ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 16 (2019) 1–17
https://doi.org/10.26493/1855-3974.1467.04b
(Also available at http://amc-journal.eu)

Splittable and unsplittable graphs and configurations*

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Received 21 August 2017, accepted 18 March 2018, published online 22 August 2018

Abstract

We prove that there exist infinitely many splittable and also infinitely many unsplittable cyclic (n_3) configurations. We also present a complete study of trivalent cyclic Haar graphs on at most 60 vertices with respect to splittability. Finally, we show that all cyclic flag-transitive configurations with the exception of the Fano plane and the Möbius-Kantor configuration are splittable.

Keywords: Configuration of points and lines, unsplittable configuration, unsplittable graph, independent set, Levi graph, Grünbaum graph, splitting type, cyclic Haar graph.

Math. Subj. Class.: 51A20, 05B30

*The work was supported in part by a grant from the Picker Institute at Colgate University and the ARRS of Slovenia, grants P1-0294, N1-0011, N1-0032, J1-6720, and J1-7051. We would like to thank István Estélyi and Jaka Kranjc for fruitful discussions during the preparation of this paper and the two anonymous referees for useful remarks and suggestions that have improved the quality of the paper.

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1 Introduction and preliminaries

The idea of *unsplittable configuration* was conceived in 2004 and formally introduced in the monograph [8] by Grünbaum. Later, it was also used in [19]. In [20], the notion was generalized to graphs. In this paper we present some constructions for splittable and unsplittable cyclic configurations. In [9], the notion of *cyclic Haar graph* was introduced. It was shown that cyclic Haar graphs are closely related to cyclic configurations. Namely, each cyclic Haar graph of girth 6 is a Levi graph of a cyclic combinatorial configuration; see also [18]. For the definition of the *Levi graph* (also called *incidence graph*) of a configuration the reader is referred to [4]. The classification of configurations with respect to splittability is a purely combinatorial problem and can be interpreted purely in terms of Levi graphs.

Let n be a positive integer, let $\mathbb{Z}_n$ be the cyclic group of integers modulo n and let $S \subseteq \mathbb{Z}_n$ be a set, called the *symbol*. The graph $H(n, S)$ with the vertex set $\{u_i \mid i \in \mathbb{Z}_n\} \cup \{v_i \mid i \in \mathbb{Z}_n\}$ and edges joining u_i to v_{i+k} for each $i \in \mathbb{Z}_n$ and each $k \in S$ is called a *cyclic Haar graph* over $\mathbb{Z}_n$ with symbol S [9]. In practice, we will simplify the notation by denoting u_i by i^+ and v_i by i^-.

Definition 1.1. A *combinatorial* (v_k) *configuration* is an incidence structure $\mathcal{C} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, where $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$, $\mathcal{P} \cap \mathcal{B} = \emptyset$ and $|\mathcal{P}| = |\mathcal{B}| = v$. The elements of $\mathcal{P}$ are called *points*, the elements of $\mathcal{B}$ are called *lines* and the relation $\mathcal{I}$ is called the *incidence* relation. Furthermore, each line is incident with k points, each point is incident with k lines and two distinct points are incident with at most one common line, i.e.,

$$\{(p_1, b_1), (p_2, b_1), (p_1, b_2), (p_2, b_2)\} \subseteq \mathcal{I}, p_1 \neq p_2 \implies b_1 = b_2. \tag{1.1}$$

If $(p, b) \in \mathcal{I}$ then we say that the line b passes through point p or that the point p lies on line b. An element of $\mathcal{P} \cup \mathcal{B}$ is called an *element of configuration* $\mathcal{C}$.

A combinatorial (v_k) configuration $\mathcal{C} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is *geometrically realisable* if the elements of $\mathcal{P}$ can be mapped to different points in the Euclidean plane and the elements of $\mathcal{B}$ can be mapped to different lines in the Euclidean plane, such that $(p, b) \in \mathcal{I}$ if and only if the point that corresponds to p lies on the line that corresponds to b. A geometric realisation of a combinatorial (v_k) configuration is called a *geometric* (v_k) *configuration*. Note that examples in Figures 2, 3 and 4 are all geometric configurations. The Fano plane (7_3) is an example of a geometrically non-realizable configuration.

An *isomorphism* between configurations $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ and $(\mathcal{P}', \mathcal{B}', \mathcal{I}')$ is a pair of bijections $\psi\colon \mathcal{P} \to \mathcal{P}'$ and $\varphi\colon \mathcal{B} \to \mathcal{B}'$, such that

$$(p, b) \in \mathcal{I} \text{ if and only if } (\psi(p), \varphi(b)) \in \mathcal{I}'. \tag{1.2}$$

The configuration $\mathcal{C}^* = (\mathcal{B}, \mathcal{P}, \mathcal{I}^*)$, where $\mathcal{I}^* = \{(b, p) \in \mathcal{B} \times \mathcal{P} \mid (p, b) \in \mathcal{I}\}$, is called the *dual configuration* of $\mathcal{C}$. A configuration that is isomorphic to its dual is called a *self-dual* configuration.

The *Levi graph* of a configuration $\mathcal{C}$ is the bipartite graph on the vertex set $\mathcal{P} \cup \mathcal{B}$ having an edge between $p \in \mathcal{P}$ and $b \in \mathcal{B}$ if and only if the elements p and b are incident in $\mathcal{C}$, i.e., if $(p, b) \in \mathcal{I}$. It is denoted $L(\mathcal{C})$. Condition (1.1) in Definition 1.1 implies that the girth of $L(\mathcal{C})$ is at least 6. Moreover, any combinatorial (v_k) configuration is completely determined by a k-regular bipartite graph of girth at least 6 with a given black-and-white vertex coloring, where black vertices correspond to points and white vertices correspond to lines. Such a graph will be called a *colored Levi graph*. Note that the reverse coloring determines the dual configuration $\mathcal{C}^* = (\mathcal{B}, \mathcal{P}, \mathcal{I}^*)$. Also, an isomorphism between configurations corresponds to color-preserving isomorphism between their respective colored Levi graphs.

A configuration $\mathcal{C}$ is said to be *connected* if its Levi graph $L(\mathcal{C})$ is connected. Similarly, a configuration $\mathcal{C}$ is said to be *k-connected* if its Levi graph $L(\mathcal{C})$ is k-connected.

Definition 1.2 ([19]). A combinatorial (v_k) configuration $\mathcal{C}$ is *cyclic* if it admits an automorphism of order v that cyclically permutes the points and lines, respectively.

In [9] the following was proved:

Proposition 1.3 ([9]). *A configuration $\mathcal{C}$ is cyclic if and only if its Levi graph is isomorphic to a cyclic Haar graph of girth* 6.

It can be shown that each cyclic configuration is self-dual, see for instance [9].

2 Splittable and unsplittable configurations (and graphs)

Let G be any graph. The *square* of G, denoted G^2, is a graph with the same vertex set as G, where two vertices are adjacent if and only if their distance in G is at most 2. In other words, $V(G^2) = V(G)$ and $E(G^2) = \{uv \mid d_G(u, v) \leq 2\}$. The square of the Levi graph $L(\mathcal{C})$ of a configuration $\mathcal{C}$ is called the *Grünbaum graph* of $\mathcal{C}$ in [19] and [20]. In [8], it is called the *independence graph*. Two elements of a configuration $\mathcal{C}$ are said to be *independent* if they correspond to independent vertices of the Grünbaum graph.

Example 2.1. The Grünbaum graph of the Heawood graph is shown in Figure 1. Its complement is the Möbius ladder M_{14}.

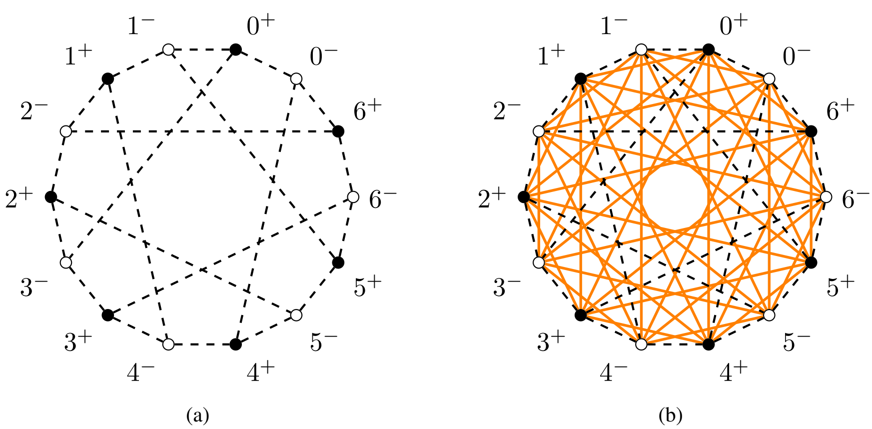


Figure 1: The Heawood graph $H = H(7, \{0, 1, 3\}) \cong \mathrm{LCF}[5, -5]^7$ (on the left) is the Levi graph of the Fano plane. Its Grünbaum graph G is on the right. Note that there is an orange solid edge between two vertices of G if and only if they are at distance 2 in H.

It is easy to see that two elements of $\mathcal{C}$ are independent if and only if one of the following hold:

(i) two points of $\mathcal{C}$ that do not lie on a common line of $\mathcal{C}$;

(ii) two lines of $\mathcal{C}$ that do not intersect in a common point of $\mathcal{C}$;

(iii) a point of $\mathcal{C}$ and a line of $\mathcal{C}$ that are not incident.

The definition of unsplittable configuration was introduced in [8] and is equivalent to the following:

Definition 2.2. A configuration $\mathcal{C}$ is *splittable* if there exists an independent set of vertices Σ in the Grünbaum graph $(L(\mathcal{C}))^2$ such that $L(\mathcal{C}) - \Sigma$, i.e., the graph obtained by removing the set of vertices Σ from the Levi graph $L(\mathcal{C})$, is disconnected. In this case the set Σ is called a *splitting set of elements*. A configuration that is not splittable is called *unsplittable*.

This definition carries over to graphs:

Definition 2.3. A connected graph G is *splittable* if there exists an independent set Σ in G^2 such that $G - \Sigma$ is disconnected.

Example 2.4. Every cycle of length at least 6 is splittable (there exists a pair of vertices at distance 3 in G).

Every graph of diameter 2 without a cut vertex is unsplittable. The square of such a graph on n vertices is the complete graph K_n. This implies that $|S| = 1$. Since there are no cut vertices, a splitting set does not exist. The Petersen graph is an example of unsplittable graph.

In [8], refinements of the above definition are also considered. Configuration $\mathcal{C}$ is *point-splittable* if it is splittable and there exists a splitting set of elements that consists of points only (i.e., only black vertices in the corresponding colored Levi graph). In a similar way *line-splittable* configurations are defined. Note that these refinements can be defined for any bipartite graph with a given black-and-white coloring. There are four possibilities, that we call *splitting types*. Any configuration may be:

(T1) point-splittable, line-splittable,

(T2) point-splittable, line-unsplittable,

(T3) point-unsplittable, line-splittable,

(T4) point-unsplittable, line-unsplittable.

Any configuration of splitting type T1, T2 or T3 is splittable. A configuration of splitting type T4 may be splittable or unsplittable. For an example of a point-splittable (T2) configuration see Figure 2. The configuration on Figure 2 is isomorphic to a configuration on Figure 5.1.11 from [8]. For an example of a line-splittable (T3) configuration see Figure 3.

Note the following:

Proposition 2.5. *If $\mathcal{C}$ is of type T1 then its dual is also of type T1. If it is of type T2 then its dual is of type T3 (and vice versa). If it is of type T4 then its dual is also of type T4.*

Since types are mutually disjoint, this has a straightforward consequence for cyclic configurations:

Corollary 2.6. *Any self-dual configuration, in particular any cyclic configuration, is either of type T1 or T4.*

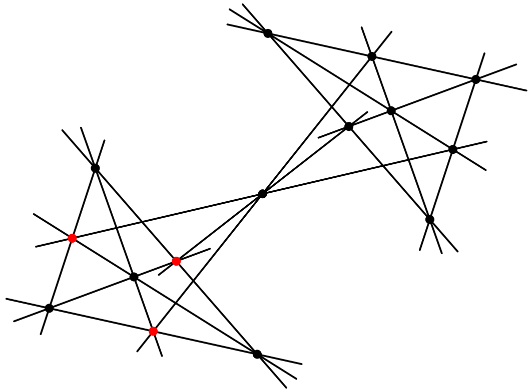


Figure 2: A point-splittable (15_3) configuration of type T2. Points that belong to a splitting set are colored red. Its dual is of type T3 (see Figure 3).

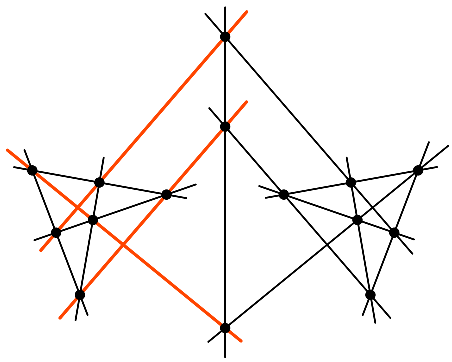


Figure 3: A line-splittable (15_3) configuration of type T3. Lines that belong to a splitting set are colored orange. Its dual is depicted in Figure 2.

Obviously, unsplittable configurations are of type T4. However, the converse is not true:

Proposition 2.7. *Any unsplittable configuration is point-unsplittable and line-unsplittable. There exist splittable configurations that are both point-unsplittable and line-unsplittable.*

Proof. The first statement of Proposition 2.7 is obviously true. An example that provides the proof of the second statement is shown in Figure 4. The splitting set is

$$\{0, 8, 10, (1, 9, 11), (6, 7, 14)\}. \qquad \square$$

Note that configuration in Figure 4 is not cyclic, but it is 3-connected. In [8], the following theorem is proven:

Theorem 2.8 ([8, Theorem 5.1.5])**.** *Any unsplittable (n_3) configuration is 3-connected.*

Our computational results show that the converse to Theorem 2.8 is not true. There exist 3-connected splittable configurations. See, for instance, the configuration in Figure 4.

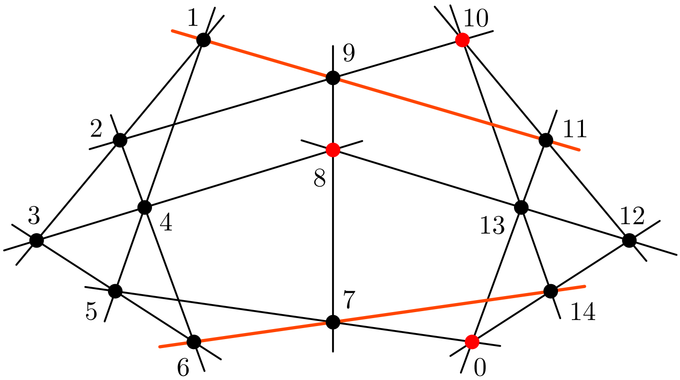


Figure 4: A splittable (15_3) configuration of type T4. Elements of a splitting set are points 0, 8 and 10 (colored red) and lines $(1, 9, 11)$ and $(6, 7, 14)$ (colored orange).

3 Splittable and unsplittable cyclic (n_3) configurations

We used a computer program to analyse all cyclic (n_3) configurations for $7 \leq n \leq 30$ (see Table 1, Table 2 and Table 3). In [9] it was shown that cyclic Haar graphs contain all information about cyclic combinatorial configurations. In trivalent case combinatorial isomorphisms of cyclic configurations are well-understood; see [11]. Namely, it is known how to obtain all sets of parameters of isomorphic cyclic Haar graphs. We would like to draw the reader's attention to the manuscript [10], where the main result of [11] is extended to cyclic (n_k) configurations for all $k > 3$. One would expect that large sparse graphs are splittable. In this sense the following result is not a surprise:

Theorem 3.1. *Let $H(n, \{0, a, b\})$ be a cyclic Haar graph, where $0 < a < b$. Let*

$$\mathcal{W} = \{0, a, b, 2b, b + a, b - a, 2b - a, 2b - 2a, 3b - a, 3b - 2a, 2b + a, 3b\} \text{ and}$$
$$\mathcal{B} = \{0, a, b, 2b, b + a, b - a, 2b - a, 2b - 2a, 3b - a, 3b - 2a, -a, b - 2a\}$$

be multisets with elements from $\mathbb{Z}_n$. If all elements of $\mathcal{W}$ are distinct and all elements of $\mathcal{B}$ are distinct (i.e. $\mathcal{W}$ and $\mathcal{B}$ are ordinary sets, $|\mathcal{W}| = |\mathcal{B}| = 12$) then $H(n, \{0, a, b\})$ is splittable and

$$\Sigma = \{0^+, 2b^+, (2b - 2a)^+, (b - a)^-, (b + a)^-, (3b - a)^-\}$$

is a splitting set for $H(n, \{0, a, b\})$.

Proof. See Figure 5. If $\mathcal{W}$ and $\mathcal{B}$ are ordinary sets then the graph in Figure 5 is a subgraph of $H(n, \{0, a, b\})$. It is easy to see that Σ is a splitting set. The set Σ is indeed an independent set in the square of the graph $H(n, \{0, a, b\})$ since no two vertices of Σ are adjacent to the same vertex. In order to see that the subgraph obtained by removing the vertices of Σ is disconnected, note that one of the connected components is the cycle determined by vertices $\{b^+, (b - a)^+, (2b - a)^+, b^-, 2b^-, (2b - a)^-\}$. □

Corollary 3.2. *Under conditions of Theorem 3.1, the girth of the graph $H(n, \{0, a, b\})$ is 6.*

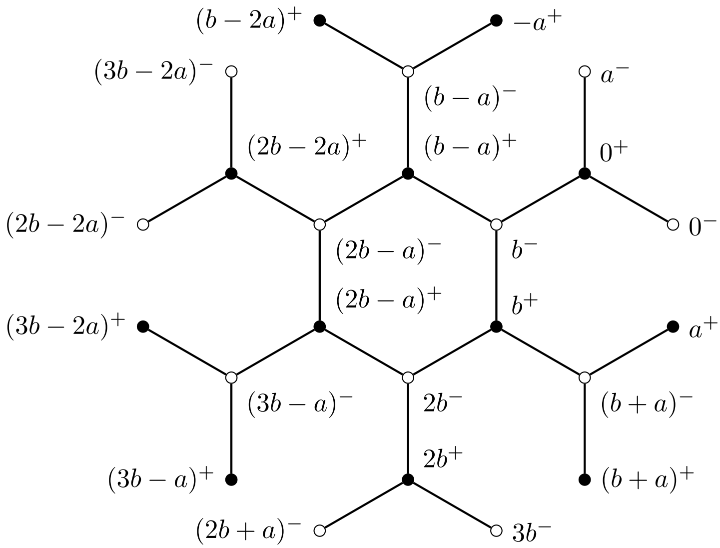


Figure 5: The set $\Sigma = \{0^+, 2b^+, (2b-2a)^+, (b-a)^-, (b+a)^-, (3b-a)^-\}$ is a splitting set for $H(n, \{0, a, b\})$.

Proof. The girth of such a graph is at most 6 because it contains a 6-cycle (see Figure 5). It is easy to see that the girth cannot be 4. Because the graph $H(n, \{0, a, b\})$ is bipartite, each 4-cycle must contain a black vertex. Consider vertex b^+ in Figure 5. Its neighborhood is $\{b^-, 2b^-, (b+a)^-\}$. None of those vertices have a common neighbor, so b^+ does not belong to any 4-cycle. Because of symmetry this argument holds for all black vertices. □

Corollary 3.3. *There exist infinitely many cyclic (n_3) configurations that are splittable. For example, the following three families of cyclic Haar graphs are splittable:*

(a) $H(n, \{0, 1, 4\})$ *for* $n \geq 13$,

(b) $H(n, \{0, 1, 5\})$ *for* $n \geq 16$, *and*

(c) $H(n, \{0, 2, 5\})$ *for* $n \geq 16$.

Proof. Corollary 3.2 implies that each graph from any of the three families has girth 6. From Theorem 3.1 it follows that $\Sigma = \{0^+, 6^+, 8^+, 3^-, 5^-, 11^-\}$ is a splitting set for $H(n, \{0, 1, 4\})$ if $n \geq 13$ (see Figure 6), $\{0^+, 8^+, 10^+, 4^-, 6^-, 14^-\}$ is a splitting set for $H(n, \{0, 1, 5\})$ if $n \geq 16$, and $\{0^+, 6^+, 10^+, 3^-, 7^-, 13^-\}$ is a splitting set for $H(n, \{0, 2, 5\})$ if $n \geq 16$. □

If $n < 13$ then conditions of Theorem 3.1 are not fulfilled. If $n = 12$ then $(n-1)^+ = 11^+$ which means that the vertices of the graph in Figure 6 are not all distinct. If $n = 9$ then $9^- = 0^-$ since we work with $\mathbb{Z}_9$. Similar arguments can be made if $n < 16$ in the case of the other two families from Corollary 3.3.

We investigated the first 100 graphs from the $H(n, \{0, 1, 4\})$ family. All but two are zero symmetric, nowadays called graphical regular representation or GRR for short (see [5]). The exceptions are for $n = 13$ and $n = 15$.

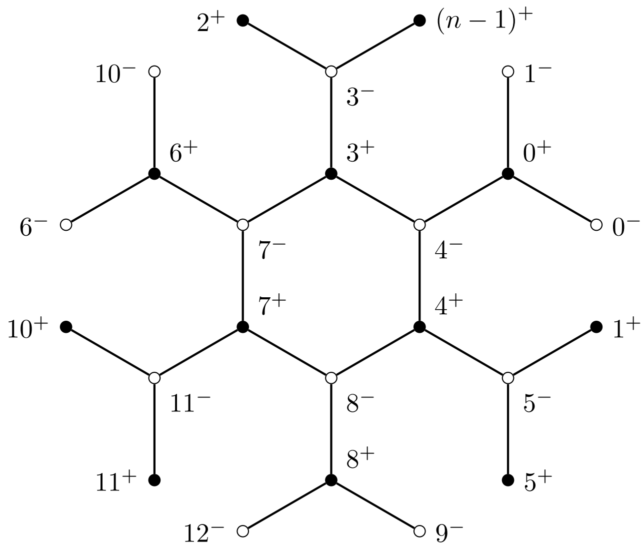


Figure 6: The set $\Sigma = \{0^+, 6^+, 8^+, 3^-, 5^-, 11^-\}$ is a splitting set for $H(n, \{0, 1, 4\})$ where $n \geq 13$.

By Corollary 3.3, there are infinitely many splittable (n_3) configurations. However, we are also able to show that there is no upper bound on the number of vertices of unsplittable (n_3) configurations:

Theorem 3.4. *There exist infinitely many cyclic (n_3) configurations that are unsplittable.*

Proof. We use the cyclic Haar graphs $X = H(n, \{0, 1, 3\})$, where $n \geq 7$. Clearly, each of them has girth 6. The graph can be written as $\mathrm{LCF}[5, -5]^n$. (For the LCF notation see [19].) This means that the edges determined by symbols 0 and 1 form a Hamiltonian cycle while the edges arising from the symbol 3 form chords of length 5. See Figure 1 for an example.

Let us assume the result does not hold. This means there exists a splitting set Σ. By removing Σ from the graph the Hamiltonial cycle breaks into paths. Each path must contain at least two vertices. Let the sequence $\Pi = (p_1, p_2, \ldots, p_k)$ denote the lengths of the consecutive paths along the Hamiltonial cycle. The rest of the proof is in two steps:

Step 1. If there are no two consecutive numbers of Π equal to 2, then the corresponding segments are connected in $X - \Sigma$ since there is a chord of length 5 joining these two segments. But this means that all paths are connected by chords, so Σ is not a splitting set.

Step 2. We can show that no two consecutive segments are of length 2. In case of two adjacent segments of length 2 we would have vertices $\{i - 3, i, i + 3\} \subseteq \Sigma$. But that is impossible, since $i - 3$ is adjacent to $i + 3$ in X^2. □

Note that this is not the only such family. Here is another one:

Theorem 3.5. *Cyclic configurations defined by $H(3n, \{0, 1, n\})$, where $n \geq 2$, are unsplittable.*

Proof. The technique used here is similar to the technique used in proof of Theorem 3.4. Let $X = H(3n, \{0, 1, n\})$. The graph X can be written as LCF$[2n-1, -(2n-1)]^{3n}$. Suppose that there exists a splitting set Σ. The edges determined by symbols 0 and 1 form a Hamiltonian cycle which breaks into paths when the splitting set Σ is removed.

We show that any two consecutive paths are connected in $X - \Sigma$. Without loss of generality (because of symmetry), we may assume that $0^+ \in \Sigma$ is the vertex adjacent to the two paths under consideration. If $0^+ \in \Sigma$ then $1^-, 0^-, n^-, n^+, 1^+, 2n^+, (2n+1)^+ \notin \Sigma$. We show that vertices 1^- and 0^- (which belong to the two paths under consideration) are connected in $X - \Sigma$.

If $(2n+1)^- \notin \Sigma$ then $2n^+$ and $(2n+1)^+$ are connected in $X - \Sigma$. Since 0^- is adjacent to $2n^+$ and 1^- is adjacent to $(2n+1)^+$, vertices 0^- and 1^- are also connected in $X - \Sigma$. Now, suppose that $(2n+1)^- \in \Sigma$. This implies that $2n^-, (n+1)^+(n+1)^- \notin \Sigma$. Then $2n^+$ is adjacent to $2n^-$, $2n^-$ is adjacent to n^+, n^+ is adjacent to $(n+1)^-$, $(n+1)^-$ is adjacent to 1^+, and 1^+ is adjacent to 1^- in $X - \Sigma$. Therefore, 1^- and 0^- are connected in $X - \Sigma$. □

Cubic symmetric bicirculants were classified in [13] and [16]. These results can be summarised as follows:

Theorem 3.6 ([13, 16])**.** *A connected cubic symmetric graph is a bicirculant if and only if it is isomorphic to one of the following graphs:*

(1) the complete graph K_4,

(2) the complete bipartite graph $K_{3,3}$,

(3) the seven symmetric generalized Petersen graphs $GP(4,1)$, $GP(5,2)$, $GP(8,3)$, $GP(10,2)$, $GP(10,3)$, $GP(12,5)$ and $GP(24,5)$,

(4) the Heawood graph $H(7, \{0, 1, 3\})$, and

(5) the cyclic Haar graph $H(n, \{0, 1, r+1\})$, where $n \geq 11$ is odd and $r \in \mathbb{Z}_n^*$ such that $r^2 + r + 1 \equiv 0 \pmod{n}$.

It is well known that an (n_3) configuration is flag-transitive if and only if its Levi graph is cubic symmetric graph of girth at least 6. From Theorem 3.6 it follows that the girth of any connected cubic symmetric bicirculant is at most 6. If the girth of such a graph is 6 or more then it is a Levi graph of a flag-transitive configuration. This enables us to characterise splittability of such configurations:

Theorem 3.7. *The Fano plane (7_3), the Möbius-Kantor configuration (8_3), and the Desargues configuration (10_3) are unsplittable. Their Levi graphs are*

$$H(7, \{0, 1, 3\}), \quad H(8, \{0, 1, 3\}) \cong GP(8, 3) \quad \text{and} \quad GP(10, 3),$$

respectively.

If $n \geq 9$, all flag-transitive (n_3) configurations, except the Desargues configuration, are splittable.

Proof. We start with the classification given in Theorem 3.6. Only bipartite graphs of girth 6 have to be considered. This rules out the complete graph K_4, the complete bipartite graph $K_{3,3}$, and the generalised Petersen graphs $GP(5,2)$, $GP(10,2)$ and $GP(4,1)$. Note that $GP(4,1)$ is isomorphic to the cube graph Q_3.

Table 1: Overview of splittable and unsplittable connected cyclic Haar graphs.

n	(a)	(b)	(c)	(d)	(e)	(f)
3	1	0	0	1	0	0
4	1	0	0	1	0	0
5	1	0	0	1	0	0
6	2	0	0	2	0	0
7	2	1	0	2	0	1
8	3	1	1	2	0	1
9	2	1	0	2	0	1
10	3	1	1	2	0	1
11	2	1	0	2	0	1
12	5	3	1	4	0	3
13	3	2	1	2	1	1
14	4	2	2	2	1	1
15	5	4	1	4	1	3
16	5	3	3	2	2	1
17	3	2	1	2	1	1
18	6	4	3	3	2	2
19	4	3	2	2	2	1
20	7	5	5	2	4	1
21	7	6	3	4	3	3
22	6	4	4	2	3	1
23	4	3	2	2	2	1
24	11	9	7	4	6	3
25	5	4	3	2	3	1
26	7	5	5	2	4	1
27	6	5	3	3	3	2
28	9	7	7	2	6	1
29	5	4	3	2	3	1
30	13	11	9	4	8	3

(a) Number of non-isomorphic connected cubic cyclic Haar graphs on $2n$ vertices.
(b) Those that have girth 6. (c) Those that are splittable. (d) Those that are unsplittable.
(e) Those that are splittable of girth 6. (f) Those that are unsplittable of girth 6.

It is well known, but one may check by computer that $GP(8,3) \cong H(8,\{0,1,3\})$. See for instance [9, Table 2].

One may also check by computer that $GP(8,3)$, $GP(10,3)$ and the Heawood graph $H(7,\{0,1,3\})$ are unsplittable.

Let

$$V(GP(n,k)) = \{0,1,\dots,n-1,0',1',\dots,(n-1)'\} \text{ and}$$
$$E(GP(n,k)) = \{\{i',((i+1) \bmod n)'\},\{i,i'\},\{i,(i+k) \bmod n\} \mid i = 0,\dots,n-1\}.$$

Note that $\Sigma = \{0',4',8',2,6,10\}$ is a splitting set for $GP(12,5)$ as shown in Figure 7. Also, $GP(12,5) - S \cong 3C_6$, i.e., a disjoint union of three copies of C_6. The splitting

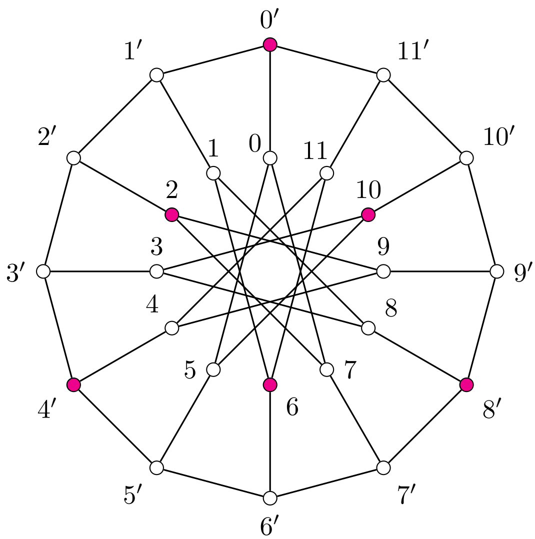


Figure 7: The magenta vertices form a splitting set for the Nauru graph $GP(12,5)$ [6, 21].

set for $GP(24,5)$ is $\Sigma = \{0',4',8',12',16',20',2,6,10,14,18,22\}$ as shown in Figure 8. Note that $GP(24,5) - S \cong 3C_{12}$. Also, note that $GP(24,5)$ is not isomorphic to a cyclic Haar graph since its girth is 8.

Using Theorem 3.1, one may verify that all graphs in item (5) of Theorem 3.6 have girth 6 and for each of them the splitting set is $\{0^+, 2r^+, (2r+2)^+, r^-, (r+2)^-, (3r+2)^-\}$. We have

$$\mathcal{W} = \{0,1,r,r+1,r+2,2r,2r+1,2r+2,2r+3,3r+1,3r+2,3r+3\},$$
$$\mathcal{B} = \{0,1,n-1,r-1,r,r+1,r+2,2r,2r+1,2r+2,3r+1,3r+2\}.$$

It is easy to verify that all elements of $\mathcal{W}$ are distinct and that all elements of $\mathcal{B}$ are distinct. For example, suppose that $r \equiv 3r+3 \pmod{n}$. This means that

$$2r \equiv -3 \pmod{n}. \tag{3.1}$$

From condition $r^2+r+1 \equiv 0 \pmod{n}$ we obtain

$$4r^2+4r+4 = (2r)^2 + 2\cdot 2r + 4 \equiv 0 \pmod{n}. \tag{3.2}$$

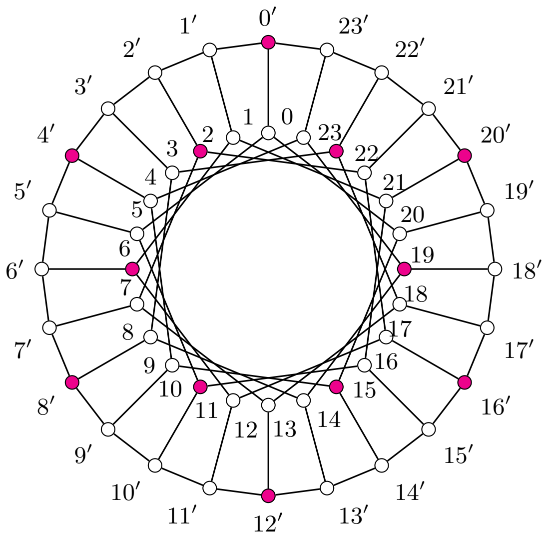


Figure 8: The magenta vertices form a splitting set for $GP(24,5)$ which was recently named the ADAM graph [14].

Equations (3.1) and (3.2) together imply that $(-3)^2 + 2 \cdot (-3) + 4 = 7 \equiv 0 \pmod{n}$, which is a contradiction since $n > 11$. All other cases can be checked in a similar way. □

From Theorem 3.7 we directly obtain the following corollary.

Corollary 3.8. *A cyclic flag-transitive* (n_3) *configuration is splittable if and only if* $n > 8$. *The only two exceptions are:*

(1) $H(7, \{0, 1, 3\})$*, i.e. the Fano plane, and*

(2) $H(8, \{0, 1, 3\})$*, i.e. the Möbius-Kantor configuration.*

4 Splittable geometric (n_k) configurations

We will now show that for any k there exist a geometric, triangle-free, (n_k) configuration which is of type T1, i.e., it is point-splittable and line-splittable.

Let us first provide a construction to obtain a geometric (n_k) configuration for any k. We start with an unbalanced $(k_1, 1_k)$ configuration, denoted $\mathcal{G}_k^{(1)}$, that consists of a single line containing k points. Let $\mathcal{G}_k^{(i)}$ be a configuration that is obtained from $\mathcal{G}_k^{(i-1)}$ by the *k-fold parallel replication* (see [19, p. 245]). The configuration $\mathcal{G}_k^{(k)}$ is a balanced $({k^k}_k)$ configuration, called a *generalised Gray configuration*; see [17].

Lemma 4.1. *Let* $\mathcal{C}$ *be an arbitrary geometric* (n_k) *configuration. There exists a geometric* (kn_k) *configuration* $\mathcal{D}$ *that is point- and line-splittable. Moreover, if* $\mathcal{C}$ *is triangle-free then* $\mathcal{D}$ *is also triangle-free.*

Proof. Let $\mathcal{C}$ be as stated. Select an arbitrary line L of $\mathcal{C}$ passing through points $p^{(1)}, p^{(2)}$, $\ldots, p^{(k)}$ of $\mathcal{C}$ as shown in Figure 9(a). Remove line L and call the resulting structure

Table 2: List of non-isomorphic connected trivalent cyclic Haar graphs $H(n, S)$ with $n \leq 25$ and some of their properties.

n	S	(a)	(b)	(c)	(d)
3	$\{0,1,2\}$	$\bot$	4	2	$\top$
4	$\{0,1,2\}$	$\bot$	4	3	$\top$
5	$\{0,1,2\}$	$\bot$	4	3	$\bot$
6	$\{0,1,2\}$	$\bot$	4	4	$\bot$
6	$\{0,1,3\}$	$\bot$	4	3	$\bot$
7	$\{0,1,2\}$	$\bot$	4	4	$\bot$
7	$\{0,1,3\}$	$\bot$	6	3	$\top$
8	$\{0,1,2\}$	$\bot$	4	5	$\bot$
8	$\{0,1,3\}$	$\bot$	6	4	$\top$
8	$\{0,1,4\}$	$\top$	4	4	$\bot$
9	$\{0,1,2\}$	$\bot$	4	5	$\bot$
9	$\{0,1,3\}$	$\bot$	6	4	$\bot$
10	$\{0,1,2\}$	$\bot$	4	6	$\bot$
10	$\{0,1,3\}$	$\bot$	6	4	$\bot$
10	$\{0,1,5\}$	$\top$	4	5	$\bot$
11	$\{0,1,2\}$	$\bot$	4	6	$\bot$
11	$\{0,1,3\}$	$\bot$	6	5	$\bot$
12	$\{0,1,2\}$	$\bot$	4	7	$\bot$
12	$\{0,1,3\}$	$\bot$	6	5	$\bot$
12	$\{0,1,4\}$	$\bot$	6	5	$\bot$
12	$\{0,1,5\}$	$\bot$	6	5	$\bot$
12	$\{0,1,6\}$	$\top$	4	6	$\bot$
13	$\{0,1,2\}$	$\bot$	4	7	$\bot$
13	$\{0,1,3\}$	$\bot$	6	5	$\bot$
13	$\{0,1,4\}$	$\top$	6	5	$\top$
14	$\{0,1,2\}$	$\bot$	4	8	$\bot$
14	$\{0,1,3\}$	$\bot$	6	6	$\bot$
14	$\{0,1,4\}$	$\top$	6	5	$\bot$
14	$\{0,1,7\}$	$\top$	4	7	$\bot$
15	$\{0,1,2\}$	$\bot$	4	8	$\bot$
15	$\{0,1,3\}$	$\bot$	6	6	$\bot$
15	$\{0,1,4\}$	$\top$	6	5	$\bot$
15	$\{0,1,5\}$	$\bot$	6	5	$\bot$
15	$\{0,1,6\}$	$\bot$	6	5	$\bot$
16	$\{0,1,2\}$	$\bot$	4	9	$\bot$
16	$\{0,1,3\}$	$\bot$	6	6	$\bot$
16	$\{0,1,4\}$	$\top$	6	5	$\bot$
16	$\{0,1,7\}$	$\top$	6	5	$\bot$
16	$\{0,1,8\}$	$\top$	4	8	$\bot$
17	$\{0,1,2\}$	$\bot$	4	9	$\bot$
17	$\{0,1,3\}$	$\bot$	6	7	$\bot$
17	$\{0,1,4\}$	$\top$	6	5	$\bot$
18	$\{0,1,2\}$	$\bot$	4	10	$\bot$
18	$\{0,1,3\}$	$\bot$	6	7	$\bot$
18	$\{0,1,4\}$	$\top$	6	6	$\bot$
18	$\{0,1,5\}$	$\top$	6	6	$\bot$
18	$\{0,1,6\}$	$\bot$	6	6	$\bot$
18	$\{0,1,9\}$	$\top$	4	9	$\bot$
19	$\{0,1,2\}$	$\bot$	4	10	$\bot$
19	$\{0,1,3\}$	$\bot$	6	7	$\bot$
19	$\{0,1,4\}$	$\top$	6	6	$\bot$
19	$\{0,1,8\}$	$\top$	6	5	$\top$
20	$\{0,1,2\}$	$\bot$	4	11	$\bot$
20	$\{0,1,3\}$	$\bot$	6	8	$\bot$
20	$\{0,1,4\}$	$\top$	6	6	$\bot$
20	$\{0,1,5\}$	$\top$	6	6	$\bot$
20	$\{0,1,6\}$	$\top$	6	6	$\bot$
20	$\{0,1,9\}$	$\top$	6	7	$\bot$
20	$\{0,1,10\}$	$\top$	4	10	$\bot$
21	$\{0,1,2\}$	$\bot$	4	11	$\bot$
21	$\{0,1,3\}$	$\bot$	6	8	$\bot$
21	$\{0,1,4\}$	$\top$	6	6	$\bot$
21	$\{0,1,5\}$	$\top$	6	6	$\top$
21	$\{0,1,7\}$	$\bot$	6	7	$\bot$
21	$\{0,1,8\}$	$\bot$	6	7	$\bot$
21	$\{0,1,9\}$	$\top$	6	6	$\bot$
22	$\{0,1,2\}$	$\bot$	4	12	$\bot$
22	$\{0,1,3\}$	$\bot$	6	8	$\bot$
22	$\{0,1,4\}$	$\top$	6	7	$\bot$
22	$\{0,1,5\}$	$\top$	6	6	$\bot$
22	$\{0,1,6\}$	$\top$	6	7	$\bot$
22	$\{0,1,11\}$	$\top$	4	11	$\bot$
23	$\{0,1,2\}$	$\bot$	4	12	$\bot$
23	$\{0,1,3\}$	$\bot$	6	9	$\bot$
23	$\{0,1,4\}$	$\top$	6	7	$\bot$
23	$\{0,1,5\}$	$\top$	6	7	$\bot$
24	$\{0,1,2\}$	$\bot$	4	13	$\bot$
24	$\{0,1,3\}$	$\bot$	6	9	$\bot$
24	$\{0,1,4\}$	$\top$	6	7	$\bot$
24	$\{0,1,5\}$	$\top$	6	7	$\bot$
24	$\{0,1,6\}$	$\top$	6	7	$\bot$
24	$\{0,1,7\}$	$\top$	6	7	$\bot$
24	$\{0,1,8\}$	$\bot$	6	8	$\bot$
24	$\{0,1,9\}$	$\bot$	6	8	$\bot$
24	$\{0,1,10\}$	$\top$	6	6	$\bot$
24	$\{0,1,11\}$	$\top$	6	7	$\bot$
24	$\{0,1,12\}$	$\top$	4	12	$\bot$
25	$\{0,1,2\}$	$\bot$	4	13	$\bot$
25	$\{0,1,3\}$	$\bot$	6	9	$\bot$
25	$\{0,1,4\}$	$\top$	6	7	$\bot$
25	$\{0,1,5\}$	$\top$	6	7	$\bot$
25	$\{0,1,10\}$	$\top$	6	7	$\bot$

(a) splittable? (b) girth (c) diameter (d) arc-transitive?

Table 3: List of non-isomorphic connected trivalent cyclic Haar graphs $H(n, S)$ with $26 \leq n \leq 30$ and some of their properties.

n	S	(a)	(b)	(c)	(d)
26	$\{0,1,2\}$	$\bot$	4	14	$\bot$
26	$\{0,1,3\}$	$\bot$	6	10	$\bot$
26	$\{0,1,4\}$	$\top$	6	8	$\bot$
26	$\{0,1,5\}$	$\top$	6	7	$\bot$
26	$\{0,1,7\}$	$\top$	6	8	$\bot$
26	$\{0,1,8\}$	$\top$	6	7	$\bot$
26	$\{0,1,13\}$	$\top$	4	13	$\bot$
27	$\{0,1,2\}$	$\bot$	4	14	$\bot$
27	$\{0,1,3\}$	$\bot$	6	10	$\bot$
27	$\{0,1,4\}$	$\top$	6	8	$\bot$
27	$\{0,1,5\}$	$\top$	6	7	$\bot$
27	$\{0,1,6\}$	$\top$	6	7	$\bot$
27	$\{0,1,9\}$	$\bot$	6	9	$\bot$
28	$\{0,1,2\}$	$\bot$	4	15	$\bot$
28	$\{0,1,3\}$	$\bot$	6	10	$\bot$
28	$\{0,1,4\}$	$\top$	6	8	$\bot$
28	$\{0,1,5\}$	$\top$	6	7	$\bot$
28	$\{0,1,6\}$	$\top$	6	8	$\bot$
28	$\{0,1,7\}$	$\top$	6	7	$\bot$
28	$\{0,1,8\}$	$\top$	6	7	$\bot$
28	$\{0,1,13\}$	$\top$	6	9	$\bot$
28	$\{0,1,14\}$	$\top$	4	14	$\bot$
29	$\{0,1,2\}$	$\bot$	4	15	$\bot$
29	$\{0,1,3\}$	$\bot$	6	11	$\bot$
29	$\{0,1,4\}$	$\top$	6	8	$\bot$
29	$\{0,1,5\}$	$\top$	6	7	$\bot$
29	$\{0,1,9\}$	$\top$	6	7	$\bot$
30	$\{0,1,2\}$	$\bot$	4	16	$\bot$
30	$\{0,1,3\}$	$\bot$	6	11	$\bot$
30	$\{0,1,4\}$	$\top$	6	9	$\bot$
30	$\{0,1,5\}$	$\top$	6	7	$\bot$
30	$\{0,1,6\}$	$\top$	6	7	$\bot$
30	$\{0,1,7\}$	$\top$	6	7	$\bot$
30	$\{0,1,8\}$	$\top$	6	9	$\bot$
30	$\{0,1,9\}$	$\top$	6	7	$\bot$
30	$\{0,1,10\}$	$\bot$	6	10	$\bot$
30	$\{0,1,11\}$	$\bot$	6	10	$\bot$
30	$\{0,1,12\}$	$\top$	6	8	$\bot$
30	$\{0,1,15\}$	$\top$	4	15	$\bot$
30	$\{0,2,5\}$	$\top$	6	8	$\bot$

(a) splittable? (b) girth (c) diameter (d) arc-transitive?

$\mathcal{C}'$. Make k copies of $\mathcal{C}'$: $\mathcal{C}'_1, \mathcal{C}'_2, \ldots, \mathcal{C}'_k$ and place them equally spaced in any direction $\vec{v}$ that is non-parallel to the direction of any line of $\mathcal{C}'$ (see Figure 9(b)). Point of $\mathcal{C}'_i$ that

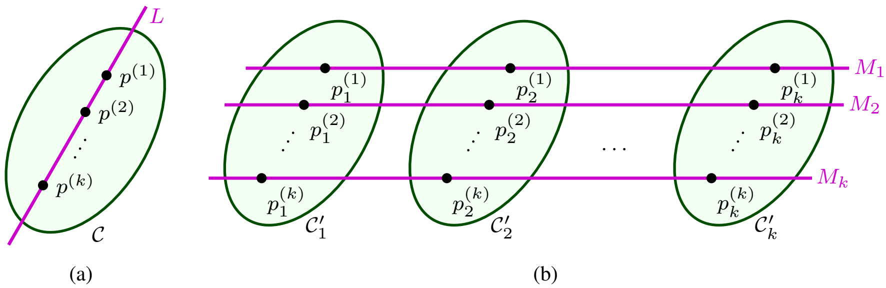


Figure 9: Construction provided by Lemma 4.1.

correspond to $p^{(j)}$ in $\mathcal{C}'$ is denoted $p_i^{(j)}$. Now add lines $M_1, M_2, \ldots, M_k$, such that M_i passes through points $p_1^{(i)}, p_2^{(i)}, \ldots, p_k^{(i)}$. The resulting structure, denoted $\mathcal{D}$, is clearly a (kn_k) configuration.

The set of lines $\{M_1, M_2, \ldots, M_k\}$ is a splitting set of $\mathcal{D}$ which proves that $\mathcal{D}$ is line-splittable. The set of points $\{p_i^{(1)}, p_i^{(2)}, \ldots, p_i^{(k)}\}$ is a splitting set for an arbitrary $1 \leq i \leq k$ which proves that $\mathcal{D}$ is also point-splittable.

It is easy to see that the resulting structure $\mathcal{D}$ is triangle-free. □

Now we can state the main result of this section.

Theorem 4.2. *For any $k > 1$ and any n_0 there exist a number $n > n_0$, such that there exists a splittable (n_k) configuration.*

Proof. Let $\mathcal{C}_0 = \mathcal{G}_k^{(k)}$, i.e. the generalised Gray $(k^k{}_k)$ configuration. Let $\mathcal{C}_i$ be a configuration obtained from $\mathcal{C}_{i-1}$ by an application of Lemma 4.1. Note that the obtained configuration $\mathcal{C}_i$ is not uniquely defined – it depends on the choice of the line L.

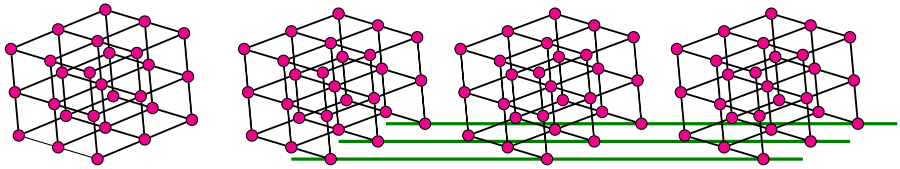
From Lemma 4.1 it follows that each $\mathcal{C}_i$, $i \geq 1$, is a point- and line-splittable configuration. Each configuration $\mathcal{C}_i$ is balanced and the number of points of $\mathcal{C}_{i+1}$ is strictly greater than the number of points of $\mathcal{C}_i$. Therefore, for increasing values of i, the number of points will eventually exceed any given number n_0. □

Since configurations $\mathcal{C}_1, \mathcal{C}_2, \ldots$ constructed in the proof of Theorem 4.2 are all of type T1, their duals are also of type T1.

Example 4.3. The generalised Gray $(k^k{}_k)$ configuration for $k = 3$ is simply called the *Gray configuration* (see Figure 10(a) and [17]). Let $\mathcal{C}_0$ be the Gray configuration. By one application of Lemma 4.1 we obtain a configuration $\mathcal{C}_1$ (see Figure 10(b)) which is point- and line-splittable.

5 Conclusion

Theorems 3.4 and 3.5, Corollary 3.3, and our experimental investigations (see periodic behaviour of the last column of Table 1 past $n = 9$) of splittability of cyclic Haar graphs led us to the following conjecture.



(a) (b)

Figure 10: The Gray (27_3) configuration $\mathcal{C}_0$ and the corresponding $\mathcal{C}_1$.

Conjecture 5.1. *A cyclic (n_3) configuration is unsplittable if and only if its Levi graph belongs to one of the following three infinite families:*

(1) $H(n, \{0, 1, 3\})$ *for* $n \geq 7$*;*

(2) $H(3n, \{0, 1, n\})$ *for* $n \geq 2$*;*

(3) $H(3n, \{0, 1, n + 1\})$ *for* $n \geq 4$ *where* $n \not\equiv 0 \pmod 3$.

To show that all other cyclic (n_3) configurations are splittable, we expect that the method used in the proof of Theorem 3.1, Corollary 3.2 and Corollary 3.3 can be extended. Nedela and Škoviera [15] showed a nice property of cubic graphs with respect to the cyclic connectivity. Their result is likely to have applications in splittablity.

In Section 4 we have shown how to construct geometric point- and line-splittable (n_k) configuration for any k. However, we were not able to obtain any splittable cyclic (n_k) configuration for $k \geq 4$ so far. Therefore, we pose the following claim.

Conjecture 5.2. *All cyclic (n_k) configurations for $k \geq 4$ are unsplittable.*

Notions of splittable and unsplittable configurations have been defined via associated graphs. Since splittability is a property of combinatorial configurations, it can be extended from bipartite graphs of girth at least 6 to more general graphs. We expect that results concerning cyclic connectivity such as those presented in [15] will play an important role in such investigations.

Note that cyclic Haar graphs have girth at most 6 and form a special class of bicirculants [16]. However, there exist other bicirculants with girth greater than 6. The corresponding configurations have been investigated in [3, 1]. One way of extending this study is on the one hand to consider splittability of these more general bicirculants and on the other hand to study tricirculants [12], tetracirculants and beyond [7]. In the language of configurations, they can be described as special classes of polycyclic configurations [2].

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