


Complete regular dessins and skew-morphisms of cyclic groups*

Yan-Quan Feng † 

*Department of Mathematics, Beijing Jiaotong University,
Beijing 100044, People's Republic of China*

Kan Hu ‡ 

*School of Mathematics, Physics and Information Science, Zhejiang Ocean University,
Zhoushan, Zhejiang 316022, People's Republic of China*

Roman Nedela § 

*University of West Bohemia, NTIS FAV, Pilsen, Czech Republic and Mathematical
Institute, Slovak Academy of Sciences, Banská Bystrica, Slovakia*

Martin Škoviera ¶ 

Department of Computer Science, Comenius University, 842 48 Bratislava, Slovakia

Na-Er Wang || 

*Key Laboratory of Oceanographic Big Data Mining & Application of Zhejiang Province,
Zhoushan, Zhejiang 316022, People's Republic of China*

Received 8 July 2018, accepted 18 January 2020, published online 21 October 2020

Abstract

A dessin is a 2-cell embedding of a connected 2-coloured bipartite graph into an orientable closed surface. A dessin is regular if its group of orientation- and colour-preserving automorphisms acts regularly on the edges. In this paper we study regular dessins whose underlying graph is a complete bipartite graph $K_{m,n}$, called (m, n) -complete regular dessins. The purpose is to establish a rather surprising correspondence between (m, n) -complete regular dessins and pairs of skew-morphisms of cyclic groups. A skew-morphism

*The authors would like to express their gratitude to the anonymous referees for their helpful comments and suggestions which have improved the content and presentation of the paper.

† National Natural Science Foundation of China (No. 11571035, 11731002).

‡ (Corresponding author.) Zhejiang Provincial Natural Science Foundation of China (No. LY16A010010).

§ APVV-15-0220; VEGA 2/0078/20; Project LO1506 of the Czech Ministry of Education, Youth and Sports.

¶ APVV-15-0220; VEGA 1/0813/18.

|| Zhejiang Provincial Natural Science Foundation of China (No. LQ17A010003) and National Natural Science Foundation of China (No. 11801507).

of a finite group A is a bijection $\varphi: A \rightarrow A$ that satisfies the identity $\varphi(xy) = \varphi(x)\varphi^{\pi(x)}(y)$ for some function $\pi: A \rightarrow \mathbb{Z}$ and fixes the neutral element of A . We show that every (m, n) -complete regular dessin \mathcal{D} determines a pair of reciprocal skew-morphisms of the cyclic groups \mathbb{Z}_n and \mathbb{Z}_m . Conversely, \mathcal{D} can be reconstructed from such a reciprocal pair. As a consequence, we prove that complete regular dessins, exact bicyclic groups with a distinguished pair of generators, and pairs of reciprocal skew-morphisms of cyclic groups are all in a one-to-one correspondence. Finally, we apply the main result to determining all pairs of integers m and n for which there exists, up to interchange of colours, exactly one isomorphism class of (m, n) -complete regular dessins. We show that the latter occurs precisely when every group expressible as a product of cyclic groups of order m and n is abelian, which eventually comes down to the condition $\gcd(m, \phi(n)) = \gcd(\phi(m), n) = 1$, where ϕ is Euler's totient function.

Keywords: Regular dessin, bicyclic group, skew-morphism, graph embedding.

Math. Subj. Class.: 05E18, 20B25, 57M15

Polne pravilne risbe in poševni morfizmi cikličnih grup*

Yan-Quan Feng † 

*Department of Mathematics, Beijing Jiaotong University,
Beijing 100044, People's Republic of China*

Kan Hu ‡ 

*School of Mathematics, Physics and Information Science, Zhejiang Ocean University,
Zhoushan, Zhejiang 316022, People's Republic of China*

Roman Nedela § 

*University of West Bohemia, NTIS FAV, Pilsen, Czech Republic and Mathematical
Institute, Slovak Academy of Sciences, Banská Bystrica, Slovakia*

Martin Škoviera ¶ 

Department of Computer Science, Comenius University, 842 48 Bratislava, Slovakia

Na-Er Wang || 

*Key Laboratory of Oceanographic Big Data Mining & Application of Zhejiang Province,
Zhoushan, Zhejiang 316022, People's Republic of China*

Prejeto 8. julija 2018, sprejeto 18. januarja 2020, objavljeno na spletu 21. oktobra 2020

Povzetek

Risba je 2-celična vložitev povezanega 2-barvnega dvodelnega grafa na orientabilno sklenjeno ploskev. Risba je pravilna, če njena grupa avtomorfizmov, ki ohranjajo orientacijo in barve, deluje pravilno na povezavah. V tem članku preučujemo pravilne risbe, katerih osnovni graf je polni dvodelni graf $K_{m,n}$, imenovane (m, n) -polne pravilne risbe.

* Avtorji bi radi izrazili svojo hvaležnost neznanim recenzentom za njihove koristne pripombe in predloge, ki so izboljšali vsebino in organizacijo članka.

† National Natural Science Foundation of China (No. 11571035, 11731002).

‡ (Kontaktirni avtor.) Zhejiang Provincial Natural Science Foundation of China (No. LY16A010010).

§ APVV-15-0220; VEGA 2/0078/20; Project LO1506 of the Czech Ministry of Education, Youth and Sports.

¶ APVV-15-0220; VEGA 1/0813/18.

|| Zhejiang Provincial Natural Science Foundation of China (No. LQ17A010003) in National Natural Science Foundation of China (No. 11801507).

Na ta način vzpostavimo precej presenetljivo korespondenco med (m, n) -polnimi pravilnimi risbami in pari poševnih morfizmov cikličnih grup. Poševni morfizem končne grupe A je bijekcija $\varphi: A \rightarrow A$, ki zadošča identiteti $\varphi(xy) = \varphi(x)\varphi^{\pi(x)}(y)$ za neko funkcijo $\pi: A \rightarrow \mathbb{Z}$ in fiksira nevtralni element grupe A . Dokažemo, da vsaka (m, n) -polna pravilna risba \mathcal{D} določa par recipročnih poševnih morfizmov cikličnih grup \mathbb{Z}_n in \mathbb{Z}_m . Velja tudi obratno, \mathcal{D} lahko rekonstruiramo iz takšnega recipročnega para. Na podlagi tega dokažemo, da so polne pravilne risbe, eksaktne biciklične grupe z izbranim parom generatorjev, ter pari recipročnih poševnih morfizmov cikličnih grup vsi v povratno enolični korespondenci. Nazadnje pa uporabimo naš glavni rezultat še za določitev vseh parov celih števil m in n , za katere obstaja, do zamenjave barv natančno, samo en izomorfnostni razred (m, n) -polnih regularnih risb. Dokažemo, da se to zgodi natanko takrat, ko je vsaka grupa, izrazljiva kot produkt cikličnih grup reda m in n , abelska, kar se naposled prevede na pogoj $\gcd(m, \phi(n)) = \gcd(\phi(m), n) = 1$, kjer je ϕ Eulerjeva funkcija.

Ključne besede: Pravilna risba, biciklična grupa, poševni morfizem, vložitev grafa.

Math. Subj. Class.: 05E18, 20B25, 57M15
