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Semiregular automorphisms in vertex-transitive graphs with a solvable group of automorphisms

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Abstract

It has been conjectured that automorphism groups of vertex-transitive (di)graphs, and more generally 2-closures of transitive permutation groups, must necessarily possess a fixed-point-free element of prime order, and thus a non-identity element with all orbits of the same length, in other words, a *semiregular* element. The known affirmative answers for graphs with primitive and quasiprimitive groups of automorphisms suggest that solvable groups need to be considered if one is to hope for a complete solution of this conjecture. It is the purpose of this paper to present an overview of known results and suggest possible further lines of research towards a complete solution of the problem.

Keywords: Solvable group, semiregular automorphism, fixed-point-free automorphism, polycirculant conjecture.

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1 Introduction

It is known that each finite transitive permutation group contains a fixed-point-free element of prime power order (see [8, Theorem 1]), but not necessarily a fixed-point-free element of prime order (which is equivalent to existence of a semiregular element) [2, 8]. In 1981 it was asked if every vertex-transitive digraph admits a semiregular automorphism (see [18, Problem 2.4]). The existence of such automorphisms plays an important role in solutions to many important open problems in algebraic graph theory, such as, for example, in the classifications of graphs satisfying certain prescribed symmetry conditions (see [15, 16,

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20, 22, 25]). Semiregular automorphisms have also proved useful in the long standing hamiltonicity problem for connected vertex-transitive graphs and in a recently explored dichotomy of even/odd automorphisms (see [1, 13, 17]).

In 1997 Klin generalized the semiregularity problem conjecturing that every transitive 2-closed permutation group contains a semiregular element (see [4]) – the term *polycirculant conjecture* is sometimes used for the semiregularity problem in this wider context. (Recall that for a finite permutation group G on a set V the 2-*closure* $G^{(2)}$ of G is the largest subgroup of the symmetric group $\mathrm{Sym}(V)$ containing G and having the same orbits as G in the induced action on $V \times V$.) Both terms will be used throughout the paper, this should cause no confusion. The problem has spurred a lot of interest in the mathematical community producing several partial results – addressing graphs with valency and/or order restrictions – with varying degrees of difficulties involved in their proofs (see for instance [2, 3, 5, 6, 7, 8, 9, 10, 12, 14, 19, 21, 23, 24]). Recently, Giudici, Potočnik and Verret [11] considered the problem in the context of graphs whose automorphism group acts transitivity on edges and not necessarily on vertices. They proved that every regular edge-transitive graph of valency three or four has a semiregular automorphism. Also, in 2003 Giudici [9] proved the polycirculant conjecture for quasiprimitive groups, and in 2007 Giudici and Xu [12] proved it for biquasiprimitive groups.

Since the automorphism group of a vertex-transitive graph is a transitive 2-closed group these results imply that the only graphs for which the semiregularity problem has not yet been settled are graphs whose automorphism groups contain a non-identity normal subgroup with at least three orbits. Clearly, since disconnected vertex-transitive graphs must contain semiregular automorphisms, we can restrict ourselves to connected graphs.

It is usually the case that algebraic graph theory problems dealing with group actions on graphs are considerably harder to address for nonsolvable groups than for the solvable ones. Such is the case, for example, with various types of classification problems for arc-transitive graphs and vertex-transitive graphs in general. Counter-intuitively, this does not seem to be the case with the polycirculant conjecture. While, as already mentioned it has been proved that the polycirculant conjecture holds for quasiprimitive groups [9], nothing of that nature is known for groups at the opposite end of the spectrum. For example, solvable groups turned out to be the steepest hill to climb in the completion of the proof that groups of square-free degree satisfy the polycirculant conjecture, see [5].

Our aim is to discuss possible ways of approaching the semiregularity problem for solvable groups, trying to single out certain idiosyncrasies of this class of groups relevant to the problem.

Problem 1.1. Does the 2-closure $G^{(2)}$ of a solvable group G contain a semiregular element?

In Proposition 2.3 a partial solution to this problem is given for groups of degree mp^2, where $m < p$ is square-free. As a consequence a partial solution to existence of semiregular automorphism in vertex-transitive graphs of order p^2q, where p and q are primes, is shown (see Theorem 2.4 and Corollary 2.5). Since disconnected vertex-transitive graphs clearly contain semiregular automorphisms, the graphs considered in Section 2 are connected.

2 Searching for semiregular elements

First let us recall the definition of a pseudometric first defined in [5] where it was used as one of the tools in proving the existence of semiregular elements in transitive permutation groups of square-free degree.

Let G be a transitive permutation group with a complete block system $\mathcal{B}$ such that $\mathrm{fix}_G(\mathcal{B})$ contains a subgroup $K \cong U^s$, for some $s \geq 1$ and such that the restriction $K^B \cong U^r$, $1 \leq r \leq s$, acts transitively on B, for each $B \in \mathcal{B}$. Then in view of [5, Proposition 3.1] we can define a *pseudometric* on $\mathcal{B}$ by letting

$$\mathrm{Dist}_K(B, B') = \log_{|U|} |K_{(B)}^{B'}|.$$

(For the proof that Dist_K is symmetric and that it satisfies the triangle inequality see [5, Proposition 3.1].)

In Proposition 2.1 below the extremal case where the possible distances in this pseudometric are only 0 and 1 is considered.

Proposition 2.1. *Let p be a prime, let $s \geq 1$ be an integer, let U be a simple group and let G be a transitive permutation group on a set V admitting an imprimitivity block system $\mathcal{B}$ with blocks of length divisible by p. If $\mathrm{fix}_G(\mathcal{B})$ contains a subgroup $K \cong U^s$ such that for each block $B \in \mathcal{B}$, the restriction K^B is isomorphic to U, acts transitively on B and contains a semiregular element of order p, then $G^{(2)}$ contains a semiregular element of order p.*

Proof. Observe that the assumptions in the statement of the proposition imply that in the above pseudometric language the possible distances between any two blocks in $\mathcal{B}$ are either 0 or 1. Namely, since $K_{(B)}^{B'}$ is a normal subgroup of $K^{B'} \cong U$ it follows that $K_{(B)}^{B'}$ is either 1 or U. In particular, for $B, B' \in \mathcal{B}$, the following holds

$$\mathrm{Dist}_K(B, B') = 1 \Leftrightarrow \ K_{(B)} \text{ is transitive on } B' \text{ and } K_{(B')} \text{ is transitive on } B. \tag{2.1}$$

This will allow us to construct a semiregular element in $G^{(2)}$ by a succession of superpositions of permutations acting independently on collections of blocks at distance 0. First, if $s = 1$ then the distance between any two blocks in $\mathcal{B}$ is equal to 0, and thus the element of K whose restriction to a block $B \in B$ is semiregular on B is semiregular on V too. We may therefore assume that the maximal distance between blocks in $\mathcal{B}$ is precisely 1. One can easily see that each of the subsets of those blocks in $\mathcal{B}$ being at mutual distance 0 forms a block of G. More precisely,

$$\mathcal{C} = \{\{B_{i_0} \cup \ldots \cup B_{i_k} \mid \mathrm{Dist}_K(B_{i_j}, B_{i_t}) = 0 \text{ for all } i_j, i_t \in \{i_0, \ldots, i_k\}\} \mid \\ i \in \{0, \ldots, e\}\},$$

where $e = |\mathcal{B}|$, is an imprimitivity block system of G. Moreover, in view of (2.1) for every block $C_i = B_{i_0} \cup \ldots \cup B_{i_k} \in \mathcal{C}$ there exists an element $\gamma_i \in K$ such that $\gamma_i^{C_i}$ is semiregular and $\gamma_i^{C_j} = 1$ for all blocks $C_j \in \mathcal{C}$, $i \neq j$. Consequently, $\gamma_0 \gamma_1 \cdots \gamma_k$ is semiregular on V, completing the proof of Proposition 2.1. □

Corollary 2.2. *Let G be a permutation group acting transitively on a set V and let M be a minimal normal subgroup of G having orbits of prime length p on V. Then $G^{(2)}$ contains a semiregular element of order p.*

Proof. Since M is a minimal normal subgroup of G it is isomorphic to a direct product of isomorphic simple groups, that is, $M \cong U^s$, $s \geq 1$, where U is a simple group. The orbits of M form an imprimitivity block system $\mathcal{B}$ consisting of blocks of length p. For $B \in \mathcal{B}$ the restriction M^B is therefore a transitive group of prime degree p, and thus $M^B \cong U$. Hence Proposition 2.1 applies. □

Proposition 2.3. *Let G be a transitive solvable group of degree mp^2, where $m < p$ is square-free. Then $G^{(2)}$ contains a semiregular element of prime order.*

Proof. Let M be a minimal normal subgroup of G. Since G is solvable we have $M \cong \mathbb{Z}_q^s$, where q is a prime and $s \geq 1$. The orbits of M give rise to an imprimitivity block system $\mathcal{B}$. If the blocks in $\mathcal{B}$ are of prime length then Corollary 2.2 applies. We may therefore assume that $\mathcal{B}$ consists of blocks of size p^2 and $M \cong \mathbb{Z}_p^s$, $s \geq 1$.

Let $\mathcal{B} = \{B_0, \ldots, B_{m-1}\}$. Assume that $G^{(2)}$ does not contain semiregular elements, and let $\alpha \in M$ be an element of order p with a minimal number of orbits of M on which the restriction of α is the identity. Without loss of generality we may assume that

$$\alpha^{B_i} = \begin{cases} 1; & 0 \leq i \leq t \\ \neq 1; & t < i \leq m-1 \end{cases}$$

where $t < m-1$. Let $B \in \mathcal{B}$ be a block for which $\alpha^B = 1$. Because of transitivity of G there exists $\beta \in M$, a conjugate of α, such that $\beta^B \neq 1$. Since $m < p$ there exists $k \in \mathbb{Z}_p$ such that $\alpha\beta^k$ is semiregular and of order p on each of the blocks B_i, $t < i \leq m-1$, as well as on the block B. Therefore, the number of blocks on which the restriction of $\alpha\beta^k$ is the identity is at least one less than the number of blocks on which the restriction of α is the identity, contradicting the minimality condition. It follows that $G^{(2)}$ must contain semiregular elements as claimed. (Note that the assumption that $m < p$ was essential in this respect.) □

With the use of Corollary 2.2 and Proposition 2.3 we can now prove the following result about existence of semiregular automorphisms in vertex-transitive graphs of order qp^2, where p and q are primes.

Theorem 2.4. *Let X be a connected vertex-transitive graph of order p^2q, where p and q are primes, and either $q \leq p$ or $p^2 < q$. Then either*

(i) X admits a semiregular automorphism, or

(ii) $2 < q < p$ and $\mathrm{Aut}(X)$ is nonsolvable with an intransitive non-abelian minimal normal subgroup whose orbits are either of length p^2 or of length pq.

Proof. First, we may assume that $p > 3$ and $q > 2$ and that $q \neq p$. Namely, if $q = 2$ or $q = p$ then the order of X equals $2p^2$ or p^3, and the existence of semiregular automorphisms was proved in [19] and [18], respectively. If $q > p^2$ then the existence of semiregular automorphisms follows from results in [18]. Therefore, we may assume that $q < p^2$.

If $\mathrm{Aut}(X)$ is primitive or quasiprimitive then, by [9], X contains a semiregular automorphism. We may therefore assume that there exists a minimal normal subgroup M of $\mathrm{Aut}(X)$ whose orbits give rise to a non-trivial imprimitivity block system $\mathcal{B}$.

If $\mathrm{Aut}(X)$ is solvable then M is abelian and isomorphic to $\mathbb{Z}_r^k$, where $r \in \{q, p\}$ and $k \geq 1$. Hence $\mathcal{B}$ consists of blocks of length q, p or p^2, and Corollary 2.2 and Proposition 2.3 imply the existence of semiregular automorphisms of X. If, however, $\mathrm{Aut}(X)$

is nonsolvable then M is non-abelian. If the orbits of M are of prime length then Corollary 2.2 applies. Otherwise the orbits of M are either of length p^2 or qp, completing the proof of Theorem 2.4. □

The following corollary is an immediate consequence of Theorem 2.4.

Corollary 2.5. *Let q and p be primes such that either $q \leq p$ or $p^2 < q$. A connected vertex-transitive graph of order p^2q admitting a transitive solvable group of automorphisms has a semiregular automorphism.*

In our search for semiregular group elements we now turn to vertex-transitive graphs admitting a solvable group of automorphisms and satisfying certain valency restrictions. For a graph X admitting a transitive action of a group G with an imprimitivity block system $\mathcal{B}$ arising from the orbits of a normal subgroup $M \leq G$, we let $X/\mathcal{B}$ denote the corresponding quotient graph having vertex set $\mathcal{B}$ with two blocks $B, B' \in \mathcal{B}$ being adjacent if there is an edge in X joining a vertex in B to a vertex in B'. Further, for $B, B' \in \mathcal{B}$ we let $[B, B']$ denote the bipartite graph induced by the edges of X joining blocks B and B', and we let $\mathrm{val}(B, B')$ denote the valency of $[B, B']$. Also let $\mathrm{val}(X)$ denote the valency of X.

Lemma 2.6. *Let X be a connected vertex-transitive graph admitting a transitive action of a solvable group G with a minimal normal subgroup $M = \mathbb{Z}_q^k$, and suppose further that $\mathrm{val}(X) < pq$, where $p > q$ is the largest prime dividing $|G|$. Then either*

(i) G contains a semiregular subgroup or

(ii) M is intransitive and there exist orbits B, B' of M such that $\mathrm{val}(B, B') \geq mq$, where mp is the smallest multiple of q exceeding p.

Proof. Assume that (i) does not hold. It follows that M is intransitive for otherwise X would be a Cayley graph of M and so M would act semiregularly on $V(X)$. Let $\mathcal{B}$ be the imprimitivity block system arising from the orbits of M, and let $\mathrm{Dist} = \mathrm{Dist}_M$ be the associated pseudometric on $\mathcal{B}$. If $\mathrm{Dist}(B, B') = 0$ for every two adjacent blocks B and B' in $X/\mathcal{B}$, then clearly M contains a semiregular element of order q. We may therefore assume that there are adjacent blocks B and B' such that $d = \mathrm{Dist}(B, B') \geq 1$. Furthermore, we may assume that G_v, for $v \in V(X)$, contains elements of order q as well as elements of order p.

Fix a vertex $v \in B$. Since $d \geq 1$, we have that $\mathrm{val}(B, B')$ is a positive multiple of q, and so is at least q. By assumption, there exists an element $\gamma \in G_v$ of order p which cyclically permutes at least p neighbors $w_0, w_1, \ldots, w_{p-1}$ of v, where $\gamma^i(w_0) = w_i$, and clearly fixes all other neighbors.

Without loss of generality let $w_0 \in B'$. We claim that w_i belongs to B' for every $i \in \{0, \ldots, p-1\}$. If that was not the case there would be p distinct blocks $\gamma^i(B')$, with $\mathrm{Dist}(B, \gamma^i(B')) \geq 1$, $i \in \{0, \ldots, p-1\}$. Consequently, v would have at least q neighbors in each of these p blocks and so the valency of X would be at least pq, which contradicts the assumption. We conclude that each $w_i \in B'$. It follows that $\mathrm{val}(B, B') \geq p$. But $\mathrm{val}(B, B')$ is a multiple of q, and hence at least mq. □

Proposition 2.7. *Let $p > q$ be primes and let X be a connected vertex-transitive graph admitting a transitive solvable $\{p, q\}$-group G, and let M be a minimal normal elementary abelian subgroup of G. Then one of the following possibilities occurs:*

(i) G contains a semiregular subgroup, or

(ii) $M \cong \mathbb{Z}_q^k$ and $\mathrm{val}(X) > mq$, where mq is the smallest multiple of q exceeding p, or

(iii) $M \cong \mathbb{Z}_p^k$ and $\mathrm{val}(X) > p$.

Proof. Assuming that G does not contain a semiregular subgroup and assuming that $M \cong \mathbb{Z}_q^k$ we have, by Lemma 2.6, that the valency $\mathrm{val}(X)$ of X is at least mq. Further if it is exactly mq then the imprimitivity block system $\mathcal{B}$ arising from the orbits of M consists of two blocks alone, that is, $\mathcal{B} = \{B, B'\}$ with valency $\mathrm{val}(B, B') = mq$. It follows that X is a bipartite graph (with bipartition $\{B, B'\}$). As the blocks of $\mathcal{B}$ have order q^j for some j it must be that either X has order a power of 2 or $p = 2$. But $q < p$ which is not possible. Therefore $\mathrm{val}(X) > mq$. Finally, suppose that $M \cong \mathbb{Z}_p^k$. Then the nonexistence of a semregular subgroup implies that there must exist a pair of adjacent blocks B and B' in $X/\mathcal{B}$ such that in the above defined pseudometrc Dist we have $\mathrm{Dist}(B, B') \geq 1$. This implies that $\mathrm{val}(X) \geq p$. But if $\mathrm{val}(X)$ was equal to p then a semiregular automorphism could be produced in an analogous way to the previous case. □

3 Conclusions

Special cases for valencies 3 and 4 of Lemma 2.6 and Proposition 2.7 played an important role in the proofs of these results, see [6, 19]. For example, in a cubic vertex-transitive graph an automorphism of prime order greater than 3 is clearly semiregular. In fact, in a vertex-transitive graph an automorphism of order greater than the valency of the graph is necessarily semiregular. Therefore, in order to complete the proof of the existence of semiregular automorphisms in such graphs it suffices to deal with graphs having a group of automorphisms which is a $\{2, 3\}$-group. Clearly, Proposition 2.7 applies. As for quartic vertex-transitive graphs again assuming that all automorphisms are of order 2 and 3, and hence a group in question is a $\{2, 3\}$-group, Proposition 2.7 implies that the minimal normal elementary abelian subgroup M has to be a 3-group, and furthermore that the quotient graph with respect to the imprimitivity block system arising from the orbits of M is a multicycle of even length obtained from a cycle with every second edge replaced by a triple of edges. A delicate analysis of this case is then needed in order to prove the existence of a semiregular automorphism (see [6]).

An obvious possible next goal would be to prove the existence of semiregular automorphisms in quintic vertex-transitive graphs. In 2007 Giudici and Xu [12] proved that all vertex-transitive locally-quasiprimitive graphs have a semiregular automorphism, implying that arc-transitive graphs of prime valency have semiregular automorphisms. This result combined together with the above remark about automorphisms of order greater than the valency of the graph allows us to assume that the group of automorphisms in question is a $\{2, 3\}$-group. Two cases may occur depending on whether the elementary abelian subgroup is a 2-group or a 3-group. If $M \cong \mathbb{Z}_2^k$ then, using Proposition 2.7, one can prove that the quotient graph with respect to the imprimitivity block system arising from the orbits of M is a multicycle of even length obtained from a cycle with every second edge replaced by a quadruple of edges. It is reasonable to expect that an approach similar to the one used in [5] for the quartic case would result in a construction of semiregular automorphisms. If on the other hand $M \cong \mathbb{Z}_3^k$, two possibilities needing further analysis may arise from Proposition 2.7. First, the quotient graph is a multicycle of even length obtained from a cycle with every other edge replaced, respectively, by a pair of edges and a triple of edges.

Second, the quotient graph is a vertex-transitive multigraph obtained from a cubic graph with every edge in a perfect matching replaced by a triple of edges. A complete solution for quintic vertex-transitive graphs depends heavily on a successful analysis of these three remaining cases.

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