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Comparing the expected number of random elements from the symmetric and the alternating groups needed to generate a transitive subgroup

Andrea Lucchini , Mariapia Moscatiello
Università degli Studi di Padova, Dipartimento di Matematica "Tullio Levi-Civita"

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Abstract

Given a transitive permutation group of degree n, we denote by $e_{\mathcal{T}}(G)$ the expected number of elements of G which have to be drawn at random, with replacement, before a set of generators of a transitive subgroup of G is found. We compare $e_{\mathcal{T}}(\mathrm{Sym}(n))$ and $e_{\mathcal{T}}(\mathrm{Alt}(n))$.

Keywords: Transitive groups, generation, expectation.

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1 Introduction

Let $n \in \mathbb{N}$ and suppose that we are in the following situation. There are two boxes, one is blue and one is red. The balls in the blue box correspond to the elements of $\mathrm{Sym}(n)$, the balls in the red box correspond to the elements of $\mathrm{Alt}(n)$. We choose one of the boxes, and then we extract balls from the chosen box, with replacement, until a transitive permutation group of degree n is generated. In order to minimize the number of extractions, is it better to choose the red box or the blue one? We are going to prove that the answer depends on the parity of n. If n is even the best choice is the blue box, if n is odd the red one.

In order to formulate and discuss this problem in an appropriate way, we need to introduce some definitions. Let G be a transitive permutation group of degree n and $x = (x_m)_{m\in\mathbb{N}}$ be a sequence of independent, uniformly distributed G-valued random variables. We may define a random variable τ_G by setting

$$\tau_G = \min\{t \geq 1 \mid \langle x_1, \dots, x_t\rangle \text{ is a transitive subgroup of } \mathrm{Sym}(n)\} \in [1, +\infty].$$

E-mail addresses: lucchini@math.unipd.it (Andrea Lucchini), moscatie@math.unipd.it (Mariapia Moscatiello)

We denote with $e_{\mathcal{T}}(G) = \sum_{t \geq 1} tP(\tau_G = t)$ the expectation of the random variable τ_G. Thus $e_{\mathcal{T}}(G)$ is the expected number of elements of G which have to be drawn at random, with replacement, before a set of generators of a transitive subgroup of G is found.

The case when $G = \mathrm{Sym}(n)$ has been studied in [2, Section 5]. Denote by Π_n the set of partitions of n, i.e. nondecreasing sequences of natural numbers whose sum is n. Given $\omega = (n_1, \dots, n_k) \in \Pi_n$ with

$$n_1 = \cdots = n_{k_1} > n_{k_1+1} = \cdots = n_{k_1+k_2} > \cdots > n_{k_1+\cdots+k_{r-1}+1} = \cdots = n_{k_1+\cdots+k_r}$$

define

$$\mu(\omega) = (-1)^{k-1}(k-1)!,$$
$$\iota(\omega) = \frac{n!}{n_1!n_2!\cdots n_k!},$$
$$\nu(\omega) = k_1!k_2!\cdots k_r!.$$

It turns out (see [2, Theorem 9]) that for every $n \geq 2$,

$$e_{\mathcal{T}}(\mathrm{Sym}(n)) = -\sum_{\omega \in \Pi_n^*} \frac{\mu(\omega)\iota(\omega)^2}{\nu(\omega)(\iota(\omega)-1)},$$

where Π_n^* is the set of partitions of n into at least two subsets. The aim of this paper is to consider the case $G = \mathrm{Alt}(n)$. Our main result is the following.

Theorem 1.1. *For every natural number $n \geq 3$*

$$e_{\mathcal{T}}(\mathrm{Sym}(n)) - e_{\mathcal{T}}(\mathrm{Alt}(n)) = \frac{(-1)^{n+1}n!(n-1)!}{(n!-1)(n!-2)}.$$

So the difference $e_{\mathcal{T}}(\mathrm{Sym}(n)) - e_{\mathcal{T}}(\mathrm{Alt}(n))$ tends to zero when n tends to infinity, but it is positive if n is odd and negative otherwise. To explain this behaviour notice that, if $G \leq \mathrm{Sym}(n)$, then $P(\tau_G = 1)$ concides with the probability $P_{\mathcal{T}}(G, 1)$ that one randomly chosen element g in G generates a transitive subgroup of $\mathrm{Sym}(n)$, i.e. that g is an n-cycle: in particular $P_{\mathcal{T}}(\mathrm{Sym}(n), 1) = 1/n$, $P_{\mathcal{T}}(\mathrm{Alt}(n), 1) = 2/n$ if n is odd and $P_{\mathcal{T}}(\mathrm{Alt}(n), 1) = 0$ if n is even.

In [2, Section 5], it is proved that $\lim_{n\to\infty} e_{\mathcal{T}}(\mathrm{Sym}(n)) = 2$ and

$$2 = e_{\mathcal{T}}(\mathrm{Sym}(2)) \leq e_{\mathcal{T}}(\mathrm{Sym}(n)) \leq e_{\mathcal{T}}(\mathrm{Sym}(4)) = \frac{7982}{3795} \sim 2.1033.$$

A similar result can be obtained in the alternating case.

Theorem 1.2. *Assume $n \geq 3$.*

1. *If n is odd, then $\frac{3}{2} = e_{\mathcal{T}}(\mathrm{Alt}(3)) \leq e_{\mathcal{T}}(\mathrm{Alt}(n)) < 2$.*
2. *If n is even, then $2 < e_{\mathcal{T}}(\mathrm{Alt}(n)) \leq e_{\mathcal{T}}(\mathrm{Alt}(4)) = \frac{394}{165} \sim 2.3879$.*

Moreover $\lim_{n\to\infty} e_{\mathcal{T}}(\mathrm{Alt}(n)) = 2$.

2 Proof of Theorem 1.1

Let $\Lambda = (X, \leq)$ be a finite poset. Recall that the Möbius function μ_Λ on the poset Λ is the unique function $\mu_\Lambda \colon X \times X \to \mathbb{Z}$, satisfying $\mu(x, y) = 0$ unless $x \leq y$ and the recursion formula

$$\sum_{x \leq y \leq z} \mu_\Lambda(y, z) = \begin{cases} 1 & \text{if } x = z, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a transitive subgroup of $\mathrm{Sym}(n)$. We denote with $P_{\mathcal{T}}(G, t)$ the probability that t randomly chosen elements of G generate a transitive subgroup of $\mathrm{Sym}(n)$. Notice that $\tau_G > t$ if and only if $\langle x_1, \dots, x_t \rangle$ is not a transitive subgroup of G, so we have

$$P(\tau_G > t) = 1 - P_{\mathcal{T}}(G, t).$$

We get that

$$\begin{aligned} e_{\mathcal{T}}(G) &= \sum_{t \geq 1} t P(\tau_G = t) = \sum_{t \geq 1} \left(\sum_{m \geq t} P(\tau_G = m) \right) \\ &= \sum_{t \geq 1} P(\tau_G \geq t) = \sum_{t \geq 0} P(\tau_G > t) = \sum_{t \geq 0} (1 - P_{\mathcal{T}}(G, t)). \end{aligned} \tag{2.1}$$

Consider the poset $\mathcal{X}_G$ of the intransitive subgroups of G, let $\mathcal{I}_G$ be the set of subgroups of G than can be obtained as intersection of maximal elements of the the poset $\mathcal{X}_G$, and let $\mathcal{J}_G = \mathcal{I}_G \cup \{G\}$. From [1, Section 2] we have that

$$P_{\mathcal{T}}(G, t) = \sum_{H \in \mathcal{L}_{\mathcal{T}}(G)} \frac{\mu_{\mathcal{T},G}(H, G)}{|G : H|^t} = \sum_{H \in \mathcal{J}_G} \frac{\mu_{\mathcal{T},G}(H, G)}{|G : H|^t},$$

where $\mu_{\mathcal{T},G}$ denotes the Möbius function on the lattice $\mathcal{L}_{\mathcal{T}}(G) = \mathcal{X}_G \cup \{G\}$. So in order to compute the function $P_{\mathcal{T}}(G, t)$ we need information about the subgroups in $\mathcal{J}_G$. Let $\mathcal{P}_n$ be the poset of partitions of $\{1, \dots, n\}$, ordered by refinement. The maximum $\hat{1}$ of $\mathcal{P}_n$ is $\{\{1, \dots, n\}\}$ (the partition into only one part), while the minimum $\hat{0}$ is $\{\{1\}, \{2\}, \dots, \{n\}\}$ (the partition into n parts of size 1). The orbit lattice of G is defined as

$$\mathcal{P}_n(G) = \{\sigma \in \mathcal{P}_n \mid \text{the orbits of some } H \leq G \text{ are the parts of } \sigma\}.$$

If $\sigma = \{\Omega_1, \dots, \Omega_k\} \in \mathcal{P}_n$, then we define

$$G(\sigma) = (\mathrm{Sym}(\Omega_1) \times \dots \times \mathrm{Sym}(\Omega_k)) \cap G.$$

If $\sigma \in \mathcal{P}_n(G)$, then $G(\sigma)$ is the maximal element in the lattice of those subgroups of G whose orbits are precisely the parts of σ. Notice that $H \in \mathcal{J}_G$ if and only if there exists $\sigma \in \mathcal{P}_n(G)$ with $H = G(\sigma)$; moreover $\mu_{\mathcal{T},G}(G(\sigma), G) = \mu_{\mathcal{P}_n(G)}(\sigma, \hat{1})$ so

$$P_{\mathcal{T}}(G, t) = \sum_{\sigma \in \mathcal{P}_n(G)} \frac{\mu_{\mathcal{P}_n(G)}(\sigma, \hat{1})}{|G : G(\sigma)|^t}. \tag{2.2}$$

We want now to use (2.2) in order to compute $P_{\mathcal{T}}(\mathrm{Sym}(n), t) - P_{\mathcal{T}}(\mathrm{Alt}(n), t)$. Let $\mathcal{P}_{2,n}$ be the subset of $\mathcal{P}_n$ consisting of the partitions of $\{1, \dots, n\}$ into $n-1$ parts (one of size 2, the others of size 1) and let $\mathcal{P}^*_{2,n} = \mathcal{P}_{2,n} \cup \{\hat{0}\}$. The following two lemmas are immediate but crucial in our computation.

Lemma 2.1. *$\mathcal{P}_n(\mathrm{Sym}(n)) = \mathcal{P}_n$ and $\mathcal{P}_n(\mathrm{Alt}(n)) = \mathcal{P}_n \setminus \mathcal{P}_{2,n}$.*

Lemma 2.2. *If $\sigma \in \mathcal{P}_n \setminus \mathcal{P}^*_{2,n}$, then*

1. $\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\sigma, \hat{1}) = \mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\sigma, \hat{1}) = \mu_{\mathcal{P}_n}(\sigma, \hat{1})$;
2. $|\,\mathrm{Sym}(n) : \mathrm{Sym}(n)(\sigma)| = |\,\mathrm{Alt}(n) : \mathrm{Alt}(n)(\sigma)|$.

Lemma 2.3. *We have*

1. $\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$;
2. $\mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)! + \frac{(-1)^{n-2}n!}{2}$.

Proof. We use the following known result (see for example [3, p. 128]):

$$\mu_{\mathcal{P}_n}(\{\Omega_1, \dots, \Omega_k\}, \hat{1}) = (-1)^{k-1}(k-1)!. \tag{2.3}$$

This immediately implies $\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\hat{0}, \hat{1}) = \mu_{\mathcal{P}_n}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$. Moreover

$$\begin{aligned}
\mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\hat{0}, \hat{1}) &= -\sum_{\sigma \in \mathcal{P}_n(\mathrm{Alt}(n)) \setminus \{\hat{0}\}} \mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\sigma, \hat{1}) \\
&= -\sum_{\sigma \in \mathcal{P}_n \setminus \mathcal{P}^*_{2,n}} \mu_{\mathcal{P}_n}(\sigma, \hat{1}) \\
&= -\sum_{\sigma \in \mathcal{P}_n \setminus \{\hat{0}\}} \mu_{\mathcal{P}_n}(\sigma, \hat{1}) + \sum_{\sigma \in \mathcal{P}_{2,n}} \mu_{\mathcal{P}_n}(\sigma, \hat{1}) \\
&= \mu_{\mathcal{P}_n}(\hat{0}, \hat{1}) + \sum_{\sigma \in \mathcal{P}_{2,n}} \mu_{\mathcal{P}_n}(\sigma, \hat{1}) \\
&= (-1)^{n-1}(n-1)! + \binom{n}{n-2}(-1)^{n-2}(n-2)! \\
&= (-1)^{n-1}(n-1)! + \frac{(-1)^{n-2}n!}{2}.
\end{aligned}$$

□

Theorem 2.4. *For every natural number $n \geq 2$*

$$P_{\mathcal{T}}(\mathrm{Sym}(n), t) - P_{\mathcal{T}}(\mathrm{Alt}(n), t) = \frac{(-1)^{n+1}(n-1)!(2^t - 1)}{(n!)^t}.$$

Proof. For every $t \in \mathbb{N}$ (and using (2.3) and Lemma 2.3) let

$$\begin{aligned}
\eta_1(n,t) &= \sum_{\sigma \in \mathcal{P}_{2,n}} \frac{\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\sigma, \hat{1})}{|\,\mathrm{Sym}(n) : \mathrm{Sym}(n)(\sigma)|^t} \\
&= \binom{n}{2} \frac{(-1)^{n-2}(n-2)!2^t}{(n!)^t} = \frac{(-1)^{n-2}(n!)2^t}{2(n!)^t}, \\
\eta_2(n,t) &= \frac{\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\hat{0}, \hat{1})}{|\,\mathrm{Sym}(n) : \mathrm{Sym}(n)(\hat{0})|^t} = \frac{(-1)^{n-1}(n-1)!}{(n!)^t}, \\
\eta_3(n,t) &= \frac{\mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\hat{0}, \hat{1})}{|\,\mathrm{Alt}(n) : \mathrm{Alt}(n)(\hat{0})|^t} = \left((-1)^{n-1}(n-1)! + \frac{(-1)^{n-2}n!}{2}\right)\left(\frac{2}{n!}\right)^t.
\end{aligned}$$

From (2.2), Lemma 2.1 and Lemma 2.2, we deduce that

$$\begin{aligned}
P_{\mathcal{T}}(\mathrm{Sym}(n),t) &= \sum_{\sigma\in\mathcal{P}_n(\mathrm{Sym}(n))} \frac{\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\sigma,\hat{1})}{|\,\mathrm{Sym}(n):\mathrm{Sym}(n)(\sigma)|^t} \\
&= \sum_{\sigma\in\mathcal{P}_n(\mathrm{Alt}(n))} \frac{\mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\sigma,\hat{1})}{|\,\mathrm{Alt}(n):\mathrm{Alt}(n)(\sigma)|^t} \\
&\qquad + \sum_{\sigma\in\mathcal{P}_{2,n}(\mathrm{Sym}(n))} \frac{\mu_{\mathcal{P}_n}(\sigma,\hat{1})}{|\,\mathrm{Sym}(n):\mathrm{Sym}(n)(\sigma)|^t} \\
&\qquad + \frac{\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\hat{0},\hat{1})}{|\,\mathrm{Sym}(n):\mathrm{Sym}(n)(\hat{0})|^t} - \frac{\mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\hat{0},\hat{1})}{|\,\mathrm{Alt}(n):\mathrm{Alt}(n)(\hat{0})|^t} \\
&= P_{\mathcal{T}}(\mathrm{Alt}(n),t)+\eta_1(n,t)+\eta_2(n,t)-\eta_3(n,t) \\
&= P_{\mathcal{T}}(\mathrm{Alt}(n),t)+\frac{(-1)^n(n-1)!(2^t-1)}{(n!)^t}. \qquad \square
\end{aligned}$$

Proof of Theorem 1.1. Using equation (2.1) we obtain that

$$\begin{aligned}
e_{\mathcal{T}}(\mathrm{Sym}(n))-e_{\mathcal{T}}(\mathrm{Alt}(n)) &= \sum_{t\geq 0}\left(P_{\mathcal{T}}(\mathrm{Alt}(n),t)-P_{\mathcal{T}}(\mathrm{Sym}(n),t)\right) \\
&= \sum_{t\geq 0}\frac{(-1)^{n+1}(n-1)!(2^t-1)}{(n!)^t} \\
&= (-1)^{n+1}(n-1)!\left(\sum_{t\geq 0}\left(\frac{2}{n!}\right)^t-\sum_{t\geq 0}\left(\frac{1}{n!}\right)^t\right) \\
&= (-1)^{n+1}(n-1)!\left(\frac{n!}{n!-2}-\frac{n!}{n!-1}\right) \\
&= \frac{(-1)^{n+1}n!(n-1)!}{(n!-1)(n!-2)}. \qquad \square
\end{aligned}$$

3 Examples

In this section we want to verify Theorem 1.1 in the particular case when $n\in\{3,4\}$ using some direct, elementary arguments to compute $e_{\mathcal{T}}(\mathrm{Sym}(n))$ and $e_{\mathcal{T}}(\mathrm{Alt}(n))$.

First assume $n=3$. Notice that $\tau_{\mathrm{Alt}(3)}$ is a geometric random variable with parameter $\frac{2}{3}$, so $e_{\mathcal{T}}(\mathrm{Alt}(3))=\frac{3}{2}$. To generate a transitive subgroup of $\mathrm{Sym}(3)$ first of all we have to search for a nontrivial element of $\mathrm{Sym}(3)$. The numbers of trials needed to obtain a nontrivial element x of $\mathrm{Sym}(3)$ is a geometric random variable of parameter $\frac{5}{6}$: its expectation is equal to $E_0=\frac{6}{5}$. If this element has order 3, we have already obtained a transitive subgroup. However, with probability $p_1=\frac{3}{5}$, the nontrivial element x is a transposition: in this case in order to generate a transitive subgroup we need to find an element $y\notin\langle x\rangle$ and the number of trials needed to find $y\notin\langle x\rangle$ is a geometric random variable with parameter $\frac{2}{3}$ and expectation $E_1=\frac{3}{2}$. Definitely

$$e_{\mathcal{T}}(\mathrm{Sym}(3))=E_0+p_1E_1=\frac{6}{5}+\frac{3}{5}\cdot\frac{3}{2}=\frac{21}{10}.$$

In particular

$$e_{\mathcal{T}}(\mathrm{Sym}(3)) - e_{\mathcal{T}}(\mathrm{Alt}(3)) = \frac{21}{10} - \frac{3}{2} = \frac{3}{5},$$

according with Theorem 1.1.

Now assume $n = 4$. The transitive subgroups of $\mathrm{Alt}(4)$ are the noncyclic subgroups. Thus the subgroup $\langle x_1, \dots, x_t \rangle$ of $\mathrm{Alt}(4)$ is transitive if and only if there exist $1 \leq i < j \leq t$ such that $x_i \neq 1$ and $x_j \notin \langle x_i \rangle$. The numbers of trials needed to obtain a nontrivial element x of $\mathrm{Alt}(4)$ is a geometric random variable of parameter $\frac{11}{12}$ and expectation $E_0 = \frac{12}{11}$. With probability $p_1 = \frac{3}{11}$ the nontrivial element x has order 2: in this case the number of trials needed to find an element $y \notin \langle x \rangle$ is a geometric random variable of parameter $\frac{10}{12}$ and expectation $E_1 = \frac{12}{10}$. On the other hand, with probability $p_2 = \frac{8}{11}$ the nontrivial element x has order 3: in this second case the number of trials needed to find an element $y \notin \langle x \rangle$ is a geometric random variable of parameter $\frac{9}{12}$ and expectation $E_2 = \frac{12}{9}$. Thus

$$e_{\mathcal{T}}(\mathrm{Alt}(4)) = E_0 + p_1 E_1 + p_2 E_2 = \frac{394}{165}.$$

The case of $\mathrm{Sym}(4)$ is more complicated. To generate a transitive subgroup of $\mathrm{Sym}(4)$ first of all we have to search for a nontrivial element of $\mathrm{Sym}(4)$. The numbers of trials needed to obtain a nontrivial element x of $\mathrm{Sym}(4)$ is a geometric random variable of parameter $\frac{23}{24}$: its expectation is equal to $E_0 = \frac{24}{23}$. If x is a 4-cycle, then we have already generated a transitive subgroup. With probability $p_1 = \frac{3}{23}$, x is a product of two disjoint transposition: in this case to generate a transitive subgroup it is sufficient to find an element $y \notin \langle x \rangle$ and the number of trials needed to find such an element is a geometric random variable of parameter $\frac{20}{24}$ and expectation $E_1 = \frac{24}{20}$. With probability $p_2 = \frac{8}{23}$, x is a 3-cycle: to generate a transitive subgroup we need an elements y which does not normalizes $\langle x \rangle$: the number of trials needed to find such an element is a geometric random variable of parameter $\frac{18}{24}$ and expectation $E_2 = \frac{24}{18}$. Finally, with probability $p_3 = \frac{6}{23}$, x is a transposition. To find an element $y \notin \langle x \rangle$ we need $E_3 = \frac{24}{22}$ trials. If y is a 4-cycle or a 3-cycle with $|\operatorname{supp}(y) \cap \operatorname{supp}(x)| = 1$ or a product of two disjoint transpositions $(a, b)(c, d)$ with $x \notin \{(a, b), (c, d)\}$, then we have already generated a transitive subgroup. With probability $q_1 = \frac{2}{22}$, $\langle x, y \rangle$ is an intransitive subgroup of order 4: to generate a transitive subgroup we need an elements $z \notin \langle x, y \rangle$. The number of trials needed to find such an element is a geometric random variable of parameter $\frac{20}{24}$ and expectation $E_1^* = \frac{24}{20}$. With probability $q_2 = \frac{8}{22}$, $\langle x, y \rangle \cong \mathrm{Sym}(3)$ and to generate a transitive subgroup we need other $E_2^* = \frac{24}{18}$ trials. Definitely

$$\begin{aligned} e_{\mathcal{T}}(\mathrm{Sym}(4)) &= E_0 + p_1 E_1 + p_2 E_2 + p_3(E_3 + q_1 E_1^* + q_2 E_2^*) \\ &= \frac{24}{23} + \frac{3}{23} \cdot \frac{24}{20} + \frac{8}{23} \cdot \frac{24}{18} + \frac{6}{23} \left(\frac{24}{22} + \frac{2}{22} \cdot \frac{24}{20} + \frac{8}{22} \cdot \frac{24}{18} \right) = \frac{7982}{3795}. \end{aligned}$$

In particular

$$e_{\mathcal{T}}(\mathrm{Sym}(4)) - e_{\mathcal{T}}(\mathrm{Alt}(4)) = \frac{7982}{3795} - \frac{394}{165} = -\frac{72}{253},$$

according with Theorem 1.1.

4 Proof of Theorem 1.2

Lemma 4.1. *Let $\epsilon = 0$ if n is even, $\epsilon = 1$ if n is odd. Then*

$$e_{\mathcal{T}}(\mathrm{Alt}(n)) \leq 2 - \frac{2\epsilon}{n} + \frac{1}{n-1} + \frac{2}{n(n-1)-2} + \frac{3n}{n(n-1)(n-2)-6}.$$

Proof. Since an element of $\mathrm{Alt}(n)$ generates a transitive subgroup if and only if it is a cycle of length n, we have that $P_{\mathcal{T}}(\mathrm{Alt}(n), 1) = 2\epsilon/n$. Let now $t \geq 2$ and let $x_1, \dots, x_t \in \mathrm{Alt}(n)$ and $Y = \langle x_1, \dots, x_t \rangle \leq \mathrm{Alt}(n)$. If Y is contained in an intransitive maximal subgroup, then Y is contained in a subgroup conjugate to $\mathrm{Sym}(k) \times \mathrm{Sym}(n-k)$ for some $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. Let $k \in \{1, \dots, n-1\}$. The probability that Y is contained in a subgroup conjugate to $\mathrm{Sym}(k) \times \mathrm{Sym}(n-k)$ is bounded by $\binom{n}{k}^{1-t}$. So

$$1 - P_{\mathcal{T}}(\mathrm{Alt}(n), t) \leq \sum_{1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor} \binom{n}{k}^{1-t}.$$

Notice that

$$\sum_{3 \leq k \leq \lfloor \frac{n-1}{2} \rfloor} \binom{n}{k}^{1-t} \leq \frac{n}{2} \binom{n}{3}^{1-t}.$$

Hence

$$\begin{aligned}
e_{\mathcal{T}}(\mathrm{Alt}(n)) &= \sum_{t \geq 0} (1 - P_{\mathcal{T}}(\mathrm{Alt}(n), t)) \\
&= (1 - P_{\mathcal{T}}(\mathrm{Alt}(n), 0)) + (1 - P_{\mathcal{T}}(\mathrm{Alt}(n), 1)) + \sum_{t \geq 2} (1 - P_{\mathcal{T}}(\mathrm{Alt}(n), t)) \\
&\leq 2 - \frac{2\epsilon}{n} + \sum_{t \geq 2} \left(n^{1-t} + \binom{n}{2}^{1-t} + \frac{n}{2} \binom{n}{3}^{1-t} \right) \\
&= 2 - \frac{2\epsilon}{n} + \frac{1}{n-1} + \frac{1}{\binom{n}{2} - 1} + \frac{n}{2} \frac{1}{\binom{n}{3} - 1} \\
&= 2 - \frac{2\epsilon}{n} + \frac{1}{n-1} + \frac{2}{n(n-1)-2} + \frac{3n}{n(n-1)(n-2)-6}.
\end{aligned}$$

□

Proof of Theorem 1.2. Let

$$f(n) = \frac{(-1)^{n+1} n!(n-1)!}{(n!-1)(n!-2)}.$$

In [2, Section 5] it has been proved that $\lim_{n \to \infty} e_{\mathcal{T}}(\mathrm{Sym}(n)) = 2$. This implies

$$\lim_{n \to \infty} e_{\mathcal{T}}(\mathrm{Alt}(n)) = \lim_{n \to \infty} (e_{\mathcal{T}}(\mathrm{Sym}(n)) - f(n)) = \lim_{n \to \infty} e_{\mathcal{T}}(\mathrm{Sym}(n)) - \lim_{n \to \infty} f(n) = 2.$$

Moreover, again by [2, Section 5], if $n \geq 2$, then

$$2 \leq e_{\mathcal{T}}(\mathrm{Sym}(n)) \leq e_{\mathcal{T}}(\mathrm{Sym}(4)) \sim 2.1033. \tag{4.1}$$

The values of $e_{\mathcal{T}}(\mathrm{Alt}(n))$ and $e_{\mathcal{T}}(\mathrm{Sym}(n))$ when $n \in \{3, 4\}$ have been discussed in the previous section. So we may assume $n \geq 5$. Notice that $|f(n)|$ is a decreasing function and that $f(n) < 0$ if n is even, $f(n) > 0$ otherwise.

Assume that n is even:

$$e_{\mathcal{T}}(\mathrm{Alt}(n)) = e_{\mathcal{T}}(\mathrm{Sym}(n)) - f(n) \geq 2 - f(n) > 2,$$
$$e_{\mathcal{T}}(\mathrm{Alt}(n)) = e_{\mathcal{T}}(\mathrm{Sym}(n)) - f(n) \leq e_{\mathcal{T}}(\mathrm{Sym}(4)) - f(4) = e_{\mathcal{T}}(\mathrm{Alt}(4)).$$

Assume that n is odd: it follows immediately from Lemma 4.1, that $e_{\mathcal{T}}(\mathrm{Alt}(n)) < 2$ if $n \geq 9$. Moreover

$$e_{\mathcal{T}}(\mathrm{Alt}(5)) = \frac{2205085}{1170324} \sim 1.8842,$$
$$e_{\mathcal{T}}(\mathrm{Alt}(7)) = \frac{14930156286199468544486}{7793163632454473580445} \sim 1.9158.$$

Finally

$$e_{\mathcal{T}}(\mathrm{Alt}(n)) = e_{\mathcal{T}}(\mathrm{Sym}(n)) - f(n) \geq 2 - f(5) \geq 2 - \frac{1440}{7021} > \frac{3}{2} = e_{\mathcal{T}}(\mathrm{Alt}(3)). \quad \square$$

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