A note on quotients of strongly regular graphs

Michael Giudici ∗, Murray R. Smith

School of Mathematics and Statistics, The University of Western Australia
35 Stirling Highway, WA 6009, Australia

Received 9 March 2010, accepted 9 September 2010, published online 21 October 2010

Abstract

We give examples of vertex-transitive strongly regular graphs with a normal quotient which is neither complete nor strongly regular.

Keywords: Strongly regular graph.

Math. Subj. Class.: 05E30

1 Results

A strongly regular graph is a graph that is not complete and for which each vertex has valency \(k\) and there exist integers \(\lambda, \mu\) such that each pair of adjacent vertices have \(\lambda\) common neighbours and each pair of non-adjacent vertices have \(\mu\) common neighbours. Such a graph is usually denoted by \(srg(v,k,\lambda,\mu)\) where \(v\) is the number of vertices.

One common method for studying graphs is by taking quotients. Given a partition \(\mathcal{B}\) of the vertex set of a graph \(\Gamma\), the quotient graph is the graph whose vertices are the parts of the partition \(\mathcal{B}\) and two parts \(B_1\) and \(B_2\) are joined by an edge if there exist \(v \in B_1\) and \(w \in B_2\) such that \(v\) is adjacent to \(w\) in the original graph \(\Gamma\). When \(\mathcal{B}\) is the set of orbits of a normal subgroup \(N\) of some group \(G\) of automorphisms of \(\Gamma\) we denote the quotient by \(\Gamma_N\) and refer to it as a normal quotient. It was shown in [5] that if \(\Gamma\) is a strongly regular graph with a group of automorphisms \(G\) which acts transitively on the vertex set and edge set of \(\Gamma\) then for a nontrivial normal subgroup \(N\) of \(G\), the normal quotient \(\Gamma_N\) is either a complete graph or a strongly regular graph. The purpose of this note is to show that edge-transitivity is indeed required.

In Example 1.1, we provide a vertex-transitive, edge-intransitive strongly regular graph \(\Gamma\) where we take \(G\) to be the full automorphism group and obtain a normal quotient which is neither strongly regular nor complete. The graph \(\Gamma\) also has the following interesting properties:

∗The first author is supported by an Australian Research Fellowship.

E-mail addresses: giudici@maths.uwa.edu.au (Michael Giudici), murray@murraysmith.id.au (Murray R. Smith)
Let \( \Gamma \) be a vertex-transitive proper subgroup of the full automorphism group and obtain for isomorphic groups are given in [1, 6].

1. \( \Gamma \) is a Cayley graph for three different isomorphism types of groups.

2. \( \text{Aut}(\Gamma) \) contains five conjugacy classes of regular subgroups, of which 4 are normal subgroups.

3. \( \text{Aut}(\Gamma) \) contains two isomorphic regular subgroups of shape \( C_3^2 \rtimes C_2^2 \) for which one is normal in \( \text{Aut}(\Gamma) \) while the other is not, that is, \( \Gamma \) is both a normal Cayley graph and a nonnormal Cayley graph for isomorphic groups.

Other examples of Cayley graphs that are both normal and nonnormal Cayley graphs for isomorphic groups are given in [1, 6].

In Example 1.2 we provide an infinite family of strongly regular graphs where we take \( G \) to be a vertex-transitive proper subgroup of the full automorphism group and obtain normal quotients which are neither strongly regular nor complete.

**Example 1.1.** Let \( \Gamma \) be the strongly regular graph with adjacency matrix \( A \) given in Figure 1 which has parameters \( srg(36, 14, 4, 6) \). The adjacency matrix was retrieved from [7].
According to a GAP [3] calculation, \( \text{Aut}(\Gamma) \) equals

\[
(31, 35)(34, 36), \\
(30, 33)(31, 34)(35, 36), \\
(1, 25, 32, 2, 26, 27)(3, 15, 5, 4, 24, 6)(7, 21, 33, 18, 13, 31)(8, 14, 28, 19, 20, 36)(9, 11, 35, 17, 23, 29) \\
(10, 22, 30, 16, 12, 34) \right\rangle
\]

which has shape \( C_6^2 \times C_2^2 \). (By a group \( G \) of shape \( H \times K \) we mean that \( G \) has a normal subgroup \( H \) and a subgroup \( K \) such that \( H \cap K = 1 \). Since this does not specify how \( K \) acts on \( H \) there may be more than one isomorphism class of groups of a given shape.) In fact, \( G \cong \mathbb{Z}_6^2 \times \langle \sigma, \tau \rangle \) acting on \( \mathbb{Z}_6^2 \), with \( \mathbb{Z}_6^2 \) acting regularly on itself and \( (a, b)^\tau = (-a, -b) \) and \( (a, b)^\sigma = (b, a) \). Thus \( \text{Aut}(\Gamma) \) is vertex-transitive and is a Cayley graph for \( H_1 = \mathbb{Z}_6^2 \). The joining set is

\[
\{(0, 5), (0, 1), (0, 3), (5, 0), (1, 0), (3, 0), (1, 3), (5, 3), (3, 1), (3, 5), \\
(1, 5), (5, 1), (2, 4), (4, 2)\}.
\]

Since \( \text{Aut}(\Gamma)_{(0, 0)} = \langle \tau, \sigma \rangle \) has five orbits on this set, \( \text{Aut}(\Gamma) \) has five orbits on edges.

Now \( \text{Aut}(\Gamma) \) has a normal subgroup \( N \) of order two generated by the element \((3, 3) \in \mathbb{Z}_6^2\) and which is the centre of \( \text{Aut}(\Gamma) \). The group \( N \) has 18 orbits of length two on the 36 vertices of \( \Gamma \) and the set of neighbours of \( (0, 0) \) contains the three \( N \)-orbits \{\((0, 3), (3, 0)\}\}, \{\((1, 5), (5, 1)\}\) and \{\((2, 4), (4, 2)\)\}. Hence, \( \Gamma_N \) is a valency 11 graph on 18 vertices of diameter 2 but is not strongly regular. Indeed there are no feasible parameters for strongly regular graphs on 18 vertices which are not complete multipartite [4, p227]. The matrix \( B \) given in Figure 2 is the adjacency matrix for \( \Gamma_N \).

Not only is \( \Gamma \) a Cayley graph for \( H_1 \), which is normal in \( \text{Aut}(\Gamma) \), we also have that \( H_2 = \langle (2, 0), (0, 2), (3, 0) \rangle, H_3 = \langle (0, 2), (2, 0), (3, 0) \rangle, H_4 = \langle (2, 0), (0, 2), (2, 5) \rangle, H_5 = \langle (2, 0), (0, 2), (1, 0) \rangle \) are normal subgroups of \( \text{Aut}(\Gamma) \) that act regularly on \( VT \). The subgroup \( H_2 \) has shape \( C_2^3 \times C_2^2 \), while \( H_3 \cong H_4 \) have shape \( C_3^2 \times C_4 \). Finally, \( H_5 = \langle (2, 0), (0, 2), (1, 0) \rangle \cong H_2 \) is a regular subgroup of \( \text{Aut}(\Gamma) \) which is not normal. Thus \( \Gamma \) is a Cayley graph for three different isomorphism types of groups. A Magma [2]
calculation shows that $H_1, H_2, H_3, H_4$ and the subgroups conjugate to $H_5$ are the only regular subgroups of $\text{Aut}(\Gamma)$.

The automorphism group of $\Gamma_N$ is isomorphic to $S_2 \times S_4 \times S_3$, which is vertex-transitive and has three orbits on edges. Note that $\text{Aut}(\Gamma)/N < \text{Aut}(\Gamma_N)$. The automorphism group contains 4 conjugacy classes of regular subgroups, none of which are normal in $\text{Aut}(\Gamma_N)$. One class is isomorphic to $C_2^3 \times C_2$, and there are three classes of subgroups with shape $C_3^2 \rtimes C_2$, with two of the classes being isomorphic to each other. Representatives of these four conjugacy classes are $H_i/N$ for $i = 1, 2, 3, 4$. Note that $H_2/N = H_5N/N$.

Example 1.2. Let $\Gamma = H(2,m)$, the Hamming graph with $m^2$ vertices and suppose that $m$ is not a prime. Then $\Gamma$ is a strongly regular graph with parameters $(m^2, 2(m-1), m-2, 2)$. Let $G = M_1 \times M_2$, with $M_1 \cong M_2 \cong C_m$, act regularly on the set of vertices of $\Gamma$. Let $N_1 \leq M_1$ and $N_2 \leq M_2$ and $N = N_1 \times N_2 < G$. Consider the graph $\Gamma_N$. Then $\Gamma_N$ is the cartesian product of $K_r$ and $K_k$ where $|M_1 : N_1| = r$ and $|M_2 : N_2| = k$. The adjacent vertices $(a, b_1), (a, b_2)$ in $\Gamma_N$ have $k - 2$ common neighbours, namely the vertices of the form $(a, b)$ with $b \neq b_1, b_2$. However, the adjacent vertices $(a_1, b), (a_2, b)$ have $r - 2$ common neighbours, these being the vertices of the form $(a, b)$ with $a \neq a_1, a_2$. Hence for $r \neq k$, the graph $\Gamma_N$ is not strongly regular.

2 Acknowledgements

The authors are grateful to the anonymous referees whose comments greatly improved the paper and led to Example 1.2.

References


