleus of a point of a unique conic $v_2(\ell)$.

If $P \in \mathcal{V}_1$, then the tangent hyperplane H to $\mathcal{V}_1$ at P is such that $H \cap \mathcal{V} = v_2(\ell^2)$, where $\ell = \langle p_1, p_2 \rangle$ if $P \notin \pi_N$ and hence $P \in \langle v_2(p_1), v_2(p_2) \rangle$, or ℓ is such that P is the nucleus of $v_2(\ell)$ if $P \in \pi_N$. The tangent plane at $v_2(p)$ to $\mathcal{V}$ is the intersection of three hyperplanes K_1, K_2, K_3 such that $K_i \cap \mathcal{V} = v_2(\ell_i \cup \ell_i')$, where ℓ_i, ℓ_i' are lines through p.

If $\mathbb{F}$ is an algebraically closed field, then any subspace of $\mathbb{P}^5$ of dimension at least 1 has non-empty intersection with $\mathcal{V}_1$. If $\mathbb{F} = \mathbb{F}_q$, then there are subspaces of larger dimension disjoint from $\mathcal{V}_1$ and, by the Chevalley-Warning Theorem, we know that they can have dimension at most 2. For q even we have the following result.

Theorem 2.4 ([5]). *Let $q \geq 4$ be even, then there exists one orbit of planes under the action of $\hat{G}$ disjoint from $\mathcal{V}_1$.*

3 Proof of the main result

Through this section, we assume q to be even. Let $\overline{\mathbb{F}_q}$ be the algebraic closure of $\mathbb{F}_q$.

We adopt the $\mathbb{F}_q$-linear representation of $\mathrm{PG}(5, q^t)$, i.e., we regard the points of $\mathrm{PG}(5, q^t)$ as elements of a Desarguesian spread of $\mathrm{PG}(6t-1, q)$ and L_U as the subset of the spread with non-empty intersection with a $(3t-1)$-dimensional subspace Π_U of $\mathrm{PG}(6t-1, q)$; also, we consider $\mathrm{PG}(6t-1, q)$ as subgeometry of $\mathrm{PG}(6t-1, q^t)$ (cf. Section 2). Let f be the polynomial with coefficients in $\mathbb{F}_2$ such that $\mathcal{V}_1 = V(f)$, hence the $\mathbb{F}_q$-linear representation of $\mathcal{V}_1$ is $V(f, f^\sigma, \dots, f^{\sigma^{t-1}}) \cap \mathrm{Fix}\,\sigma$. Let $\mathcal{V}_1^t$ be $V(f, f^\sigma, \dots, f^{\sigma^{t-1}})$.

We have that $\mathcal{V}_1 \cap L_U = \emptyset \Leftrightarrow \mathcal{V}_1^t \cap \Pi_U = \emptyset \Leftrightarrow \mathcal{V}_1^t \cap \mathrm{Fix}\,\sigma \cap \Pi_U(\mathbb{F}_{q^t}) = \emptyset$. Let $\mathcal{W}$ be $\Pi_U(\overline{\mathbb{F}_q}) \cap \mathcal{V}_1^t$. We observe that $\mathcal{W} = \mathcal{W}^\sigma$, hence $\dim \mathcal{W} = \dim \mathcal{W} \cap \mathrm{Fix}\,\sigma$. We stress out that $\mathcal{W}$ is defined by polynomials in $\mathbb{F}_{q^t}[x_0, x_1, \ldots, x_{6t-1}]$ but it might not contain any $\mathbb{F}_{q^t}$-rational point. The linear representation of π_N is the $(3t-1)$-dimensional subspace Π_N of $\mathrm{Fix}\,\sigma$ that is partitioned by the spread elements $\{\Pi_P : P \in \pi_N\}$. As $L_U \cap \pi_N = \emptyset$, we must have $\Pi_U \cap \Pi_N = \emptyset$ and hence, by Lemma 2.1, $\Pi_U(\mathbb{F}_{q^t}) \cap \Pi_N(\mathbb{F}_{q^t}) = \emptyset$ and $\Pi_U(\overline{\mathbb{F}_q}) \cap \Pi_N(\overline{\mathbb{F}_q}) = \emptyset$.

Theorem 3.1. *Let* $P \in \mathcal{W}$, *then* $\dim T_P(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q}) = \dim T_P(\mathcal{V}_1^t) - 3t$.

Proof. The subspace $\Pi_U(\overline{\mathbb{F}_q})$ has codimension $3t$, hence

$$\dim T_P(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q}) \geq \dim T_P(\mathcal{V}_1^t) - 3t.$$

Let $P \in \langle P_0, P_1, \ldots, P_{t-1} \rangle$ with $P_i \in \Pi_i(\overline{\mathbb{F}_q})$. We have that

$$T_P(\mathcal{V}_1^t) = \langle T_{P_0}(\mathcal{V}_1), T_{P_1}(\mathcal{V}_1^\sigma), \ldots, T_{P_{t-1}}(\mathcal{V}_1^{\sigma^{t-1}}) \rangle$$

and $\pi_N^{\sigma^i} \subset T_{P_i}(\mathcal{V}_1^{\sigma^i})\ \forall i$, hence $\Pi_N(\overline{\mathbb{F}_q}) \subset T_P(\mathcal{V}_1^t)$. Since $\Pi_U(\overline{\mathbb{F}_q}) \cap \Pi_N(\overline{\mathbb{F}_q}) = \emptyset$ and $\dim \Pi_N(\overline{\mathbb{F}_q}) = 3t-1$, we have $\dim T_P(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q}) \leq \dim T_P(\mathcal{V}_1^t) - 3t$, hence the statement follows. □

Corollary 3.2. *We have* $\dim \mathcal{W} = 2t-1$, *hence* $\mathcal{W}$ *is a complete intersection.*

Proof. If P is non-singular for $\mathcal{V}_1^t$, then $\dim T_P(\mathcal{V}_1^t) = \dim(\mathcal{V}_1^t) = 5t-1$, whereas $\dim T_P(\mathcal{V}_1^t) > 5t-1$ for $P \in \mathrm{Sing}(\mathcal{V}_1^t)$. As $\mathcal{W} = \mathcal{V}_1^t \cap \Pi_U(\overline{\mathbb{F}_q})$, $T_P(\mathcal{W}) = T_P(\mathcal{V}_1^t) \cap \Pi_U(\mathbb{F}_{q^t})$. By Theorem 3.1,

$$\dim T_P(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q}) = \dim T_P(\mathcal{V}_1^t) - 3t \geq 2t-1,$$

and

$$\dim T_P(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q}) > 2t-1$$

only if $P \in \mathrm{Sing}(\mathcal{V}_1^t)$. Hence $\dim \mathcal{W} = 2t-1$. We observe that $2t-1 = \dim \Pi_U(\overline{\mathbb{F}_q}) - t$, hence $\mathcal{W}$ is a complete intersection. □

Corollary 3.3. $\mathrm{Sing}(\mathcal{W}) = \mathrm{Sing}(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q})$.

Proof. By Theorem 3.1, $\dim T_P(\mathcal{W}) = \dim T_P(\mathcal{V}_1^t) - 3t$, hence $\dim T_P(\mathcal{W}) > \dim \mathcal{W} = 2t-1$ if and only if $\dim T_P(\mathcal{V}_1^t) > 5t-1 = \dim(\mathcal{V}_1^t)$, i.e., $P \in \mathrm{Sing}(\mathcal{V}_1^t)$. □

If a variety $\mathcal{Y}$ is a complete intersection and $\dim \mathcal{Y} - \dim \mathrm{Sing}(\mathcal{Y}) \geq 2$, then $\mathcal{Y}$ is *normal* (see [19, Chapter 2, Section 5.1] for the general definition of normal varieties). An important tool for our proof is the following reformulation of the Hartshorne connectedness theorem ([9]).

Theorem 3.4 ([3, Theorem 2.1])**.** *If* $\mathcal{Y}$ *is a normal complete intersection, then* $\mathcal{Y}$ *is absolutely irreducible.*

Theorem 3.5. *If* $\mathcal{W}$ *is reducible and* $L_U \cap \mathcal{V}_1 = \emptyset$, *then* L_U *is a plane which is isomorphic to* $\mathrm{PG}(2, q^t)$ *disjoint from* $\mathcal{V}_1$.

Proof. If $\mathcal{W}$ is reducible, then $\mathcal{W}$ is not normal and hence $\dim \mathrm{Sing}(\mathcal{W}) = \dim \mathcal{W} - 1 = 2t-2$. A point $P \in \mathcal{W}$ is singular if and only if $P \in \mathrm{Sing}(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q})$. We have $\mathrm{Sing}(\mathcal{V}_1^t) = \bigcup_{i=0}^{t-1} S_i$, with

$$S_i = \mathrm{Join}(\mathrm{Sing}(\mathcal{V}_1^{\sigma^i}), \mathcal{V}_1^{\sigma^j}, j \neq i) = \mathrm{Join}(\mathcal{V}^{\sigma^i}, \mathcal{V}_1^{\sigma^j}, j \neq i)$$

(see Section 2), so $S_0^{\sigma^i} = S_i$ and hence

$$\dim \mathrm{Sing}(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q}) = \dim S_0 \cap \Pi_U(\overline{\mathbb{F}_q}) = 2t-2.$$

Let $P \in S_0 \cap \Pi_U(\overline{\mathbb{F}_q})$ with $P = \langle P_0, P_1, \ldots, P_{t-1}\rangle$, $P_0 \in \mathcal{V}$, $P_i \in \mathcal{V}_1^{\sigma^i}$, $i = 1, 2, \ldots, t-1$, then the tangent space $T_P(S_0 \cap \Pi_U(\overline{\mathbb{F}_q}))$ is

$$\begin{aligned}\langle T_{P_0}(\mathcal{V}), T_{P_1}(\mathcal{V}_1^{\sigma}), \ldots, T_{P_{t-1}}(\mathcal{V}_1^{\sigma^{t-1}})\rangle \cap \Pi_U(\overline{\mathbb{F}_q})\\ = K_1^* \cap K_2^* \cap K_3^* \cap H_1^* \cap \cdots \cap H_{t-1}^* \cap \Pi_U(\overline{\mathbb{F}_q}),\end{aligned}$$

with K_i^*, H_j^* hyperplanes of $\mathrm{PG}(6t-1, q^t)$ such that K_i^* projects on the hyperplane K_i of Π_0 for $i = 1, 2, 3$, H_j^* projects on the hyperplane H_j of Π_j $\forall j = 1, 2, \ldots, t-1$, $K_1 \cap K_2 \cap K_3 = T_{P_0}(\mathcal{V})$ and $H_j = T_{P_j}(\mathcal{V}_1^{\sigma^j})$. We can take K_1, K_2, K_3 such that $K_1 \cap \mathcal{V} = v_2(\ell_1^2)$, $K_2 \cap \mathcal{V} = v_2(\ell_2^2)$ and $K_3 \cap \mathcal{V} = v_2(\ell_1 \cup \ell_2)$. Hence, $K_1^* \cap K_2^* \cap H_1^* \cap \cdots \cap H_{t-1}^*$ contains Π_N and so $\dim K_2^* \cap K_3^* \cap H_1^* \cap \cdots \cap H_{t-1}^* \cap \Pi_U(\overline{\mathbb{F}_q})$ is the smallest possible, i.e., $2t-2$. Hence,

$$K_1^* \supseteq K_2^* \cap K_3^* \cap H_1^* \cap \cdots \cap H_{t-1}^* \cap \Pi_U(\overline{\mathbb{F}_q})$$

and the projection of $\Pi_U(\overline{\mathbb{F}_q})$ on Π_0 is a subspace Π_0' such that the tangent space of P_0 at $\mathcal{V} \cap \Pi_0'$ has codimension 2 in Π_0'. So either the codimension of $\Pi_0' \cap \mathcal{V}$ in Π_0' is 2 or $\Pi_0' \cap \mathcal{V}$ has codimension 3 in Π_0' but it has singular points. Suppose we are in the latter case. The Veronese variety $\mathcal{V}$ is smooth, hence Π_0' can by a 3 or 4-dimensional subspace of Π_0. If Π_0' is a hyperplane of Π_0 and $\Pi_0' \cap \mathcal{V}(\overline{\mathbb{F}_q})$ has singular points, then $\Pi_0' \cap \mathcal{V}$ is is either $v_2(\ell^2)$ or $v_2(\ell_1 \cup \ell_2)$. In the first case, Π_0' contains π_N. A plane $\cong \mathrm{PG}(2, q^t)$ is a $\mathbb{F}_q$-linear set of rank $3t$, so $\Pi_0'(\mathbb{F}_{q^t}) \cong \mathrm{PG}(4, q^t)$ contains two linear sets of rank $3t$ that must intersect by Grassmann, i.e., $L_U \cap \mathcal{V}_1 \neq \emptyset$. If $\Pi_0' \cap \mathcal{V} = v_2(\ell_1 \cup \ell_2)$, then Π_0' contains the tangent space at $\mathcal{V}$ of the point $P = v_2(\ell_1 \cap \ell_2)$ and it is the unique tangent space at $\mathcal{V}$ contained in Π_0'. Let τ be the collineation induced by the field automorphism $x \mapsto x^{q^t}$, then both Π_0' and $\mathcal{V}(\overline{\mathbb{F}_q})$ are fixed by τ, hence $T_P(\mathcal{V})^\tau = T_P(\mathcal{V})$ and, by Lemma 2.1, $T_P(\mathcal{V})$ contains a $\mathrm{PG}(2, q^t)$. Again, by Grassmann, $L_U \cap \mathcal{V}_1 \neq \emptyset$. Suppose that Π_0' is a 3-dimensional space, so it contains 4 points counted with their multiplicity and at least one of them is multiple. If P is a multiple point and it is $\mathbb{F}_{q^t}$-rational, i.e., $P = P^\tau$, then Π_0' contains a line tangent to $\mathcal{V}$ at P that it is fixed by τ and hence contains a $\mathrm{PG}(1, q^t)$, so, by Grassmann, $L_U \cap \mathcal{V}_1 \neq \emptyset$. So a multiple point P must be $\mathbb{F}_{q^{st}}$-rational, but also $P^\tau \in \Pi_0' \cap \mathcal{V}$ would be, hence $s = 2$ and we have $\Pi_0' \cap \mathcal{V} = \{P, P^\tau\}$, with $P \in \Pi_0'(q^{2t})$. The line joining P and P^τ is set-wise fixed by τ and so it contains a $\mathrm{PG}(1, q^t)$, yielding again $L_U \cap \mathcal{V}_1 \neq \emptyset$. So suppose that the codimension of $\Pi_0' \cap \mathcal{V}(\overline{\mathbb{F}_q})$ in Π_0' is 2. Hence Π_0' is either a 3-dimensional space or a plane. Suppose that Π_0' is a 3-dimensional space and so $\dim \Pi_0' \cap \mathcal{V}(\overline{\mathbb{F}_q}) = 3 - 2 = 1$. Since $\Pi_0' \cap \mathcal{V}(\overline{\mathbb{F}_q})$ is the Veronese embedding of the intersection of two distinct conics, Π_0' contains the Veronese embedding of a line ℓ and it cannot contain the embedding of any other line. Hence $v_2(\ell)^\tau \subset \Pi_0'$ implies $v_2(\ell)^\tau = v_2(\ell)$ and so $\langle v_2(\ell)\rangle$ contains a plane $\cong \mathrm{PG}(2, q^t)$. By Grassmann, $L_U \cap \mathcal{V}_1 \neq \emptyset$. Hence $\Pi_0'(q^t)$ is a plane and so $L_U = \Pi_0'(q^t)$. □

Theorem 3.6. *If* $\mathcal{W}$ *is absolutely irreducible and* $q > 2 \cdot 3^{4t}$*, then* $\mathcal{W} \cap \operatorname{Fix} \sigma$ *has at least one point.*

Proof. By [2, Corollary 7.4], an absolutely irreducible algebraic variety of $\mathrm{PG}(n-1, q)$ with dimension r and degree δ for $q > \max\{2(r+1)\delta^2, 2\delta^4\}$ has at least one $\mathbb{F}_q$-rational point. By $r = 2t - 1$ and $\delta \leq 3^t = \deg \mathcal{V}_1^t$, we have the statement. □

We conclude the section with our main result.

Theorem 3.7. *Let* $q > 2 \cdot 3^{4t}$ *be even. The only symplectic semifield spread of* $\mathrm{PG}(5, q^t)$ *whose associate semifield has center containing* $\mathbb{F}_q$*, is the Desarguesian spread.*

Proof. By Theorems 3.6 and 3.5, we have that the only $\mathbb{F}_q$-linear set of rank $3t$ disjoint from $\mathcal{V}_1$ is a plane. The planes disjoint from $\mathcal{V}_1$ form a unique orbit under the action of $\hat{G}$ (see Theorem 2.4). In this case, the linear set is $\mathbb{F}_{q^t}$-linear as well, hence the semifield associated to the spread is 3-dimensional over its center. By [17], in even characteristic this implies that the semifield is a field, hence the spread is Desarguesian. □

Corollary 3.8. *Let* $q > 2 \cdot 3^{4t}$ *be even. Then a commutative semifield of order* q^{3t}*, with middle nucleus containing* $\mathbb{F}_{q^t}$ *and center containing* $\mathbb{F}_q$*, is a field.*

Remark 3.9. We emphasize that the hypothesis of even characteristic is crucial for all our arguments: only for even q the variety $\mathcal{V}_1$ contains the plane π_N, and using $L \cap \pi_N = \emptyset$ we can prove that $\mathcal{W}$ is a complete intersection, i.e. $\mathcal{W}$ has codimension t, and the singular points of $\mathcal{W}$ are just the ones coming from $\mathcal{V}$.

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