ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 16 (2019) 257–276
https://doi.org/10.26493/1855-3974.1619.a03
(Also available at http://amc-journal.eu)

Pappus's Theorem in Grassmannian $Gr(3, \mathbb{C}^n)$

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Received 26 February 2018, accepted 13 August 2018, published online 6 January 2019

Abstract

In this paper we study intersections of quadrics, components of the hypersurface in the Grassmannian $Gr(3, \mathbb{C}^n)$ introduced by S. Sawada, S. Settepanella and S. Yamagata in 2017. This lead to an alternative statement and proof of Pappus's Theorem retrieving Pappus's and Hesse configurations of lines as special points in the complex projective Grassmannian. This new connection is obtained through a third purely combinatorial object, the intersection lattice of Discriminantal arrangement.

Keywords: Discriminantal arrangements, intersection lattice, Grassmannian, Pappus's Theorem.

Math. Subj. Class.: 52C35, 05B35, 14M15

1 Introduction

Pappus's hexagon Theorem, proved by Pappus of Alexandria in the fourth century A.D., began a long development in algebraic geometry.

> *In its changing expressions one can see reflected the changing concerns of the field, from synthetic geometry to projective plane curves to Riemann surfaces to the modern development of schemes and duality.*
>
> (D. Eisenbud, M. Green and J. Harris [4])

There are several knowns proofs of Pappus's Theorem including its generalizations such as Cayley Bacharach Theorem (see Chapter 1 of [9] for a collection of proofs of Pappus's Theorem and [4] for proofs and conjectures in higher dimension).

In this paper, by mean of recent results in [6] and [10], we connect Pappus's hexagon configuration to intersections of well defined quadrics in the Grassmannian providing a new statement and proof of Pappus's Theorem as an original result on dependency conditions for defining polynomials of those quadrics. This result enlightens a new connection

*The second named author was supported by JSPS Kakenhi Grant Number 26610001.

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between special configurations of points (lines) in the projective plane and hypersurfaces in the projective Grassmannian $Gr(3, \mathbb{C}^n)$. This connection is made through a third combinatorial object, the intersection lattice of the *Discriminantal arrangement*. Introduced by Manin and Schechtman in 1989, it is an arrangement of hyperplanes generalizing classical braid arrangement (cf. [7, p. 209]). Fixed a generic arrangement $\mathcal{A} = \{H_1^0, \dots, H_n^0\}$ in $\mathbb{C}^k$, the Discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$, $n, k \in \mathbb{N}$ for $k \geq 2$ ($k = 1$ corresponds to Braid arrangement), consists of parallel translates $H_1^{t_1}, \dots, H_n^{t_n}$, $(t_1, \dots, t_n) \in \mathbb{C}^n$, of $\mathcal{A}$ which fail to form a generic arrangement in $\mathbb{C}^k$. The combinatorics of $\mathcal{B}(n, k, \mathcal{A})$ is known in the case of *very generic arrangements*, i.e. $\mathcal{A}$ belongs to an open Zariski set $\mathcal{Z}$ in the space of generic arrangements H_i^0, $i = 1, \dots, n$ (see [7], [1] and [2]), but still almost unknown for $\mathcal{A} \notin \mathcal{Z}$. In 2016, Libgober and Settepanella (cf. [6]) gave a sufficient geometric condition for an arrangement $\mathcal{A}$ not to be very generic, i.e. $\mathcal{A} \notin \mathcal{Z}$. In particular in the case $k = 3$, their result shows that multiplicity 3 codimension 2 intersections of hyperplanes in $\mathcal{B}(n, 3, \mathcal{A})$ appears if and only if collinearity conditions for points at infinity of lines, intersections of certain planes in $\mathcal{A}$, are satisfied (Theorem 3.8 in [6]). More recently (see [10]) authors applied this result to show that points in a specific degree 2 hypersurface in the Grassmannian $Gr(3, \mathbb{C}^n)$ correspond to generic arrangements of n hyperplanes in $\mathbb{C}^3$ with associated discriminantal arrangement having intersections of multiplicity 3 in codimension 2 (Theorem 5.4 in [10]). In this paper we look at Pappus's configuration (see Figure 1) as a generic arrangement of 6 lines in $\mathbb{P}^2$ which intersection points satisfy certain collinearity conditions (see Figure 2). This allows us to apply results on [6] and [10] to restate and re-prove Pappus's Theorem.

More in details, let $\mathcal{A}$ be a generic arrangement in $\mathbb{C}^3$ and $\mathcal{A}_\infty$ the arrangement of lines in $H_\infty \simeq \mathbb{P}^2$ directions at infinity of planes in $\mathcal{A}$. The space of generic arrangements of n lines in $(\mathbb{P}^2)^n$ is Zariski open set U in the space of all arrangements of n lines in $(\mathbb{P}^2)^n$. On the other hand in $Gr(3, \mathbb{C}^n)$ there is open set U' consisting of 3-spaces intersecting each coordinate hyperplane transversally (i.e. having dimension of intersection equal 2). One has also one set $\tilde{U}$ in $\mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^n)$ consisting of embeddings with image transversal to coordinate hyperplanes and $\tilde{U}/\,\mathrm{GL}(3) = U'$ and $\tilde{U}/(\mathbb{C}^*)^n = U$. Hence generic arrangements in $\mathbb{C}^3$ can be regarded as points in $Gr(3, \mathbb{C}^n)$. Let $\{s_1 < \cdots < s_6\} \subset \{1, \dots, n\}$ be a set of indices of a generic arrangement $\mathcal{A} = \{H_1^0, \dots, H_n^0\}$ in $\mathbb{C}^3$, α_i the normal vectors of H_i^0's and $\beta_{ijl} = \det(\alpha_i, \alpha_j, \alpha_l)$. For any permutation $\sigma \in \mathbf{S_6}$ denote by $[\sigma] = \{\{i_1, i_2\}, \{i_3, i_4\}, \{i_5, i_6\}\}$, $i_j = s_{\sigma(j)}$, and by Q_σ the quadric in $Gr(3, \mathbb{C}^n)$ of equation $\beta_{i_1 i_3 i_4}\beta_{i_2 i_5 i_6} - \beta_{i_2 i_3 i_4}\beta_{i_1 i_5 i_6} = 0$. The following theorem, equivalent to the Pappus's hexagon Theorem, holds.

Theorem 5.3 (Pappus's Theorem)**.** *For any disjoint classes $[\sigma_1]$ and $[\sigma_2]$, there exists a unique class $[\sigma_3]$ disjoint from $[\sigma_1]$ and $[\sigma_2]$ such that $\{\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}\}$ is a Pappus configuration, i.e.*

$$\mathrm{Q}_{\sigma_{i_1}} \cap \mathrm{Q}_{\sigma_{i_2}} = \bigcap_{i=1}^{3} \mathrm{Q}_{\sigma_i}$$

for any $\{i_1, i_2\} \subset [3]$.

In the rest of the paper, we retrieve the Hesse configuration of lines studying intersections of six quadrics of the form Q_σ for opportunely chosen $[\sigma]$. This lead to a better understanding of differences in the combinatorics of Discriminantal arrangement in the complex and real case. Indeed it turns out that this difference is connected with existence of the Hesse arrangement (see [8]) in $\mathbb{P}^2(\mathbb{C})$, but not in $\mathbb{P}^2(\mathbb{R})$.

From above results it seems very likely that a deeper understanding of combinatorics of Discriminantal arrangements arising from non very generic arrangements of hyperplanes in $\mathbb{C}^k$ (i.e. $\mathcal{A} \notin \mathcal{Z}$), could lead to new connections between higher dimensional special configurations of hyperplanes (points) in the projective space and Grassmannian. Vice versa, known results in algebraic geometry could help in understanding the combinatorics of Discriminantal arrangements in the non very generic case. Moreover we conjecture that regularity in the geometry of Discriminantal arrangement could lead to results on hyperplanes arrangements with high multiplicity intersections, e.g., in the case $k = 3$, line arrangements in $\mathbb{P}^2$ with high number of triple points (see Remark 6.6). This will be object of further studies.

The content of the paper is the following. In Section 2 we recall definition of Discriminantal arrangement from [7], basic notions on Grassmannian, and definitions and results from [10]. In Section 3 we provide an example of the case of 6 hyperplanes in $\mathbb{C}^3$. In Section 4 we define and study Pappus hypersurface. Section 5 contains Pappus's theorem in $Gr(3, \mathbb{C}^n)$ and its proof. In the last section we study intersections of higher numbers of quadrics and Hesse configuration.

2 Preliminaries

2.1 Discriminantal arrangement

Let $H_i^0, i = 1, \ldots, n$ be a generic arrangement in $\mathbb{C}^k$, $k < n$ i.e. a collection of hyperplanes such that $\operatorname{codim} \bigcap_{i \in K, |K|=p} H_i^0 = p$. Space of parallel translates $(H_1^0, \ldots, H_n^0)$ (or simply when dependence on H_i^0 is clear or not essential) is the space of n-tuples $H_1, \ldots, H_n$ such that either $H_i \cap H_i^0 = \emptyset$ or $H_i = H_i^0$ for any $i = 1, \ldots, n$. One can identify with n-dimensional affine space $\mathbb{C}^n$ in such a way that $(H_1^0, \ldots, H_n^0)$ corresponds to the origin. In particular, an ordering of hyperplanes in $\mathcal{A}$ determines the coordinate system in (see [6]).

We will use the compactification of $\mathbb{C}^k$ viewing it as $\mathbb{P}^k(\mathbb{C}) \setminus H_\infty$ endowed with collection of hyperplanes $\bar{H}_i^0$ which are projecive closures of affine hyperplanes H_i^0. Condition of genericity is equivalent to $\bigcup_i \bar{H}_i^0$ being a normal crossing divisor in $\mathbb{P}^k(\mathbb{C})$.

Given a generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$ formed by hyperplanes $H_i, i = 1, \ldots, n$ *the trace at infinity*, denoted by $\mathcal{A}_\infty$, is the arrangement formed by hyperplanes $H_{\infty,i} = \bar{H}_i^0 \cap H_\infty$ in the space $H_\infty \simeq \mathbb{P}^{k-1}(\mathbb{C})$. The trace $\mathcal{A}_\infty$ of an arrangement $\mathcal{A}$ determines the space of parallel translates $\mathbb{S}$ (as a subspace in the space of n-tuples of hyperplanes in $\mathbb{P}^k$).

Fixed a generic arrangement $\mathcal{A}$, consider the closed subset of $\mathbb{S}$ formed by those collections which fail to form a generic arrangement. This subset of $\mathbb{S}$ is a union of hyperplanes $D_L \subset \mathbb{S}$ (see [7]). Each hyperplane D_L corresponds to a subset $L = \{i_1, \ldots, i_{k+1}\} \subset [n] := \{1, \ldots, n\}$ and it consists of n-tuples of translates of hyperplanes $H_1^0, \ldots, H_n^0$ in which translates of $H_{i_1}^0, \ldots, H_{i_{k+1}}^0$ fail to form a general position arrangement. The arrangement $\mathcal{B}(n, k, \mathcal{A})$ of hyperplanes D_L is called *Discriminantal arrangement* and has been introduced by Manin and Schechtman in [7]. Notice that $\mathcal{B}(n, k, \mathcal{A})$ depends on the trace at infinity $\mathcal{A}_\infty$ hence it is sometimes more properly denoted by $\mathcal{B}(n, k, \mathcal{A}_\infty)$.

2.2 Good $3s$-partitions

Given $s \geq 2$ and $n \geq 3s$, *a good $3s$-partition* (see [10]) is a set $\mathbb{T} = \{L_1, L_2, L_3\}$, with L_i subsets of $[n]$ such that $|L_i| = 2s$, $|L_i \cap L_j| = s$ $(i \neq j)$, $L_1 \cap L_2 \cap L_3 = \emptyset$ (in particular $|\bigcup L_i| = 3s$), i.e. $L_1 = \{i_1, \ldots, i_{2s}\}, L_2 = \{i_1, \ldots, i_s, i_{2s+1}, \ldots, i_{3s}\}, L_3 =$

$\{i_{s+1}, \dots, i_{3s}\}$.

Notice that given a generic arrangement $\mathcal{A}$ in $\mathbb{C}^{2s-1}$, subsets L_i define hyperplanes D_{L_i} in the Discriminantal arrangement $\mathcal{B}(n, 2s-1, \mathcal{A}_\infty)$. In this paper we are mainly interested in the case $s = 2$ corresponding to generic arrangements in $\mathbb{C}^3$.

2.3 Matrices $A(\mathcal{A}_\infty)$ and $A_{\mathbb{T}}(\mathcal{A}_\infty)$

Let $\alpha_i = (a_{i1}, \dots, a_{ik})$ be the normal vectors of hyperplanes H_i, $1 \le i \le n$, in the generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$. Normal here is intended with respect to the usual dot product

$$(a_1, \dots, a_k) \cdot (v_1, \dots, v_k) = \sum_i a_i v_i.$$

Then the normal vectors to hyperplanes D_L, $L = \{s_1 < \dots < s_{k+1}\} \subset [n]$ in $\mathbb{S} \simeq \mathbb{C}^n$ are nonzero vectors of the form

$$\alpha_L = \sum_{i=1}^{k+1} (-1)^i \det(\alpha_{s_1}, \dots, \hat{\alpha_{s_i}}, \dots, \alpha_{s_{k+1}}) e_{s_i}, \tag{2.1}$$

where $\{e_j\}_{1 \le j \le n}$ is the standard basis of $\mathbb{C}^n$ (cf. [2]).

Let $\mathcal{P}_{k+1}([n]) = \{L \subset [n] \mid |L| = k+1\}$ be the set of cardinality $k+1$ subsets of $[n]$. Following [10] we denote by

$$A(\mathcal{A}_\infty) = (\alpha_L)_{L \in \mathcal{P}_{k+1}([n])}$$

the matrix having in each row the entries of vectors α_L normal to hyperplanes D_L and by $A_{\mathbb{T}}(\mathcal{A}_\infty)$ the submatrix of $A(\mathcal{A}_\infty)$ with rows α_L, $L \in \mathbb{T}$, $\mathbb{T} \subset \mathcal{P}_{k+1}([n])$. In this paper we are mainly interested in the matrix $A_{\mathbb{T}}(\mathcal{A}_\infty)$ in the case of $\mathbb{T}$ good 6-partition.

2.4 Grassmannian $Gr(k, \mathbb{C}^n)$

Let $Gr(k, \mathbb{C}^n)$ be the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$ and

$$\begin{aligned} \gamma\colon\ Gr(k, \mathbb{C}^n) &\to \mathbb{P}(\textstyle\bigwedge^k \mathbb{C}^n) \\ \langle v_1, \dots, v_k \rangle &\mapsto [v_1 \wedge \dots \wedge v_k], \end{aligned}$$

the Plücker embedding. Then $[x] \in \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ is in $\gamma(Gr(k, \mathbb{C}^n))$ if and only if the map

$$\begin{aligned} \varphi_x\colon \mathbb{C}^n &\to \textstyle\bigwedge^{k+1} \mathbb{C}^n \\ v &\mapsto x \wedge v \end{aligned}$$

has kernel of dimension k, i.e. $\ker \varphi_x = \langle v_1, \dots, v_k \rangle$. If $e_1, \dots, e_n$ is a basis of $\mathbb{C}^n$ then $e_I = e_{i_1} \wedge \dots \wedge e_{i_k}$, $I = \{i_1, \dots, i_k\} \subset [n]$, $i_1 < \dots < i_k$, is a basis for $\bigwedge^k \mathbb{C}^n$ and $x \in \bigwedge^k \mathbb{C}^n$ can be written uniquely as

$$x = \sum_{\substack{I \subseteq [n] \\ |I| = k}} \beta_I e_I = \sum_{1 \le i_1 < \dots < i_k \le n} \beta_{i_1 \dots i_k} (e_{i_1} \wedge \dots \wedge e_{i_k})$$

where homogeneous coordinates β_I are the Plücker coordinates on $\mathbb{P}(\bigwedge^k \mathbb{C}^n) \simeq \mathbb{P}^{\binom{n}{k}-1}(\mathbb{C})$ associated to the ordered basis $e_1, \dots, e_n$ of $\mathbb{C}^n$. With this choice of basis for $\mathbb{C}^n$ the matrix M_x associated to φ_x is a $\binom{n}{k+1} \times n$ matrix with rows indexed by subsets $I = \{i_1, \dots, i_k\} \subset [n]$ and entries

$$b_{ij} = \begin{cases} (-1)^l \beta_{I \setminus \{j\}} & \text{if } j = i_l \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Plücker relations, i.e. conditions for $\dim(\ker \varphi_x) = k$, are vanishing conditions of all $(n-k+1) \times (n-k+1)$ minors of M_x. It is well known (see for instance [5]) that Plücker relations are degree 2 relations and they can also be written as

$$\sum_{l=0}^{k} (-1)^l \beta_{p_1 \dots p_{k-1} q_l} \beta_{q_0 \dots \hat{q_l} \dots q_k} = 0 \tag{2.2}$$

for any $2k$-tuple $(p_1, \dots, p_{k-1}, q_0, \dots, q_k)$.

Remark 2.1. Notice that vectors α_L in the equation (2.1) normal to hyperplanes D_L correspond to rows indexed by L in the Plücker matrix M_x, that is

$$A(\mathcal{A}_\infty) = M_x,$$

up to permutation of rows. Notice that, in particular, $\det(\alpha_{s_1}, \dots, \hat{\alpha_{s_i}}, \dots, \alpha_{s_{k+1}})$ is the Plücker coordinate β_I, $I = \{s_1, s_2, \dots, s_{k+1}\} \setminus \{s_i\}$.

2.5 Relation between intersections of lines in $\mathcal{A}_\infty$ and quadrics in $Gr(3, \mathbb{C}^n)$

Let $\mathcal{A} = \{H_1^0, \dots, H_n^0\}$ be a generic arrangement in $\mathbb{C}^3$. If there exist $L_1, L_2, L_3 \subset [n]$ subsets of indices of cardinality 4, such that codimension of $D_{L_1} \cap D_{L_2} \cap D_{L_3}$ is 2 then $\mathcal{A}$ is *non very generic arrangement* (see [2]).

Let $\mathbb{T} = \{L_1, L_2, L_3\}$ be a good 6-partition of indices $\{s_1, \dots, s_6\} \subset [n]$. In [6], authors proved that the codimension of $D_{L_1} \cap D_{L_2} \cap D_{L_3}$ is 2 if and only if points

$$\bigcap\nolimits_{t \in L_1 \cap L_2} H_{\infty,t}, \quad \bigcap\nolimits_{t \in L_1 \cap L_3} H_{\infty,t} \quad \text{and} \quad \bigcap\nolimits_{t \in L_2 \cap L_3} H_{\infty,t}$$

are collinear in H_∞ ([6, Lemma 3.1]).

Since α_{L_i} is vector normal to D_{L_i}, the codimension of $D_{L_1} \cap D_{L_2} \cap D_{L_3}$ is 2 if and only if rank $A_{\mathbb{T}}(\mathcal{A}_\infty) = 2$, i.e. all 3×3 minors of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ vanish. In [10] authors proved the following Lemma.

Lemma 2.2 ([10, Lemma 5.3])**.** *Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{C}^3$ and*

$$\sigma.\mathbb{T} = \{\{i_1, i_2, i_3, i_4\}, \{i_1, i_2, i_5, i_6\}, \{i_3, i_4, i_5, i_6\}\}$$

a good 6-partition of indices $s_1 < \dots < s_6 \in [n]$ such that $i_j = s_{\sigma(j)}$, σ permutation in $\mathbf{S}_6$. Then rank $A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty) = 2$ *if and only if $\mathcal{A}$ is a point in the quadric of Grassmannian $Gr(3, \mathbb{C}^n)$ of equation*

$$\beta_{i_1 i_3 i_4} \beta_{i_2 i_5 i_6} - \beta_{i_2 i_3 i_4} \beta_{i_1 i_5 i_6} = 0. \tag{2.3}$$

As consequence of above results, we obtain correspondence between points

$$x = \sum_{\substack{I \subset [n] \\ |I|=3}} \beta_I e_I, \; \beta_I \neq 0,$$

in the quadric of equation (2.3) and generic arrangements of n hyperplanes $\mathcal{A}$ in $\mathbb{C}^3$ such that $H_{\infty,i_1} \cap H_{\infty,i_2}$, $H_{\infty,i_3} \cap H_{\infty,i_4}$ and $H_{\infty,i_5} \cap H_{\infty,i_6}$ are collinear in H_∞. Notice that condition $\beta_I \neq 0$ is direct consequence of $\mathcal{A}$ being generic arrangement.

3 Motivating example of Pappus's Theorem for quadrics in $Gr(3, \mathbb{C}^n)$

In classical projective geometry the following theorem is known as Pappus's theorem or Pappus's hexagon theorem.

Theorem 3.1 (*Pappus*)**.** *On a projective plane, consider two lines l_1 and l_2, and a couple of triple points A, B, C and A', B', C' which are on l_1 and l_2 respectively. Let X, Y, Z be points of $AB' \cap A'B$, $AC' \cap A'C$ and $BC' \cap B'C$ respectively. Then there exists a line l_3 passing through the three points X, Y, Z (see Figure 1).*

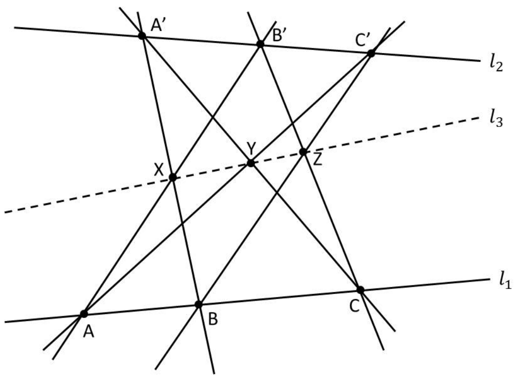

Figure 1: Original Pappus's Theorem.

This theorem was originally stated by Pappus of Alexandria around 290–350 A.D.

In this section, we restate this classical theorem in terms of quadrics in the Grassmannian. Indeed the six lines AB', $A'B$, BC', $B'C$, AC', $A'C \in \mathbb{P}^2(\mathbb{C})$ correspond to lines in the trace at infinity $\mathcal{A}_\infty$ of a generic arrangement $\mathcal{A}$ in $\mathbb{C}^3$ and lines l_1, l_2 and l_3 correspond to collinearity conditions for intersection points of lines in $\mathcal{A}_\infty$.

Consider a generic arrangement $\mathcal{A} = \{H_1, \ldots, H_6\}$ of 6 hyperplanes in $\mathbb{C}^3$, $\mathcal{A}_\infty$ its trace at infinity and $\mathbb{T} = \{L_1, L_2, L_3\}$ the good 6-partition defined by $L_1 = \{1, 2, 3, 4\}$, $L_2 = \{1, 2, 5, 6\}$, $L_3 = \{3, 4, 5, 6\}$. By Lemma 2.2 we get that the triple points

$$\bigcap\nolimits_{i \in L_1 \cap L_2} \bar{H}_i \cap H_\infty, \quad \bigcap\nolimits_{i \in L_1 \cap L_3} \bar{H}_i \cap H_\infty, \quad \bigcap\nolimits_{i \in L_2 \cap L_3} \bar{H}_i \cap H_\infty$$

are collinear if and only if $\mathcal{A}$ is a point of the quadric

$$\mathrm{Q}_1 \colon \beta_{134}\beta_{256} - \beta_{234}\beta_{156} = 0$$

in $Gr(3, \mathbb{C}^6)$.

Analogously if

$$\mathbb{T}' = \{L'_1, L'_2, L'_3\},\ L'_1 = \{4,6,2,5\},\ L'_2 = \{4,6,1,3\},\ L'_3 = \{2,5,1,3\}$$

and

$$\mathbb{T}'' = \{L''_1, L''_2, L''_3\},\ L''_1 = \{2,4,1,6\},\ L''_2 = \{2,4,3,5\},\ L''_3 = \{1,6,3,5\}$$

are different good 6-partitions then triple points

$$\bigcap\nolimits_{i\in L'_1\cap L'_2} \bar{H}_i \cap H_\infty, \quad \bigcap\nolimits_{i\in L'_1\cap L'_3} \bar{H}_i \cap H_\infty, \quad \bigcap\nolimits_{i\in L'_2\cap L'_3} \bar{H}_i^{t_i} \cap H_\infty$$

and

$$\bigcap\nolimits_{i\in L''_1\cap L''_2} \bar{H}_i \cap H_\infty, \quad \bigcap\nolimits_{i\in L''_1\cap L''_3} \bar{H}_i \cap H_\infty, \quad \bigcap\nolimits_{i\in L''_2\cap L''_3} \bar{H}_i \cap H_\infty$$

are collinear if and only if $\mathcal{A}$ is, respectively, a point of quadrics

$$\mathrm{Q}_2 \colon \beta_{425}\beta_{613} - \beta_{625}\beta_{413} = 0 \quad \text{and}$$
$$\mathrm{Q}_3 \colon \beta_{216}\beta_{435} - \beta_{416}\beta_{235} = 0.$$

With above remarks and notations we can restate Pappus's Theorem as follows (see Figure 2).

Theorem 3.2 (Pappus's Theorem)**.** *Let $\mathcal{A} = \{H_1, \ldots, H_6\}$ be a generic arrangement of hyperplanes in $\mathbb{C}^3$. If $\mathcal{A}$ is a point of two of three quadrics $\mathrm{Q}_1, \mathrm{Q}_2$ and Q_3 in the Grassmannian $Gr(3, \mathbb{C}^6)$, then $\mathcal{A}$ is also a point of the third. In other words*

$$\mathrm{Q}_{i_1} \cap \mathrm{Q}_{i_2} = \bigcap_{i=1}^{3} \mathrm{Q}_i, \quad \{i_1, i_2\} \subset [3].$$

We develop this argument in the following sections providing in Theorem 5.3 a general statement on quadrics in the Grassmannian which implies Pappus hexagon Theorem in the projective plane.

4 Pappus Variety

In this section, we consider a generic arrangement $\{H_1, \ldots, H_n\}$ in $\mathbb{C}^3$ ($n \geq 6$). Let's introduce basic notations that we will use in the rest of the paper.

Notation. *Let $\{s_1, \ldots, s_6\}$ be a subset of indices $\{1, \ldots, n\}$ and $\mathbb{T} = \{L_1, L_2, L_3\}$ be the good 6-partition given by*

$$L_1 = \{s_1, s_2, s_3, s_4\},\ L_2 = \{s_1, s_2, s_5, s_6\} \text{ and } L_3 = \{s_3, s_4, s_5, s_6\}.$$

Then for any permutation $\sigma \in \boldsymbol{S}_6$ we denote by $\sigma.\mathbb{T} = \{\sigma.L_1, \sigma.L_2, \sigma.L_3\}$ the good 6-partition given by subsets

$$\sigma.L_1 = \{i_1, i_2, i_3, i_4\},\ \sigma.L_2 = \{i_1, i_2, i_5, i_6\} \text{ and } \sigma.L_3 = \{i_3, i_4, i_5, i_6\}$$

with $i_j = s_{\sigma(j)}$. Accordingly, we denote by Q_σ the quadric in $Gr(3, \mathbb{C}^n)$ of equation

$$\mathrm{Q}_\sigma \colon \beta_{i_1 i_3 i_4}\beta_{i_2 i_5 i_6} - \beta_{i_2 i_3 i_4}\beta_{i_1 i_5 i_6} = 0.$$

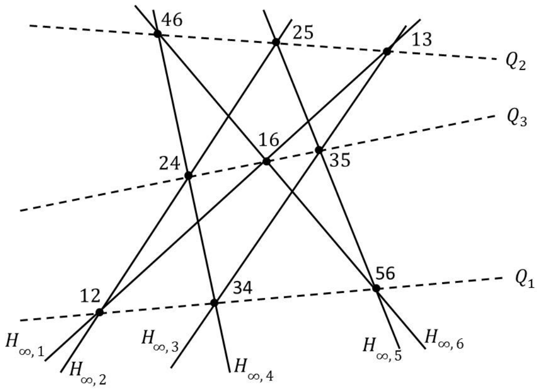

Figure 2: Trace at infinity of $\mathcal{A} \in \bigcap_{i=1}^{3} \mathrm{Q}_i$. In the figure ij denotes $H_{\infty,i} \cap H_{\infty,j}$.

The following lemma holds.

Lemma 4.1. *Let $\sigma, \sigma' \in \boldsymbol{S}_6$ be distinct permutations, then $\mathrm{Q}_\sigma = \mathrm{Q}_{\sigma'}$ if and only if there exists $\tau \in \boldsymbol{S}_3$ such that $\sigma.L_i \cap \sigma.L_j = \sigma'.L_{\tau(i)} \cap \sigma'.L_{\tau(j)}$ $(1 \leq i < j \leq 3)$.*

Proof. By definition of good 6-partition we have that

$$L_1 = (L_1 \cap L_2) \cup (L_1 \cap L_3),$$
$$L_2 = (L_2 \cap L_1) \cup (L_2 \cap L_3),$$
$$L_3 = (L_3 \cap L_1) \cup (L_3 \cap L_2).$$

Then there exists $\tau \in \mathbf{S}_3$ such that σ and σ' satisfy $\sigma.L_i \cap \sigma.L_j = \sigma'.L_{\tau(i)} \cap \sigma'.L_{\tau(j)}$ $(1 \leq i < j \leq 3)$ if and only if $\sigma.L_l = \sigma'.L_{\tau(l)}$ for $l = 1, 2, 3$, that is $A_{\sigma'.\mathbb{T}}(\mathcal{A}_\infty)$ is obtained by permuting rows of $A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty)$. It follows that $\operatorname{rank} A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty) = 2$ if and only if $\operatorname{rank} A_{\sigma'.\mathbb{T}}(\mathcal{A}_\infty) = 2$ and hence by Lemma 2.2 this is equivalent to $\mathrm{Q}_\sigma \cap N_{s_1,\ldots,s_6} = \mathrm{Q}_{\sigma'} \cap N_{s_1,\ldots,s_6}$, where

$$N_{s_1,\ldots,s_6} = \{x = \sum_{\substack{I \subseteq [n] \\ |I|=3}} \beta_I e_I \mid \beta_I \neq 0 \text{ for any } I \subset \{s_1, \ldots, s_6\}\}.$$

Since $N_{s_1,\ldots,s_6}$ is dense open set in $\gamma(Gr(3, \mathbb{C}^n))$, $\mathrm{Q}_\sigma \cap N_{s_1,\ldots,s_6} = \mathrm{Q}_{\sigma'} \cap N_{s_1,\ldots,s_6}$ if and only if $\mathrm{Q}_\sigma = \mathrm{Q}_{\sigma'}$. Vice versa if $\mathrm{Q}_\sigma \cap N_{s_1,\ldots,s_6} = \mathrm{Q}_{\sigma'} \cap N_{s_1,\ldots,s_6}$, then any generic arrangement $\mathcal{A}$ corresponding to a point in $\mathrm{Q}_\sigma \cap N_{s_1,\ldots,s_6}$ corresponds to a point in $\mathrm{Q}_{\sigma'} \cap N_{s_1,\ldots,s_6}$, that is $\operatorname{rank} A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty) = 2$ if and only if $\operatorname{rank} A_{\sigma'.\mathbb{T}}(\mathcal{A}_\infty) = 2$. It follows that $A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty)$ and $A_{\sigma'.\mathbb{T}}(\mathcal{A}_\infty)$ are submatrices of $A(\mathcal{A}_\infty)$ defined by the same three rows, i.e. $\sigma.L_l = \sigma'.L_{\tau(l)}$ for $l = 1, 2, 3$. □

Definition 4.2. For any 6 fixed indices $T = \{s_1, \ldots, s_6\} \subset [n]$ the Pappus Variety is the hypersurface in $Gr(3, \mathbb{C}^n)$ given by

$$\mathcal{P}_T = \bigcup_{\sigma \in \mathbf{S}_6} \mathrm{Q}_\sigma.$$

Notice that all the content of this section and the following section is based on the choice of six indices $\{s_1 < \cdots < s_6\} \subset [n]$. This is related to result in Theorem 3.8 in [6] and, consequently, Lemma 5.3 in [10] (Lemma 2.2 in this paper). Indeed Theorem 3.8 in [6] states that in order to study special configurations of n lines in $\mathbb{P}^2$, that is non very generic arrangements of n lines in $\mathbb{P}^2$, it is sufficient to study subsets of six lines out of n. On the other hand since Pappus Variety can be defined inside $Gr(3,\mathbb{C}^n)$, we decided to keep the discussion more general picking six indices $\{s_1 < \cdots < s_6\} \subset [n]$ instead of simply study the case $Gr(3,\mathbb{C}^6)$ (see also Remark 6.7).

For $\sigma, \sigma' \in \mathbf{S}_6$ we define the equivalence relation $\sigma.\mathbb{T} \sim \sigma'.\mathbb{T}$ corresponding to $\mathrm{Q}_\sigma = \mathrm{Q}_{\sigma'}$ as following:

$$\sigma.\mathbb{T} \sim \sigma'.\mathbb{T} \Leftrightarrow \exists\, \tau \in \mathbf{S}_3 \text{ such that } \sigma.L_i \cap \sigma.L_j = \sigma'.L_{\tau(i)} \cap \sigma'.L_{\tau(j)} \ (1 \leq i < j \leq 3).$$

We denote by $[\sigma]$ the equivalence class containing $\sigma.\mathbb{T}$ and by Q_σ the corresponding quadric (notice that σ in the notation Q_σ can be any representative of $[\sigma]$). By Lemma 4.1 $[\sigma]$ only depends on couples $L_i \cap L_j$ hence for each class $[\sigma]$ we can choice a representative

$$\tilde{\sigma}.\mathbb{T}_0 = \big\{\{j_1, j_2, j_3, j_4\},\ \{j_1, j_2, j_5, j_6\},\ \{j_3, j_4, j_5, j_6\}\big\}$$

such that $j_1 < j_2$, $j_3 < j_4$, $j_5 < j_6$ and $j_1 < j_3 < j_5$ and we can equivalently define

$$[\sigma] = \big\{\{j_1, j_2\},\ \{j_3, j_4\},\ \{j_5, j_6\}\big\}.$$

Since the number of choices of $[\sigma]$ is $\frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!} = 15$, Pappus Variety is composed by 15 quadrics. Finally remark that

$$[\sigma] = \big\{\{j_1, j_2\},\ \{j_3, j_4\},\ \{j_5, j_6\}\big\} \text{ and}$$
$$[\sigma'] = \big\{\{j'_1, j'_2\},\ \{j'_3, j'_4\},\ \{j'_5, j'_6\}\big\}$$

are *disjoint*, i.e. $[\sigma] \cap [\sigma'] = \emptyset$, if and only if $\{j_{2l-1}, j_{2l}\} \neq \{j'_{2l'-1}, j'_{2l'}\}$ for any $1 \leq l, l' \leq 3$.

Definition 4.3 (Pappus configuration)**.** Let $[\sigma_1], [\sigma_2]$ and $[\sigma_3]$ be disjoint classes, a Pappus configuration is a set $\{\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}\}$ of quadrics in $Gr(3,\mathbb{C}^n)$ such that

$$\mathrm{Q}_{\sigma_{i_1}} \cap \mathrm{Q}_{\sigma_{i_2}} = \bigcap_{i=1}^{3} \mathrm{Q}_{\sigma_i}$$

for any $\{i_1, i_2\} \subset [3]$.

Quadrics $\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}$ are said to be in Pappus configuration if $\{\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}\}$ is a Pappus configuration.

Remark 4.4. Fixed a class of good 6-partition $[\sigma] = \big\{\{j_1, j_2\},\ \{j_3, j_4\},\ \{j_5, j_6\}\big\}$, we shall count the number of disjoint classes.

First let's count the number of classes $[\sigma'] = \big\{\{j'_1, j'_2\},\ \{j'_3, j'_4\},\ \{j'_5, j'_6\}\big\}$ not disjoint and distinct from $[\sigma]$. Since $[\sigma]$ and $[\sigma']$ are distinct, only one couple $\{j'_l, j'_{l+1}\}$ is contained in $[\sigma]$. Without lost of generality we can assume $\{j_l, j_{l+1}\} = \{j'_1, j'_2\}$ (l is either $1, 3$ or 5) then pairs $\{j'_3, j'_4\}$ and $\{j'_5, j'_6\}$ are not in the same set, i.e. we have two possibilities:

$$\{j'_3, j'_5\} \text{ and } \{j'_4, j'_6\} \in [\sigma],$$

or

$$\{j'_3, j'_6\} \text{ and } \{j'_4, j'_5\} \in [\sigma].$$

Hence there are $2 \cdot 3 + 1 = 7$ not disjoint classes from $[\sigma]$ and, since the number of all classes is 15, we get that any fixed $[\sigma]$ admits exactly $15 - 7 = 8$ disjoint classes.

5 Pappus's Theorem

In this section we restate Pappus's Theorem for quadrics in $Gr(3, \mathbb{C}^n)$ by using notation introduced in the previous section. For a fixed class $[\sigma] = \big\{\{j_1, j_2\},\ \{j_3, j_4\},\ \{j_5, j_6\}\big\}$ let's denote by $G_{[\sigma]}$ the free group generated by permutations of elements in each subset of $[\sigma]$, that is

$$G_{[\sigma]} = \big\langle (j_{2l-1}\, j_{2l}) \in \mathbf{S}_6 \mid l = 1, 2, 3 \big\rangle,$$

and, for any class, $[\sigma']$ let's define the set

$$\operatorname{orbit}_{G_{[\sigma]}}([\sigma']) = \big\{\tau[\sigma'] \mid \tau \in G_{[\sigma]}\big\}$$

where τ acts naturally as permutation of entries of each set in $[\sigma']$.

Remark 5.1. The action of $G_{[\sigma]}$ on class $[\sigma']$ disjoint from $[\sigma]$ is faithful. Indeed let $\tau, \tau' \in G_{[\sigma]}$ be such that $\tau[\sigma'] = \tau'[\sigma']$ then $\tau^{-1}\tau'[\sigma'] = [\sigma']$, i.e. $\tau^{-1}\tau' \in G_{[\sigma']}$. Thus we get $\tau^{-1}\tau' \in G_{[\sigma]} \cap G_{[\sigma']}$. Since $[\sigma]$ and $[\sigma']$ are disjoint, $G_{[\sigma]} \cap G_{[\sigma']} = \{e\}$, i.e., $\tau = \tau'$. Remark that $|\operatorname{orbit}_{G_{[\sigma]}}([\sigma'])| = |G_{[\sigma]}| = 8$ and $\tau[\sigma] = [\sigma]$ for any $\tau \in G_{[\sigma]}$.

Lemma 5.2. *Let $[\sigma]$ and $[\sigma']$ be disjoint classes, then*

$$\operatorname{orbit}_{G_{[\sigma]}}([\sigma']) = \big\{[\sigma''] \mid [\sigma] \cap [\sigma''] = \emptyset\big\}.$$

Proof. First we prove that $\operatorname{orbit}_{G_{[\sigma]}}([\sigma']) \subset \big\{[\sigma''] \mid [\sigma] \cap [\sigma''] = \emptyset\big\}$. Let

$$[\sigma] = \big\{\{j_1, j_2\},\ \{j_3, j_4\},\ \{j_5, j_6\}\big\} \text{ and}$$
$$[\sigma'] = \big\{\{j'_1, j'_2\},\ \{j'_3, j'_4\},\ \{j'_5, j'_6\}\big\}$$

be disjoint, then $|\{j_{2l-1}, j_{2l}\} \cap \{j'_{2m-1}, j'_{2m}\}| \leq 1$. Since $\tau \in G_{[\sigma]}$ permutes only j_{2l-1} and j_{2l} then $\tau[\sigma'] \cap [\sigma] = \emptyset$, that is $\tau[\sigma']$ is disjoint from $[\sigma]$, i.e. $\tau[\sigma'] \in \big\{[\sigma''] \mid [\sigma] \cap [\sigma''] = \emptyset\big\}$. Since $G_{[\sigma]}$ is faithful, $|\operatorname{orbit}_{G_{[\sigma]}}([\sigma'])| = 8$ and, by calculations in the Remark 4.4, $|\big\{[\sigma''] \mid [\sigma] \cap [\sigma''] = \emptyset\big\}| = 8$, it follows that $\operatorname{orbit}_{G_{[\sigma]}}([\sigma']) = \big\{[\sigma''] \mid [\sigma] \cap [\sigma''] = \emptyset\big\}$. □

The following theorem holds.

Theorem 5.3 (Pappus's Theorem)**.** *For any disjoint classes $[\sigma]$ and $[\sigma']$, there exists a unique class $[\sigma'']$ disjoint from $[\sigma]$ and $[\sigma']$ such that $\{\mathrm{Q}_\sigma, \mathrm{Q}_{\sigma'}, \mathrm{Q}_{\sigma''}\}$ is a Pappus configuration.*

Remark 5.4. Let $[\sigma_1]$ and $[\sigma_2]$, $[\sigma_i] = \{\{j_{1,i}, j_{2,i}\},\ \{j_{3,i}, j_{4,i}\},\ \{j_{5,i}, j_{6,i}\}\}$, $i = 1, 2$ be classes of indices in $\{1, \ldots, 6\}$. Recall the following facts (see Section 2 and Lemma 2.2):

i) If

$$x = \sum_{\substack{I \subseteq [6] \\ |I| = 3}} \beta_I e_I,\ \ \beta_I \neq 0$$

is a point in Q_{σ_i} then any arrangement $\mathcal{A} \in \mathbb{C}^3$ such that $A(\mathcal{A}_\infty) = M_x$ is an arrangement of 6 planes in general position in $\mathbb{C}^3$ with lines in $\mathcal{A}_\infty$ such that points

$$H_{\infty,j_{1,i}} \cap H_{\infty,j_{2,i}}, \quad H_{\infty,j_{3,i}} \cap H_{\infty,j_{4,i}} \quad \text{and} \quad H_{\infty,j_{5,i}} \cap H_{\infty,j_{6,i}}$$

are collinear.

ii) Vice versa if $\mathcal{A}$ is an arrangement of 6 lines in general position in $\mathbb{C}^3$ with the intersection points

$$H_{\infty,j_{1,i}} \cap H_{\infty,j_{2,i}}, \quad H_{\infty,j_{3,i}} \cap H_{\infty,j_{4,i}} \quad \text{and} \quad H_{\infty,j_{5,i}} \cap H_{\infty,j_{6,i}}$$

collinear, then any point

$$x = \sum_{\substack{I \subseteq [6] \\ |I|=3}} \beta_I e_I$$

such that $M_x = A(\mathcal{A}_\infty)$ verifies $\beta_I \neq 0$ and $x \in \mathrm{Q}_{\sigma_i}$.

From ii) it follows that if $\mathcal{A}_\infty$ is an arrangement of 6 lines in general position in $\mathbb{P}^2$ such that

$$H_{\infty,j_{1,i}} \cap H_{\infty,j_{2,i}}, \quad H_{\infty,j_{3,i}} \cap H_{\infty,j_{4,i}} \quad \text{and} \quad H_{\infty,j_{5,i}} \cap H_{\infty,j_{6,i}}$$

are collinear for $i = 1, 2$, then any point

$$x = \sum_{\substack{I \subseteq [6] \\ |I|=3}} \beta_I e_I$$

such that $M_x = A(\mathcal{A}_\infty)$ belongs to $\mathrm{Q}_{\sigma_1} \cap \mathrm{Q}_{\sigma_2}$. Moreover $[\sigma_1]$ and $[\sigma_2]$ are disjoint classes. By Theorem 5.3 there exists a third class

$$[\sigma_3] = \{\{j_{1,3}, j_{2,3}\}, \{j_{3,3}, j_{4,3}\}, \{j_{5,3}, j_{6,3}\}\}$$

such that $\{\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}\}$ is a Pappus configuration. Then $x \in \bigcap_{1\le i\le 3} \mathrm{Q}_{\sigma_i}$ which implies, by i) that also

$$H_{\infty,j_{1,3}} \cap H_{\infty,j_{2,3}}, \quad H_{\infty,j_{3,3}} \cap H_{\infty,j_{4,3}} \quad \text{and} \quad H_{\infty,j_{5,3}} \cap H_{\infty,j_{6,3}}$$

have to be collinear. That is Theorem 5.3 implies Pappus hexagon Theorem in the plane (see Figure 2).

Notice that Theorem 5.3 is slightly more general than Pappus hexagon Theorem since it also applies to the case in which some $\beta_I = 0$.

Proof of Theorem 5.3. Following example in Section 3, for any class

$$[\omega_1] = \big\{\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}\big\}$$

let's consider disjoint classes

$$[\omega_2] = \big\{\{j_1, j_3\}, \{j_2, j_5\}, \{j_4, j_6\}\big\} \text{ and}$$
$$[\omega_3] = \big\{\{j_1, j_6\}, \{j_2, j_4\}, \{j_3, j_5\}\big\}.$$

The corresponding quadrics have equations:

$$\begin{aligned} \mathrm{Q}_{\omega_1}: \beta_{j_1 j_3 j_4}\beta_{j_2 j_5 j_6} - \beta_{j_2 j_3 j_4}\beta_{j_1 j_5 j_6} &= 0,\\ \mathrm{Q}_{\omega_2}: \beta_{j_4 j_2 j_5}\beta_{j_6 j_1 j_3} - \beta_{j_6 j_2 j_5}\beta_{j_4 j_1 j_3} &= 0,\\ \mathrm{Q}_{\omega_3}: \beta_{j_5 j_1 j_6}\beta_{j_3 j_2 j_4} - \beta_{j_3 j_1 j_6}\beta_{j_5 j_2 j_4} &= 0. \end{aligned}$$

By definition of β_{ijk}, equations of Q_{ω_2} and Q_{ω_3} can equivalently be written as

$$\begin{aligned} \mathrm{Q}_{\omega_2}: \beta_{j_2 j_4 j_5}\beta_{j_1 j_3 j_6} + \beta_{j_2 j_5 j_6}\beta_{j_1 j_3 j_4} &= 0,\\ \mathrm{Q}_{\omega_3}: \beta_{j_1 j_5 j_6}\beta_{j_2 j_3 j_4} + \beta_{j_1 j_3 j_6}\beta_{j_2 j_4 j_5} &= 0. \end{aligned}$$

If we denote left side of defining equations of Q_{ω_i} by P_{ω_i} then

$$P_{\omega_2} - P_{\omega_1} = P_{\omega_3},$$

that is zeros of any two polynomials $P_{\omega_{i_1}}, P_{\omega_{i_2}}$ are zeros of $P_{\omega_{i_3}}$, $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. We get

$$\mathrm{Q}_{\omega_{i_1}} \cap \mathrm{Q}_{\omega_{i_2}} = \bigcap_{i=1}^{3} \mathrm{Q}_{\omega_i}$$

for any $\{i_1, i_2\} \subset [3]$, i.e. $\mathrm{Q}_{\omega_1}, \mathrm{Q}_{\omega_2}$ and Q_{ω_3} are in Pappus configuration.

By Lemma 5.2, since $[\omega_1] \cap [\omega_2] = \emptyset$, the set of disjoint classes from $[\omega_1]$ is given by

$$\big\{[\sigma_0] \mid [\omega_1] \cap [\sigma_0] = \emptyset\big\} = \big\{\tau_0[\omega_2] \mid \tau_0 \in G_{[\omega_1]}\big\}.$$

Then if $[\sigma']$ is disjoint from $[\omega_1]$, there exists a unique element $\tau \in G_{[\omega_1]}$ such that $[\sigma'] = \tau[\omega_2]$. That is, for a generic class $[\omega_1]$, any disjoint couple $([\omega_1], [\sigma'])$ is of the form $([\omega_1], \tau[\omega_2]) = (\tau[\omega_1], \tau[\omega_2])$ and we have

$$\begin{aligned} \mathrm{Q}_{\omega_1} = \mathrm{Q}_{\tau\omega_1}: \beta_{\tau(j_1)\tau(j_3)\tau(j_4)}\beta_{\tau(j_2)\tau(j_5)\tau(j_6)} - \beta_{\tau(j_2)\tau(j_3)\tau(j_4)}\beta_{\tau(j_1)\tau(j_5)\tau(j_6)} &= 0,\\ \mathrm{Q}_{\sigma'} = \mathrm{Q}_{\tau\omega_2}: \beta_{\tau(j_4)\tau(j_2)\tau(j_5)}\beta_{\tau(j_6)\tau(j_1)\tau(j_3)} - \beta_{\tau(j_6)\tau(j_2)\tau(j_5)}\beta_{\tau(j_4)\tau(j_1)\tau(j_3)} &= 0. \end{aligned}$$

By antisymmetric property of indices of β_{ijk}, if we denote by P_{ω_1} and $P_{\sigma'}$ the left side of above equations, i.e.

$$\begin{aligned} P_{\omega_1} &= \beta_{\tau(j_1)\tau(j_3)\tau(j_4)}\beta_{\tau(j_2)\tau(j_5)\tau(j_6)} - \beta_{\tau(j_2)\tau(j_3)\tau(j_4)}\beta_{\tau(j_1)\tau(j_5)\tau(j_6)},\\ P_{\sigma'} &= \beta_{\tau(j_4)\tau(j_2)\tau(j_5)}\beta_{\tau(j_6)\tau(j_1)\tau(j_3)} - \beta_{\tau(j_6)\tau(j_2)\tau(j_5)}\beta_{\tau(j_4)\tau(j_1)\tau(j_3)} \end{aligned}$$

then

$$P_{\sigma''} := P_{\sigma'} - P_{\omega_1} = \beta_{\tau(j_5)\tau(j_1)\tau(j_6)}\beta_{\tau(j_3)\tau(j_2)\tau(j_4)} - \beta_{\tau(j_3)\tau(j_1)\tau(j_6)}\beta_{\tau(j_5)\tau(j_2)\tau(j_4)}$$

is the defining polynomial of $\mathrm{Q}_{\tau\omega_3}$. That is $[\sigma'']$ is uniquely determined by disjoint couple $([\omega_1], [\sigma'])$. □

From proof of Theorem 5.3 we get that for any class

$$[\omega_1] = \big\{\{j_1, j_2\},\ \{j_3, j_4\},\ \{j_5, j_6\}\big\}$$

if we denote

$$[\omega_2] = \{\{j_1, j_3\}, \ \{j_2, j_5\}, \ \{j_4, j_6\}\} \text{ and}$$
$$[\omega_3] = \{\{j_1, j_6\}, \ \{j_2, j_4\}, \ \{j_3, j_5\}\},$$

then all Pappus configurations are of the form $\{\mathrm{Q}_{\tau\omega_1}, \mathrm{Q}_{\tau\omega_2}, \mathrm{Q}_{\tau\omega_3}\}$, $\tau \in G_{[\omega_1]}$ and the following Corollary holds.

Notice that the proof of Theorem 5.3 only uses equations of quadrics Q_σ and hence provides alternative proof to Pappus hexagon Theorem. In particular it is also alternative to classical proof based on Grassmann-Plücker relations. Indeed the latter proof uses the fact that points in Pappus configurations verify the Grassmann-Plücker relations while, in our cases, quadrics Q_σ are proper quadrics in the Grassmannian, i.e. equations of quadrics Q_σ are not Grassmann-Plücker relations.

Corollary 5.5. *The number of Pappus configurations $\{\mathrm{Q}_\sigma, \mathrm{Q}_{\sigma'}, \mathrm{Q}_{\sigma''}\}$ in $Gr(3, \mathbb{C}^6)$ is* 20.

Proof. By Remark 4.4 the number of $[\sigma]$ is 15 and by Lemma 5.2 each fixed class $[\sigma]$ admits 8 disjoint classes. By Theorem 5.3 if $[\sigma]$ and $[\sigma']$ are fixed, $[\sigma'']$ is uniquely determined, thus the number of the sets $\{[\sigma], [\sigma'], [\sigma'']\}$ is $15 \times 8/3! = 20$. □

Corollary 5.5 establishes that for any given 6 lines in $\mathbb{P}^2$ there are 20 possible combinations of their intersections that give rise to a Pappus's configuration like the one in Figure 2.

6 Intersections of quadrics

In this section we study intersections of quadrics in $Gr(3, \mathbb{C}^n)$. In particular we are interested in the intersection of sets

$$\mathrm{Q}_\sigma^\circ = \mathrm{Q}_\sigma \cap \Big\{x = \sum_{\substack{I \subset [n] \\ |I|=3}} \beta_I e_I \mid \beta_I \neq 0 \text{ for any } I \subset \{s_1, \dots, s_6\}\Big\}$$

of points in quadrics Q_σ that correspond to arrangements of lines in $\mathbb{P}^2(\mathbb{C})$ with subarrangement $\{H_{s_1}, \dots, H_{s_6}\}$ generic. The following lemma holds.

Lemma 6.1. *If $[\sigma_1], [\sigma_2], [\sigma_3]$ are distinct and pairwise not disjoint classes then*

$$\mathrm{Q}_{\sigma_1}^\circ \cap \mathrm{Q}_{\sigma_2}^\circ \cap \mathrm{Q}_{\sigma_3}^\circ = \emptyset.$$

Proof. If $[\sigma_1], [\sigma_2], [\sigma_3]$ are not disjoint then either

(1) $|[\sigma_1] \cap [\sigma_2] \cap [\sigma_3]| = 1$ or

(2) $|[\sigma_{i_1}] \cap [\sigma_{i_2}]| = 1 \, (1 \leq i_1 < i_2 \leq 3)$ and $[\sigma_1] \cap [\sigma_2] \cap [\sigma_3] = \emptyset$.

(1) Assume $[\sigma_1] \cap [\sigma_2] \cap [\sigma_3] = \{i_1, i_2\}$. Let $[\sigma_1] = \{\{i_1, i_2\}, \{i_3, i_4\}, \{i_5, i_6\}\}$, $[\sigma_2] = \{\{i_1, i_2\}, \{i_3, i_5\}, \{i_4, i_6\}\}$, and $[\sigma_3] = \{\{i_1, i_2\}, \{i_3, i_6\}, \{i_4, i_5\}\}$ then we obtain the following quadrics

$$\mathrm{Q}_{\sigma_1} : \beta_{i_1 i_3 i_4}\beta_{i_2 i_5 i_6} - \beta_{i_2 i_3 i_4}\beta_{i_1 i_5 i_6} = 0,$$
$$\mathrm{Q}_{\sigma_2} : \beta_{i_1 i_3 i_5}\beta_{i_2 i_4 i_6} - \beta_{i_2 i_3 i_5}\beta_{i_1 i_4 i_6} = 0,$$
$$\mathrm{Q}_{\sigma_3} : \beta_{i_1 i_3 i_6}\beta_{i_2 i_4 i_5} - \beta_{i_2 i_3 i_6}\beta_{i_1 i_4 i_5} = 0.$$

Any point $x \in \mathrm{Q}^\circ_{\sigma_1} \cap \mathrm{Q}^\circ_{\sigma_2}$ belongs to $Gr(3, \mathbb{C}^n)$, that is x satisfies Plücker relations in (2.2). In particular $x \in Pl_1 \cap Pl_2$ where Pl_1 and Pl_2 are the quadrics:

$$Pl_1\colon \beta_{i_1i_3i_2}\beta_{i_4i_5i_6} - \beta_{i_1i_3i_4}\beta_{i_2i_5i_6} + \beta_{i_1i_3i_5}\beta_{i_2i_4i_6} - \beta_{i_1i_3i_6}\beta_{i_2i_4i_5} = 0,$$
$$Pl_2\colon \beta_{i_2i_3i_1}\beta_{i_4i_5i_6} - \beta_{i_2i_3i_4}\beta_{i_1i_5i_6} + \beta_{i_2i_3i_5}\beta_{i_1i_4i_6} - \beta_{i_2i_3i_6}\beta_{i_1i_4i_5} = 0.$$

Notice that Pl_1 and Pl_2 can be obtained from equations in (2.2) considering the 6-tuples $(p_1, p_2, q_0, q_1, q_2, q_3) = (i_1, i_3, i_2, i_4, i_5, i_6)$ and $(i_2, i_3, i_1, i_4, i_5, i_6)$ respectively. We get

$$\mathrm{Q}_{\sigma_2} - \mathrm{Q}_{\sigma_1} - Pl_1 + Pl_2\colon \beta_{i_1i_3i_6}\beta_{i_2i_4i_5} - \beta_{i_2i_3i_6}\beta_{i_1i_4i_5} + 2(\beta_{i_1i_2i_3}\beta_{i_4i_5i_6}) = 0.$$

Since $\beta_{i_1i_2i_3} \neq 0$ and $\beta_{i_4i_5i_6} \neq 0$ then $\beta_{i_1i_2i_3}\beta_{i_4i_5i_6} \neq 0$ and hence

$$\beta_{i_1i_3i_6}\beta_{i_2i_4i_5} - \beta_{i_2i_3i_6}\beta_{i_1i_4i_5} \neq 0,$$

that is $x \notin \mathrm{Q}^\circ_{\sigma_3}$.

(2) Assume $[\sigma_1] \cap [\sigma_2] = \{i_1, i_2\}$, $[\sigma_1] \cap [\sigma_3] = \{i_3, i_4\}$ and $[\sigma_2] \cap [\sigma_3] = \{i_5, i_6\}$ and name $P_1 = \{i_1, i_2\}$, $P_2 = \{i_3, i_4\}$, $P_3 = \{i_5, i_6\}$. To any point $x \in \mathrm{Q}^\circ_{\sigma_1} \cap \mathrm{Q}^\circ_{\sigma_2} \cap \mathrm{Q}^\circ_{\sigma_3}$ corresponds the existence of an arrangement with a generic sub-arrangement indexed by $\{i_1, \ldots, i_6\}$ which trace at infinity $\{H_{\infty,i_1}, \ldots, H_{\infty,i_6}\}$ satisfyies collinearity conditions as in Figure 3. That is there exist couples $P_4 \in [\sigma_1], P_5 \in [\sigma_2]$ and $P_6 \in [\sigma_3]$ that correspond, respectively, to intersection points p_4, p_5 and p_6 of lines in $\{H_{\infty,i_1}, \ldots, H_{\infty,i_6}\}$ (see Figure 3).

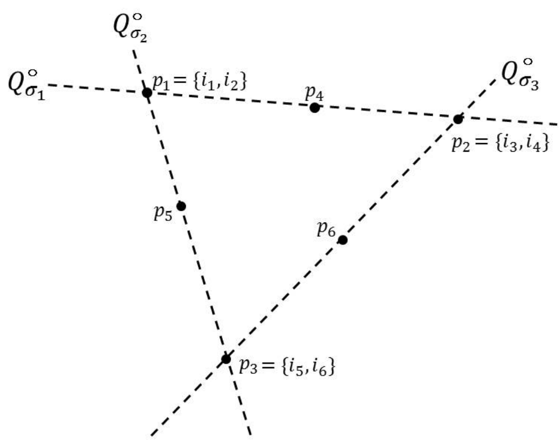

Figure 3: Case (2) trace at infinity of $\mathcal{A} \in \bigcap_{i=1}^3 \mathrm{Q}^\circ_{\sigma_i}$, $\{i, j\}$ corresponds to $H_{\infty,i} \cap H_{\infty,j}$.

By definition of P_1, P_2 and P_3 we have

$$P_3 = \{i_5, i_6\} \in \big(\{i_1, \ldots, i_6\} \setminus P_1\big) \cap \big(\{i_1, \ldots, i_6\} \setminus P_2\big).$$

On the other hand, if P_4 is different from P_1 and P_2 in $\mathrm{Q}^\circ_{\sigma_1}$ then $P_4 = \big(\{i_1, \ldots, i_6\} \setminus P_1\big) \cap \big(\{i_1, \ldots, i_6\} \setminus P_2\big)$. Thus we get $P_3 = P_4$ and, similarly, $P_5 = P_2$ and $P_6 = P_1$, that is $\mathrm{Q}^\circ_{\sigma_1} = \mathrm{Q}^\circ_{\sigma_2} = \mathrm{Q}^\circ_{\sigma_3}$ which contradict hypothesis. □

Lemma 6.2. *For any three pairwise disjoint classes* $[\sigma_1], [\sigma_2], [\sigma_3]$, *either* $\{Q_{\sigma_1}, Q_{\sigma_2}, Q_{\sigma_3}\}$ *is a Pappus configuration or*

$$\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i} = \emptyset.$$

Proof. By Pappus's Theorem, for any two disjoint classes $[\sigma_i], [\sigma_j]$, there exists $[\sigma_{ij}]$ such that $\{Q_{\sigma_i}, Q_{\sigma_j}, Q_{\sigma_{ij}}\}$ is Pappus configuration. If $[\sigma_{ij}] = [\sigma_k]$ for some $k \in [3]$, then $\{Q_{\sigma_1}, Q_{\sigma_2}, Q_{\sigma_3}\}$ is a Pappus configuration. Thus assume all $[\sigma_{ij}] \neq [\sigma_k]$ for any $k = 1, 2, 3$. Moreover $[\sigma_{12}], [\sigma_{13}], [\sigma_{23}]$ are distinct since if $[\sigma_{ij}] = [\sigma_{ik}]$ then $[\sigma_j] = [\sigma_k]$.

If $[\sigma_{12}] \cap [\sigma_{13}] \neq \emptyset$, $[\sigma_{12}] \cap [\sigma_{23}] \neq \emptyset$ and $[\sigma_{13}] \cap [\sigma_{23}] \neq \emptyset$, then

$$\bigcap_{1 \le l_1 < l_2 \le 3} Q^{\circ}_{\sigma_{l_1 l_2}} = \emptyset$$

by Lemma 6.1 and

$$\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i} = (\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i}) \cap (\bigcap_{1 \le l_1 < l_2 \le 3} Q^{\circ}_{\sigma_{l_1 l_2}}) = \emptyset.$$

Otherwise assume $[\sigma_{12}] \cap [\sigma_{13}] = \emptyset$, we get a new Pappus configuration. Since the number of disjoint classes is finite, iterating the process, we will eventually get 3 classes $[\sigma_{l_1}], [\sigma_{l_2}], [\sigma_{l_3}]$ pairwise not disjoint and

$$\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i} = (\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i}) \cap Q^{\circ}_{\sigma_{l_1}} \cap Q^{\circ}_{\sigma_{l_2}} \cap Q^{\circ}_{\sigma_{l_3}} = \emptyset.$$ □

Lemma 6.3. *If* $[\sigma_1], [\sigma_2], [\sigma_3]$ *are distinct classes such that* $[\sigma_1] \cap [\sigma_2] \neq \emptyset$ *and* $[\sigma_i] \cap [\sigma_3] = \emptyset$ *for* $i = 1, 2$, *then*

$$\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i} = \emptyset.$$

Proof. Since $[\sigma_1], [\sigma_3]$ and $[\sigma_2], [\sigma_3]$ are disjoint, there exist $[\sigma_4]$ and $[\sigma_5]$ such that $\{Q_{\sigma_1}, Q_{\sigma_3}, Q_{\sigma_4}\}$ and $\{Q_{\sigma_2}, Q_{\sigma_3}, Q_{\sigma_5}\}$ are Pappus configurations and

$$[\sigma_1] \cap [\sigma_5] \neq \emptyset, \quad [\sigma_2] \cap [\sigma_4] \neq \emptyset, \quad [\sigma_4] \cap [\sigma_5] \neq \emptyset.$$

Indeed if one of them is empty, we obtain 3 disjoint classes not in Pappus configuration and by Lemma 6.2, it follows

$$\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i} = \bigcap_{i=1}^{5} Q^{\circ}_{\sigma_i} = \emptyset.$$

Since $[\sigma_1] \cap [\sigma_2] \neq \emptyset$, we can assume $\{i_1, i_2\} = [\sigma_1] \cap [\sigma_2]$ and we can set

$$[\sigma_1] = \{\{i_1, i_2\}, \{i_3, i_4\}, \{i_5, i_6\}\},$$
$$[\sigma_2] = \{\{i_1, i_2\}, \{i'_3, i'_4\}, \{i'_5, i'_6\}\},$$
$$[\sigma_3] = \{\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}\}.$$

To any point

$$x \in \bigcap_{i=1}^{3} \mathrm{Q}^{\circ}_{\sigma_i} \neq \emptyset$$

corresponds an arrangement $\mathcal{A}$ with generic subarrangement $\{H_{i_1}, \dots, H_{i_6}\}$ with trace at infinity $\{H_{\infty,i_1}, \dots, H_{\infty,i_6}\}$ intersecting as in Figures 4 and 5 (up to rename). It follows

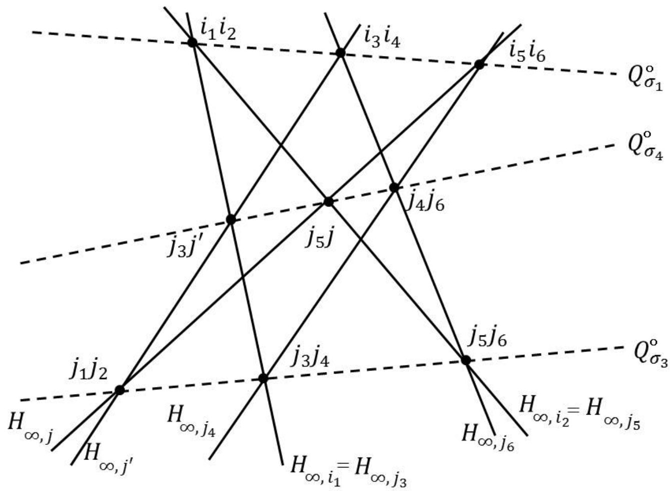

Figure 4: Each j, j' is j_1 or j_2.

that $\{j_4, j_6\} \in [\sigma_4]$ and since $\{j_3, j_5\} = \{i_1, i_2\} \in [\sigma_1]$ and $[\sigma_1] \cap [\sigma_4] = \emptyset$ (see Figure 4), there are two possibilities:

$$[\sigma_4] = \big\{\{j_4, j_6\},\ \{j_1, j_3\},\ \{j_2, j_5\}\big\}$$

or

$$[\sigma_4] = \big\{\{j_4, j_6\},\ \{j_1, j_5\},\ \{j_2, j_3\}\big\}.$$

Analogously (see Figure 5) class $[\sigma_5]$ is of the form

$$[\sigma_5] = \big\{\{j_4, j_6\}, \{j_1, j_3\}, \{j_2, j_5\}\big\}$$

or

$$[\sigma_5] = \big\{\{j_4, j_6\}, \{j_1, j_5\}, \{j_2, j_3\}\big\}.$$

Since $[\sigma_1] \cap [\sigma_5] \neq \emptyset$ and $[\sigma_5] \not\ni \{j_3, j_5\} = \{i_1, i_2\}$, we deduce that $\{j_4, j_6\} = \{i_3, i_4\}$ or $\{i_5, i_6\}$, which is not possible by $[\sigma_1] \cap [\sigma_4] = \emptyset$. Hence

$$\bigcap_{i=1}^{3} \mathrm{Q}^{\circ}_{\sigma_i} = \emptyset. \qquad \square$$

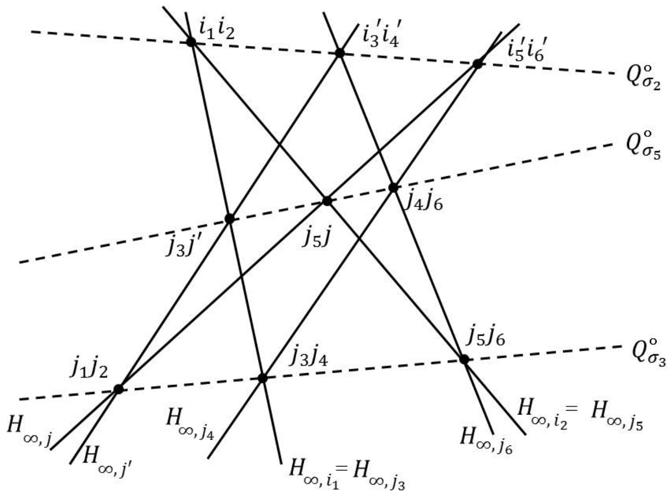

Figure 5: Each j, j' is j_1 or j_2.

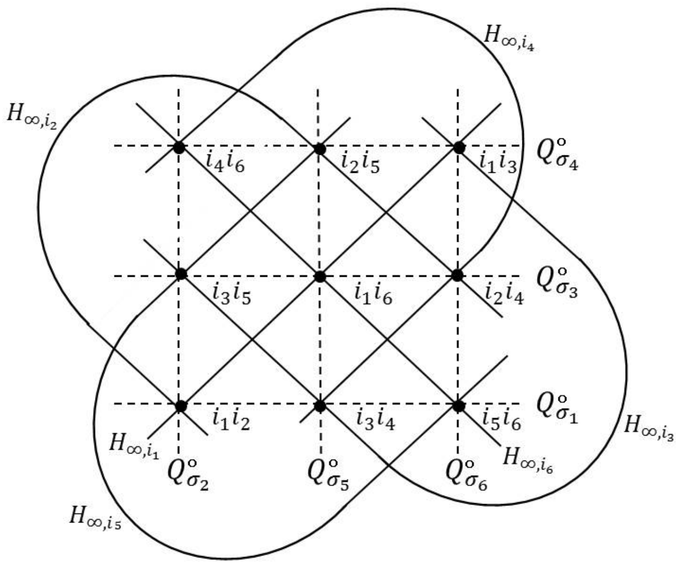

Figure 6: Hesse arrangement with $H_{\infty i_1}, \ldots, H_{\infty i_6}$ and $\bigcap_{i=1}^{6} \mathrm{Q}^\circ_{\sigma_i} \neq \emptyset$.

Notice that the Hesse arrangement in $\mathbb{P}^2(\mathbb{C})$ (see Figure 6) can be regarded as a generic arrangement of 6 lines which intersection points satisfy 6 collinearity conditions.

Definition 6.4 (Hesse configuration)**.** Let $[\sigma_i]$, $1 \leq i \leq 6$ be distinct classes, we call Hesse configuration a set $\{\mathrm{Q}_{\sigma_1}, \dots, \mathrm{Q}_{\sigma_6}\}$ of quadrics in $Gr(3, \mathbb{C}^n)$ such that there exist disjoint sets $I, J \subset [6]$, $|I| = |J| = 3$ such that $\{\mathrm{Q}_{\sigma_i}\}_{i \in I}, \{\mathrm{Q}_{\sigma_j}\}_{j \in J}$ are Pappus configurations and $[\sigma_i] \cap [\sigma_j] \neq \emptyset$ for any $i \in I$, $j \in J$.

With above notations, the following classification Theorem holds.

Theorem 6.5. *For any choice of indices $\{s_1, \dots, s_6\} \subset [n]$ sets Q°_σ, $\sigma \in \boldsymbol{S}_6$, in the Grassmannian $Gr(3, \mathbb{C}^n)$ intersect as follows.*

(1) For any disjoint classes $[\sigma_1]$ and $[\sigma_2]$, there exist $[\sigma_3], \dots, [\sigma_6]$ such that $\{\mathrm{Q}_{\sigma_1}, \dots, \mathrm{Q}_{\sigma_6}\}$ is an Hesse configuration for $I = \{1, 2, 3\}$, $J = \{4, 5, 6\}$ and

$$\bigcap_{i=1}^{2} \mathrm{Q}^\circ_{\sigma_i} = \bigcap_{i=1}^{3} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \bigcap_{i=1}^{4} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \bigcap_{i=1}^{6} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \emptyset.$$

(2) For any not disjoint classes $[\sigma_1]$ and $[\sigma_2]$, there exist $[\sigma_3], \dots, [\sigma_6]$ such that $\{\mathrm{Q}_{\sigma_1}, \dots, \mathrm{Q}_{\sigma_6}\}$ is an Hesse configuration for $I = \{1, 3, 4\}$, $J = \{2, 5, 6\}$ and

$$\bigcap_{i=1}^{2} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \bigcap_{i=1}^{3} \mathrm{Q}^\circ_{\sigma_i} = \bigcap_{i=1}^{4} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \bigcap_{i=1}^{6} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \emptyset.$$

All other intersections are empty.

Remark 6.6. Notice that, since Hesse configuration only exists in the complex case, in $Gr(3, \mathbb{C}^n)$ we can find 6 quadrics $\{\mathrm{Q}_{\sigma_1}, \dots, \mathrm{Q}_{\sigma_6}\}$ such that

$$\bigcap_{i=1}^{6} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \emptyset,$$

while in $Gr(3, \mathbb{R}^n)$,

$$\bigcap_{\substack{j \in J \subset [6] \\ |J| > 4}} \mathrm{Q}^\circ_{\sigma_j} = \emptyset.$$

It follows that in the real case, for any choice of indices $\{s_1, \dots, s_6\} \subset [n]$, we have at most 4 collinearity conditions (see Figure 7) corresponding to 15 hyperplanes in the Discriminantal arrangement with 4 multiplicity 3 intersections in codimension 2 (see Figure 8). While in the complex case Hesse configuration (see Figure 6) gives rise to a Discriminantal arrangement containing 15 hyperplanes intersecting in 6 multiplicity 3 spaces in codimension 2.

This remark allows a better understanding of differences in the combinatorics of Discriminantal arrangement in the real and complex cases. Indeed the existence of a discriminantal arrangement of 15 hyperplanes intersecting in 6 multiplicity 3 spaces in codimension 2 in $\mathbb{C}$ but not in $\mathbb{R}$ implies that there exist combinatorics of Discriminantal arrangements that cannot be realised in any field. This is especially interesting since in the case known until now, i.e. in the case of very generic arrangements $\mathcal{A}$, the combinatorics of Discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$ is independent from the field (see [1]).

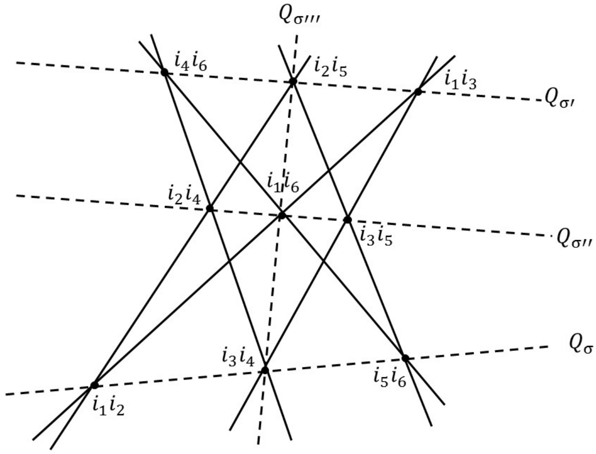

Figure 7: Generic arrangement $\mathcal{A}$ in $\mathbb{R}^3$ containing 6 lines satisfying 4 collinearity conditions.

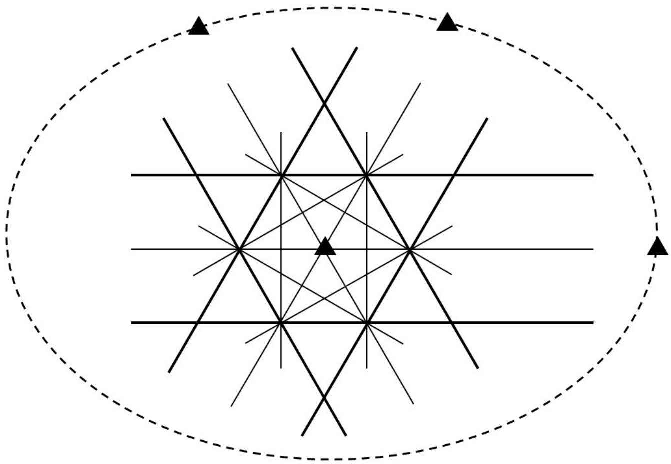

Figure 8: Codimension 2 intersections of 15 hyperplanes in $\mathcal{B}(n, 3, \mathcal{A}_\infty)$ indexed in $\{s_1, \ldots, s_6\} \subset [n]$ with 4 multiplicity 3 points ▲ corresponding to intersections $\bigcap_{i=1}^{3} D_{\sigma.L_i}, \bigcap_{i=1}^{3} D_{\sigma'.L_i}, \bigcap_{i=1}^{3} D_{\sigma''.L_i}$ and $\bigcap_{i=1}^{3} D_{\sigma'''.L_i}$, $\sigma, \sigma', \sigma'', \sigma'''$ as in Figure 7.

Remark 6.7. Finally Theorem 6.5 implies that the maximum number of intersections of multiplicity 3 in codimension 2 in the complex case is strictly higher than the one in the real case. This agrees with results on maximum number of triple points in an arrangement of lines in $\mathbb{P}^2$ (see [3] for a discussion on line arrangements with maximal number of triple points over arbitrary fields). Those observations suggest that special configurations of lines in the projective plane intersecting in a big number of triple points could be understood by studying Discriminantal arrangements with maximum number of multiplicity 3 intersections in codimension 2. Indeed each multiplicity 3 intersection in codimension 2 of $\mathcal{B}(n, 3, \mathcal{A}_\infty)$ corresponds to a collinearity condition for lines in $\mathcal{A}_\infty$ which is equivalent to the possibility to add a line that gives rise to "higher" number of triple points. It seems hence interesting to study exact number of intersections of type (1) and (2) in Theorem 6.5 in the Grassmannian $Gr(3, \mathbb{C}^n)$. This will be object of further studies.

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