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# A note on the 4-girth-thickness of $K_{n,n,n}$*

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**Abstract**

The 4-girth-thickness $\theta(4, G)$ of a graph $G$ is the minimum number of planar subgraphs of girth at least four whose union is $G$. In this paper, we obtain that the 4-girth-thickness of complete tripartite graph $K_{n,n,n}$ is $\lceil \frac{n+1}{2} \rceil$ except for $\theta(4, K_{1,1,1}) = 2$. And we also show that the 4-girth-thickness of the complete graph $K_{10}$ is three which disprove the conjecture posed by Rubio-Montiel concerning to $\theta(4, K_{10})$.

*Keywords: Thickness, 4-girth-thickness, complete tripartite graph.*

*Math. Subj. Class.: 05C10*

## 1 Introduction

The *thickness* $\theta(G)$ of a graph $G$ is the minimum number of planar subgraphs whose union is $G$. It was defined by W. T. Tutte [10] in 1963. Then, the thicknesses of some graphs have been obtained when the graphs are hypercube [7], complete graph [1, 2, 11], complete bipartite graph [3] and some complete multipartite graphs [6, 12, 13].

In 2017, Rubio-Montiel [9] defined the $g$-girth-thickness $\theta(g, G)$ of a graph $G$ as the minimum number of planar subgraphs whose union is $G$ with the girth of each subgraph is at least $g$. It is a generalization of the usual thickness in which the 3-girth-thickness $\theta(3, G)$ is the usual thickness $\theta(G)$. He also determined the 4-girth-thickness of the complete graph $K_n$ except $K_{10}$ and he conjectured that $\theta(4, K_{10}) = 4$. Let $K_{n,n,n}$ denote a complete tripartite graph in which each part contains $n$ $(n \geq 1)$ vertices. In [13], Yang obtained $\theta(K_{n,n,n}) = \lceil \frac{n+1}{3} \rceil$ when $n \equiv 3 \pmod 6$.

In this paper, we determine $\theta(4, K_{n,n,n})$ for all values of $n$ and we also give a decomposition of $K_{10}$ with three planar subgraphs of girth at least four, which shows $\theta(4, K_{10}) = 3$.

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## 2 The 4-girth-thickness of $K_{n,n,n}$

**Lemma 2.1** ([4]). *A planar graph with $n$ vertices and girth $g$ has at most $\frac{g}{g-2}(n-2)$ edges.*

**Theorem 2.2.** *The 4-girth-thickness of $K_{n,n,n}$ is*

$$\theta(4, K_{n,n,n}) = \left\lceil \frac{n+1}{2} \right\rceil$$

*except for $\theta(4, K_{1,1,1}) = 2$.*

*Proof.* It is trivial for $n = 1$, $\theta(4, K_{1,1,1}) = 2$. When $n > 1$, because $|E(K_{n,n,n})| = 3n^2$, $|V(K_{n,n,n})| = 3n$, from Lemma 2.1, we have

$$\theta(4, K_{n,n,n}) \geq \left\lceil \frac{3n^2}{2(3n-2)} \right\rceil = \left\lceil \frac{n}{2} + \frac{1}{3} + \frac{2}{3(3n-2)} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil.$$

In the following, we give a decomposition of $K_{n,n,n}$ into $\lceil \frac{n+1}{2} \rceil$ planar subgraphs of girth at least four to complete the proof. Let the vertex partition of $K_{n,n,n}$ be $(U, V, W)$, where $U = \{u_1, \ldots, u_n\}$, $V = \{v_1, \ldots, v_n\}$ and $W = \{w_1, \ldots, w_n\}$. In this proof, all the subscripts of vertices are taken modulo $2p$.

**Case 1**: When $n = 2p$ $(p \geq 1)$. Let $G_1, \ldots, G_{p+1}$ be the graphs whose edge set is empty and vertex set is the same as $V(K_{2p,2p,2p})$.

**Step 1**: For each $G_i$ $(1 \leq i \leq p)$, arrange all the vertices $u_1, v_{3-2i}, u_2, v_{4-2i}, u_3, v_{5-2i}, \ldots, u_{2p}, v_{2p-2i+2}$ on a circle and join $u_j$ to $v_{j+2-2i}$ and $v_{j+1-2i}$, $1 \leq j \leq 2p$. Then we get a cycle of length $4p$, denote it by $G_i^1$ $(1 \leq i \leq p)$.

**Step 2**: For each $G_i^1$ $(1 \leq i \leq p)$, place the vertex $w_{2i-1}$ inside the cycle and join it to $u_1, \ldots, u_{2p}$, place the vertex $w_{2i}$ outside the cycle and join it to $v_1, \ldots, v_{2p}$. Then we get a planar graph $G_i^2$ $(1 \leq i \leq p)$.

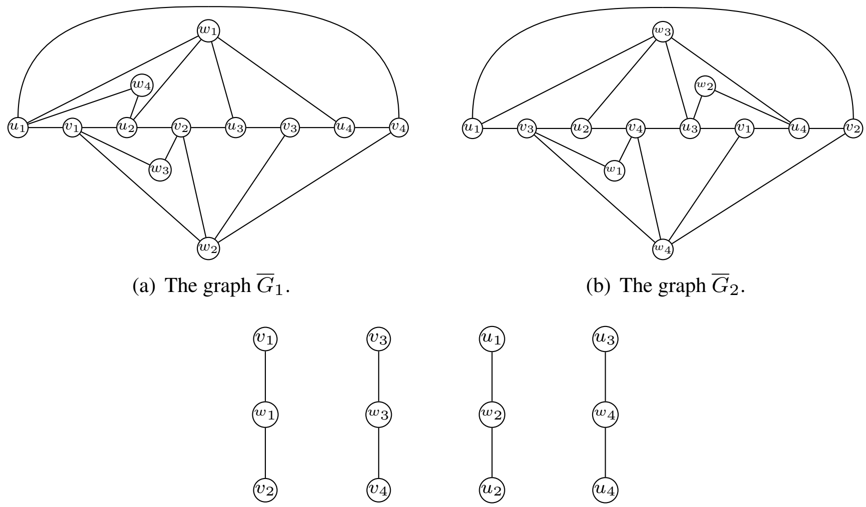
**Step 3**: For each $G_i^2$ $(1 \leq i \leq p)$, place vertices $w_{2j}$ for $1 \leq j \leq p$ and $j \neq i$, inside of the quadrilateral $w_{2i-1}u_{2i-1}v_1u_{2i}$ and join each of them to vertices $u_{2i-1}$ and $u_{2i}$. Place vertices $w_{2j-1}$, for $1 \leq j \leq p$ and $j \neq i$, inside of the quadrilateral $w_{2i}v_{2i-1}u_kv_{2i}$, in which $u_k$ is some vertex from $U$. Join each of them to vertices $v_{2i-1}$ and $v_{2i}$. Then we get a planar graph $\overline{G}_i$ $(1 \leq i \leq p)$.

**Step 4**: For $G_{p+1}$, join $w_{2i-1}$ to both $v_{2i-1}$ and $v_{2i}$, join $w_{2i}$ to both $u_{2i-1}$ and $u_{2i}$, for $1 \leq i \leq p$, then we get a planar graph $\overline{G}_{p+1}$.

For $\overline{G}_1 \cup \cdots \cup \overline{G}_{p+1} = K_{n,n,n}$, and the girth of $\overline{G}_i$ $(1 \leq i \leq p+1)$ is at least four, we obtain a 4-girth planar decomposition of $K_{2p,2p,2p}$ with $p+1$ planar subgraphs. Figure 1 shows a 4-girth planar decomposition of $K_{4,4,4}$ with three planar subgraphs.

**Case 2**: When $n = 2p + 1$ $(p > 1)$. Base on the 4-girth planar decomposition $\{\overline{G}_1, \ldots, \overline{G}_{p+1}\}$ of $K_{2p,2p,2p}$, by adding vertices and edges to each $\overline{G}_i$ $(1 \leq i \leq p+1)$ and some other modifications on it, we will get a 4-girth planar decomposition of $K_{2p+1,2p+1,2p+1}$ with $p+1$ subgraphs.

**Step 1**: (Add $u$ to $\overline{G}_i, 1 \leq i \leq p$.) For each $\overline{G}_i$ $(1 \leq i \leq p)$, we notice that the order of the $p-1$ interior vertices $w_{2j}$, $1 \leq j \leq p$, and $j \neq i$ in the quadrilateral



(a) The graph $\overline{G}_1$.

(b) The graph $\overline{G}_2$.

(c) The graph $\overline{G}_3$.

Figure 1: A 4-girth planar decomposition of $K_{4,4,4}$.

$w_{2i-1}u_{2i-1}v_1u_{2i}$ of $\overline{G}_i$ has no effect on the planarity of $\overline{G}_i$. We adjust the order of them, such that $w_{2i-1}u_{2i-1}w_{2p-2i+2}u_{2i}$ is a face of a plane embedding of $\overline{G}_i$. Place the vertex $u$ in this face and join it to both $w_{2i-1}$ and $w_{2p-2i+2}$. We denote the planar graph we obtain by $\widehat{G}_i$ $(1 \le i \le p)$.

**Step 2**: (Add $v$ and $w$ to $\widehat{G}_1$.) Delete the edge $v_1u_2$ in $\widehat{G}_1$, put both $v$ and $w$ in the face $w_ku_1v_1w_tv_2u_2$ in which $w_k$ is some vertex from $\{w_{2j} \mid 1 < j \le p\}$ and $w_t$ is some vertex from $\{w_{2j-1} \mid 1 < j \le p\}$. Join $v$ to $w$, join $v$ to $u_1, u_2$, and join $w$ to $v_1, v_2$, we get a planar graph $\widetilde{G}_1$.

**Step 3**: (Add $v$ and $w$ to $\widehat{G}_i, 2 \le i \le p$.) For each $\widehat{G}_i$ $(2 \le i \le p)$, place the vertex $v$ in the face $w_ku_{2i-1}v_1u_{2i}$ in which $w_k$ is some vertex from $\{w_{2j} \mid 1 \le j \le p \text{ and } j \ne i\}$, and join it to $u_{2i-1}$ and $u_{2i}$. Place the vertex $w$ in the face $w_kv_{2i-1}u_tv_{2i}$ in which $w_k$ is some vertex from $\{w_{2j-1} \mid 1 \le j \le p \text{ and } j \ne i\}$ and $u_t$ is some vertex from $U$. Join $w$ to both $v_{2i-1}$ and $v_{2i}$, we get a planar graph $\widetilde{G}_i$ $(2 \le i \le p)$.

**Step 4**: (Add $u, v$ and $w$ to $\overline{G}_{p+1}$.) We add $u, v$ and $w$ to $\overline{G}_{p+1}$. For $1 \le i \le 2p$, join $u$ to each $v_i$, join $v$ to each $w_i$, join $w$ to each $u_i$, join $u$ to both $v$ and $w$, and join $v_1$ to $u_2$, then we get a planar graph $\widetilde{G}_{p+1}$. Figure 2 shows a plane embedding of $\widetilde{G}_{p+1}$.

For $\widetilde{G}_1 \cup \cdots \cup \widetilde{G}_{p+1} = K_{n,n,n}$, and the girth of $\widetilde{G}_i$ $(1 \le i \le p+1)$ is at least four, we obtain a 4-girth planar decomposition of $K_{2p+1,2p+1,2p+1}$ with $p+1$ planar subgraphs. Figure 3 shows a 4-girth planar decomposition of $K_{5,5,5}$ with three planar subgraphs.

**Case 3**: When $n = 3$, Figure 4 shows a 4-girth planar decomposition of $K_{3,3,3}$ with two planar subgraphs.

Summarizing the above, the theorem is obtained. □

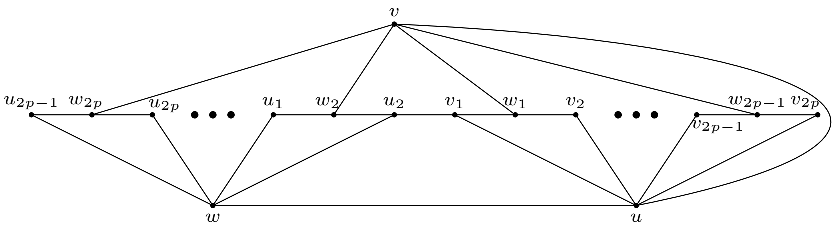
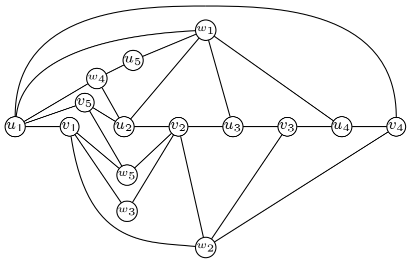
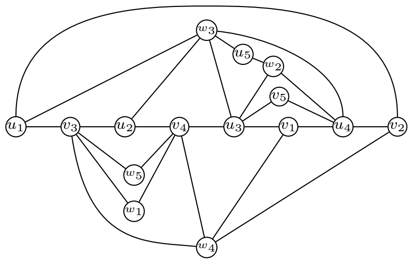


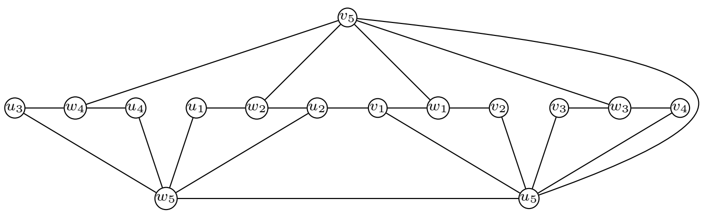

Figure 2: The graph $\widetilde{G}_{p+1}$.



(a) The graph $\widetilde{G}_1$.



(b) The graph $\widetilde{G}_2$.



(c) The graph $\widetilde{G}_3$.

Figure 3: A 4-girth planar decomposition of $K_{5,5,5}$.

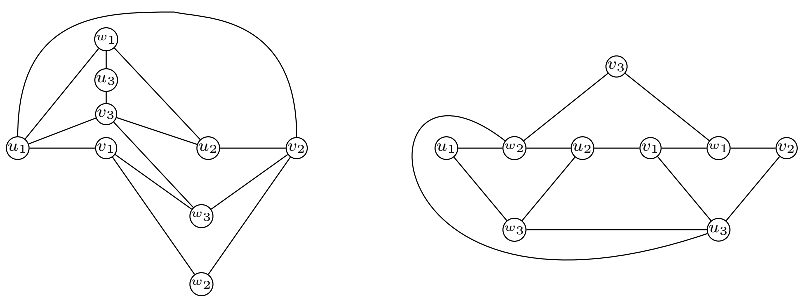


Figure 4: A 4-girth planar decomposition of $K_{3,3,3}$.

## 3 The 4-girth-thickness of $K_{10}$

In [9], the author posed the question whether $\theta(4, K_{10}) = 3$ or 4, and conjectured that it is four. We disprove his conjecture by showing $\theta(4, K_{10}) = 3$.

**Theorem 3.1.** *The 4-girth-thickness of $K_{10}$ is three.*

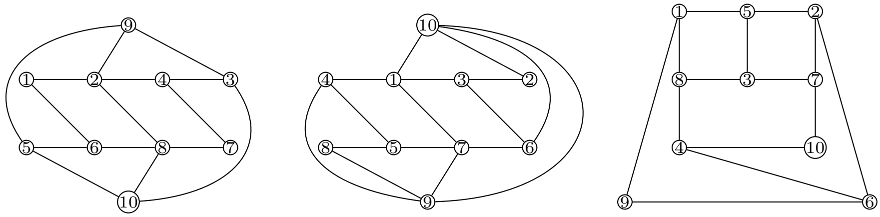



Figure 5: A 4-girth planar decomposition of $K_{10}$.

*Proof.* From [9], we have $\theta(4, K_{10}) \geq 3$. We draw a 4-girth planar decomposition of $K_{10}$ with three planar subgraphs in Figure 5, which shows $\theta(4, K_{10}) \leq 3$. The theorem follows. □

We would like to state that after submitting this paper for review, we notice that there exist two results regarding the 4-girth-thickness of $K_{2p,2p,2p}$ and $K_{10}$. Rubio-Montiel [8] obtained the exact value of the 4-girth-thickness of the complete multipartite graph when each part has an even number of vertices. And by computer, Castañeda-López et al. [5] found the other two decompositions of $K_{10}$ into three planar subgraphs of girth at least four. In this paper, we give these results in a constructive way.

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