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Edge-transitive bi-p-metacirculants of valency p

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Abstract

Let p be an odd prime. A graph is called a bi-p-metacirculant on a metacyclic p-group H if admits a metacyclic p-group H of automorphisms acting semiregularly on its vertices with two orbits. A bi-p-metacirculant on a group H is said to be abelian or non-abelian according to whether or not H is abelian.

By the results of Malnič et al. in 2004 and Feng et al. in 2006, we see that up to isomorphism, the Gray graph is the only cubic edge-transitive non-abelian bi-p-metacirculant on a group of order p^3. This motivates us to consider the classification of cubic edge-transitive bi-p-metacirculants. Previously, we have proved that a cubic edge-transitive non-abelian bi-p-metacirculant exists if and only if $p = 3$. In this paper, we give a classification of connected edge-transitive non-abelian bi-p-metacirculants of valency p, and consequently, we complete the classification of connected cubic edge-transitive non-abelian bi-p-metacirculants.

Keywords: Bi-p-metacirculant, edge-transitive, inner-abelian p-group.

Math. Subj. Class.: 05C25, 20B25

1 Introduction

Given a group H, let $\mathcal{R}$, $\mathcal{L}$ and S be three subsets of H such that $\mathcal{R}^{-1} = \mathcal{R}$, $\mathcal{L}^{-1} = \mathcal{L}$ and $\mathcal{R} \cup \mathcal{L}$ does not contain the identity element of H. The *bi-Cayley graph* over H with respect to the triple $(\mathcal{R}, \mathcal{L}, S)$, denoted by $\mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$, is the graph having vertex set the union $H_0 \cup H_1$ of two copies of H, and edges of the form $\{h_0, (xh)_0\}$, $\{h_1, (yh)_1\}$ and $\{h_0, (zh)_1\}$ with $x \in \mathcal{R}, y \in \mathcal{L}, z \in S$ and $h_0 \in H_0$, $h_1 \in H_1$ representing a given $h \in H$. It is easy to see that a graph is a bi-Cayley graph over a group H if and only if it admits H as a semiregular automorphism group with two orbits.

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Let $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$. For $g \in H$, define a permutation $R(g)$ on the vertices of Γ by the rule

$$h_i^{R(g)} = (hg)_i, \quad \forall i \in \mathbb{Z}_2,\ h \in H.$$

Then $R(H) = \{R(g) \mid g \in H\}$ is a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ which is isomorphic to H and has H_0 and H_1 as its two orbits. When $R(H)$ is normal in $\mathrm{Aut}(\Gamma)$, the bi-Cayley graph $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ is said to be *normal* (see [24]). When $N_{\mathrm{Aut}(\Gamma)}(R(H))$ is transitive on the edge set of Γ, we say that Γ is *normal edge-transitive* (see [7]).

Bi-Cayley graphs are useful in constructing edge-transitive graphs (see [7, 24]). However, it is difficult in general to decide whether a bi-Cayley graph is edge-transitive. So it is natural to investigate the edge-transitive bi-Cayley graphs over some given groups. Note that metacylic groups are widely used in constructing graphs with some kinds of symmetry, see, for example, [1, 11, 12, 13, 14, 18]. (A group G is called *metacyclic* if it contains a cyclic normal subgroup N such that G/N is cyclic.) In this paper, we shall focus on the bi-Cayley graphs over a metacyclic p-group with p an odd prime. For convenience, a bi-Cayley graph over a (resp. non-abelian or abelian) metacyclic p-group is simply called a (resp. *non-abelian* or *abelian*) *bi-p-metacirculant*.

Note that the Gray graph [6], the smallest cubic semisymmetric graph, is a non-abeian bi-3-metacirculant of order $2\cdot 3^3$. Malnič et al. in [8, 17] gave a classification of cubic edge-transitive graphs of order $2p^3$ for each prime p. Actually, it is easy to prove that every cubic edge-transitive graphs of order $2p^3$ is a bi-Cayley graph over a group of order p^3. Rather than describe the classification in detail, we would simply like to point out one striking feature: except the Gray graph, there do not exist other cubic edge-transitive non-abelian bi-p-metacirculants of order $2 \cdot p^3$ for every odd prime p. This seems to suggest that cubic edge-transitive non-abelian bi-p-metacirculants are rare. Motivated by this, we are going to consider the following problem:

Problem 1.1. Classify cubic edge-transitive non-abelian bi-p-metacirculants for every odd prime p.

In [19], we gave a partial answer to this problem. We first proved that a cubic edge-transitive non-abelian bi-p-metacirculant exists if and only if $p = 3$, and then we gave a classification of cubic edge-transitive bi-Cayley graphs over an inner-abelian metacyclic p-group for each odd prime p. (A non-abelian group is called an *inner-abelian group* if all of its proper subgroups are abelian.) In view of this, to solve Problem 1.1, it suffices to classify cubic edge-transitive non-abelian bi-3-metacirculants. Naturally, the following problem arises.

Problem 1.2. Classify edge-transitive non-abelian bi-p-metacirculants of valency p for every odd prime p.

The following is the main result of this paper which gives a solution of Problem 1.2.

Theorem 1.3. *Let p be an odd prime, and let Γ be a connected edge-transitive non-abelian bi-p-metacirculants of valency p. Then $p = 3$ and Γ is isomorphic to one of the following graphs:*

(i)

$$\Gamma_r = \mathrm{BiCay}(\mathcal{G}_r, \emptyset, \emptyset, \{1, a, a^{-1}b\}),$$
$$\mathcal{G}_r = \left\langle a, b \mid a^{3^{r+1}} = b^{3^r} = 1, b^{-1}ab = a^{1+3^r} \right\rangle,$$

(ii)

$$\Sigma_r = \mathrm{BiCay}(\mathcal{H}_r, \emptyset, \emptyset, \{1, b, b^{-1}a\}),$$
$$\mathcal{H}_r = \left\langle a, b \mid a^{3^{r+1}} = b^{3^{r+1}} = 1, b^{-1}ab = a^{1+3^r} \right\rangle,$$

where r is a positive integer.

Remark 1.4. The graphs Γ_r and Σ_r are actually those graphs what we have found in [19]. By [19], Γ_r is semisymmetric while Σ_r is symmetric. To the best of our knowledge, the graphs Γ_r form the first known infinite family of cubic semisymmetric graphs of order twice a power of 3.

From the above theorem and [19, Theorem 1], we may immediately obtain the following result which gives a solution of Problem 1.1.

Corollary 1.5. *Let p be an odd prime. A connected cubic non-abelian bi-p-metacirculant is edge-transitive if and only if it is isomorphic to one the graphs given in Theorem 1.3.*

Remark 1.6. The classification of cubic edge-transitive bi-Cayley graphs on abelian groups has been given in [10, 23]. So our result actually completes the classification of all cubic edge-transitive bi-p-metacirculants for each odd prime p.

2 Preliminaries

2.1 Definitions and notation

Throughout this paper, groups are assumed to be finite, and graphs are assumed to be finite, connected, simple and undirected. For the group-theoretic and the graph-theoretic terminology not defined here we refer the reader to [4, 21].

Let G be a permutation group on a set Ω and take $\alpha \in \Omega$. The stabilizer G_α of α in G is the subgroup of G fixing the point α. The group G is said to be *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular.

For a positive integer n, denote by $\mathbb{Z}_n$ the cyclic group of order n and by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n. For a finite group G, the full automorphism group and the derived subgroup of G will be denoted by $\mathrm{Aut}(G)$ and G', respectively. Denote by $\exp(G)$ the exponent of G. For any $x \in G$, denote by $o(x)$ the order of x. For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. A non-abelian group is called an *inner-abelian group* if all of its proper subgroups are abelian.

For a graph Γ, we denote by $V(\Gamma)$ the set of all vertices of Γ, by $E(\Gamma)$ the set of all edges of Γ, by $A(\Gamma)$ the set of all arcs of Γ, and by $\mathrm{Aut}(\Gamma)$ the full automorphism group of Γ. For $u, v \in V(\Gamma)$, denote by $\{u, v\}$ the edge incident to u and v in Γ. If a subgroup G of $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$, $E(\Gamma)$ or $A(\Gamma)$, we say that Γ is *G-vertex-transitive*, *G-edge-transitive* or *G-arc-transitive*, respectively. In the special case when $G = \mathrm{Aut}(\Gamma)$ we say that Γ is *vertex-transitive*, *edge-transitive* or *arc-transitive*, respectively. An arc-transitive graph is also called a *symmetric graph*. A graph Γ is said to be *semisymmetric* if Γ is regular and is edge- but not vertex-transitive.

2.2 Quotient graph

Let Γ be a connected graph with an edge-transitive group G of automorphisms and let N be a normal subgroup of G. The *quotient graph* Γ_N of Γ relative to N is defined as the graph with vertices the orbits of N on $V(\Gamma)$ and with two orbits adjacent if there exists an edge in Γ between the vertices lying in those two orbits. Below we introduce two propositions of which the first is a result of [15, Theorem 9].

Proposition 2.1. *Let p be an odd prime and Γ be a graph of valency p, and let $G \leq \mathrm{Aut}(\Gamma)$ be arc-transitive on Γ. Then G is an s-arc-regular subgroup of $\mathrm{Aut}(\Gamma)$ for some integer s. If $N \trianglelefteq G$ has more than two orbits in $V(\Gamma)$, then N is semiregular on $V(\Gamma)$, Γ_N is a symmetric graph of valency p with G/N as an s-arc-regular subgroup of automorphisms.*

In view of [16, Lemma 3.2], we have the following proposition.

Proposition 2.2. *Let p be an odd prime and Γ be a graph of valency p, and let $G \leq \mathrm{Aut}(\Gamma)$ be transitive on $E(\Gamma)$ but intransitive on $V(\Gamma)$. Then Γ is a bipartite graph with two partition sets, say V_0 and V_1. If $N \trianglelefteq G$ is intransitive on each of V_0 and V_1, then N is semiregular on $V(\Gamma)$, Γ_N is a graph of valency p with G/N as an edge- but not vertex-transitive group of automorphisms.*

2.3 Bi-Cayley graphs

Proposition 2.3 ([23, Lemma 3.1])**.** *Let $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ be a connected bi-Cayley graph over a group H. Then the following hold:*

(1) H is generated by $\mathcal{R} \cup \mathcal{L} \cup S$.

(2) Up to graph isomorphism, S can be chosen to contain the identity of H.

(3) For any automorphism α of H, $\mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S) \cong \mathrm{BiCay}(H, \mathcal{R}^\alpha, \mathcal{L}^\alpha, S^\alpha)$.

(4) $\mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S) \cong \mathrm{BiCay}(H, \mathcal{L}, \mathcal{R}, S^{-1})$.

Let $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ be a bi-Cayley graph over a group H. Recall that for each $g \in H$, $R(g)$ is a permutation on $V(\Gamma)$ defined by the rule

$$h_i^{R(g)} = (hg)_i, \quad \forall i \in \mathbb{Z}_2,\ h, g \in H,$$

and $R(H) = \{R(g) \mid g \in H\} \leq \mathrm{Aut}(\Gamma)$. For an automorphism α of H and $x, y, g \in H$, define two permutations on $V(\Gamma) = H_0 \cup H_1$ as following:

$$\delta_{\alpha,x,y} \colon h_0 \mapsto (xh^\alpha)_1,\ h_1 \mapsto (yh^\alpha)_0,\ \forall h \in H,$$
$$\sigma_{\alpha,g} \colon h_0 \mapsto (h^\alpha)_0,\ h_1 \mapsto (gh^\alpha)_1,\ \forall h \in H.$$

Set

$$I = \{\delta_{\alpha,x,y} \mid \alpha \in \mathrm{Aut}(H) \text{ s.t. } \mathcal{R}^\alpha = x^{-1}\mathcal{L}x, \mathcal{L}^\alpha = y^{-1}\mathcal{R}y, S^\alpha = y^{-1}S^{-1}x\},$$
$$F = \{\sigma_{\alpha,g} \mid \alpha \in \mathrm{Aut}(H) \text{ s.t. } \mathcal{R}^\alpha = \mathcal{R}, \mathcal{L}^\alpha = g^{-1}\mathcal{L}g, S^\alpha = g^{-1}S\}.$$

Proposition 2.4 ([24, Theorem 3.4])**.** *Let $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ be a connected bi-Cayley graph over the group H. Then $N_{\mathrm{Aut}(\Gamma)}(R(H)) = R(H) \rtimes F$ if $I = \emptyset$ and $N_{\mathrm{Aut}(\Gamma)}(R(H)) = R(H)\langle F, \delta_{\alpha,x,y}\rangle$ if $I \neq \emptyset$ and $\delta_{\alpha,x,y} \in I$. Furthermore, for any $\delta_{\alpha,x,y} \in I$, we have the following:*

(1) $\langle R(H), \delta_{\alpha,x,y}\rangle$ *acts transitively on* $V(\Gamma)$*;*

(2) *if* α *has order* 2 *and* $x = y = 1$*, then* Γ *is isomorphic to the Cayley graph* $\mathrm{Cay}(\bar{H}, \mathcal{R} \cup \alpha S)$*, where* $\bar{H} = H \rtimes \langle\alpha\rangle$*.*

3 Some basic properties of metacyclic p-groups

In this section, we will give some properties of metacyclic p-groups.

Proposition 3.1. *Any metacyclic* p*-group* G *(*p *an odd prime) has the following presentation:*

$$G = \left\langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, a^b = a^{1+p^r} \right\rangle,$$

where r, s, t, u *are non-negative integers with* $u \leq r$*. Different values of the parameters* r, s, t, u *with the above conditions give non-isomorphic metacyclic* p*-groups. Furthermore, the following hold:*

(1) *If* $|G'| = p^n$*, then for any* $m \geq n$*, we have*

$$(xy)^{p^m} = x^{p^m} y^{p^m}, \quad \forall x, y \in G.$$

(2) *For any positive integer* k *and for any* $x, y \in G$*,*

$$x^{p^k} = y^{p^k} \iff (x^{-1}y)^{p^k} = 1 \iff (xy^{-1})^{p^k} = 1.$$

Proof. By [22, Theorem 2.1], it suffices to prove the items (1) and (2). Since G' is cyclic, (1) follows from [9, Chapter 3, §10, Theorem 10.2 (c) and Theorem 10.8 (g)]. Item (2) follows from [9, Chapter 3, §10, Theorem 10.2 (c) and Theorem 10.6 (a)]. □

Lemma 3.2. *Let* p *be an odd prime, and let* H *be a metacyclic* p*-group generated by* a, b *with the following defining relations:*

$$a^{p^m} = b^{p^n} = 1, \qquad b^{-1}ab = a^{1+p^r},$$

where m, n, r *are positive integers such that* $r < m \leq n + r$*. Then the following hold:*

(1) *For any* $i \in \mathbb{Z}_{p^m}$*,* $j \in \mathbb{Z}_{p^n}$*, we have*

$$a^i b^j = b^j a^{i(1+p^r)^j}.$$

(2) *For any positive integer* k *and for any* $i \in \mathbb{Z}_{p^m}$*,* $j \in \mathbb{Z}_{p^n}$*, we have*

$$(b^j a^i)^k = b^{kj} a^{i\sum_{s=0}^{k-1}(1+p^r)^{sj}}.$$

(3) *For any positive integers* t, k *and any element* x *of* H*, if* $x^{p^{2t}} = 1$*, then*

$$x^{(1+p^t)^k} = x^{1+k\cdot p^t}.$$

(4) *The subgroup of* H *of order* p *is one of the following groups:*

$$\left\langle a^{p^{m-1}} \right\rangle, \quad \left\langle b^{p^{n-1}} a^{i' p^{m-1}} \right\rangle \ (i' \in \mathbb{Z}_p).$$

Proof. From [19, Lemma 14 (1) – (2)], we have the items (1) – (2).

For (3), the result is clearly true if $k = 1$. In what follows, assume $k \geq 2$. Since $x^{p^{2t}} = 1$, we have $x^{p^{kt}} = 1$. Then

$$\begin{aligned} x^{(1+p^t)^k} &= x^{[C_k^0 \cdot 1^k \cdot (p^t)^0 + C_k^1 \cdot 1^{k-1} \cdot (p^t)^1 + C_k^2 \cdot 1^{k-2} \cdot (p^t)^2 + \cdots + C_k^k \cdot 1^0 \cdot (p^t)^k]} \\ &= x^{C_k^0 \cdot (p^t)^0} \cdot x^{C_k^1 \cdot (p^t)^1} \cdot x^{C_k^2 \cdot (p^t)^2} \cdots x^{C_k^k \cdot (p^t)^k} \\ &= x \cdot (x^{p^t})^{C_k^1} \cdot (x^{p^{2t}})^{C_k^2} \cdots (x^{p^{kt}})^{C_k^k} \\ &= x \cdot x^{k \cdot p^t} \\ &= x^{1+k\cdot p^t}, \end{aligned}$$

and so (3) holds. (Here for any integers $N \geq l \geq 0$, we denote by C_N^l the binomial coefficient, that is, $C_N^l = \frac{N!}{l!(N-l)!}$.)

For (4), let $\Omega_1(H) = \langle x \in H \mid o(x) = p \rangle$. Since H is a metacyclic p-group, by [2, Exercise 85], we have $\Omega_1(H) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. It implies that H has $p+1$ subgroups of order p. Furthermore, the subgroup of H of order p is one of the following groups:

$$\left\langle a^{p^{m-1}} \right\rangle, \quad \left\langle b^{p^{n-1}} a^{i' p^{m-1}} \right\rangle \ (i' \in \mathbb{Z}_p),$$

as required. □

4 Inner-abelian bi-p-metacirculants of valency p

In this section, we focus on edge-transitive bi-Cayley graphs over inner-abelian metacyclic p-groups of valency p. For convenience, a bi-Cayley graph over an inner-abelian metacyclic p-group is simply called an *inner-abelian bi-p-metacirculant.*

In [19, Theorem 2], we gave a classification of cubic edge-transitive inner-abelian bi-p-metacirculants.

Proposition 4.1 ([19, Theorem 2])**.** *Let Γ be a connected cubic edge-transitive bi-Cayley graph over an inner-abelian metacyclic 3-group H. Then $H \cong \mathcal{G}_r$ or $\mathcal{H}_r$, and $\Gamma \cong \Gamma_r$ or Σ_r, where the groups $\mathcal{G}_r$, $\mathcal{H}_r$, and the graphs Γ_r, Σ_r are defined as in Theorem 1.3. In particular, $H/H' \cong \mathbb{Z}_{3^r} \times \mathbb{Z}_{3^r}$ or $\mathbb{Z}_{3^r} \times \mathbb{Z}_{3^{r+1}}$.*

In this section, we shall prove the following theorem.

Theorem 4.2. *Let H be an inner-abelian metacyclic p-group with p an odd prime, and let Γ be a connected edge-transitive bi-Cayley graph over H of valency p. Then $p = 3$, and Γ is isomorphic to one of the graphs given in Theorem 1.3.*

4.1 Two technical lemmas

Lemma 4.3. *Let p be an odd prime and let Γ be a connected edge-transitive graph of valency p. If $G \leq \mathrm{Aut}(\Gamma)$ is transitive on the edges of Γ, then for each $v \in V(\Gamma)$, $|G_v| = pm$ with $(m, p) = 1$.*

Proof. Since G is transitive on the edges of Γ, for each $v \in V(\Gamma)$, the order of a vertex stabilizer G_v must be divisible by p. Suppose, by way of contradiction, that $|G_v|$ is divisible by p^2. Let G_v^* be the subgroup of G_v fixing the neighborhood $\Gamma(v)$ of v in Γ pointwise.

Then $G_v/G_v^* \lesssim S_p$, forcing that $p \mid |G_v^*|$. Then G_v^* contains an element α of order p. Note that each orbit of $\langle\alpha\rangle$ has length either 1 or p. Since $\langle\alpha\rangle$ fixes v and each vertex in $\Gamma(v)$, the connectedness of Γ implies that each orbit of $\langle\alpha\rangle$ has length 1, and so $\alpha = 1$, a contradiction. □

Lemma 4.4. *Let H be a p-group with p an odd prime, and let $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ be a connected edge-transitive bi-Cayley graph of valency p. Then*

(1) Γ is normal edge-transitive, $\mathcal{R} = \mathcal{L} = \emptyset$, and $S = \{1, h, hh^\alpha, \ldots, hh^\alpha \cdots h^{\alpha^{p-2}}\}$ for some $1 \neq h \in H$ and $\alpha \in \mathrm{Aut}(H)$ satisfying $hh^\alpha h^{\alpha^2} \cdots h^{\alpha^{p-1}} = 1$ and $o(\alpha) \mid p$;

(2) if H has a characteristic subgroup K such that H/K is isomorphic to $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, then $|m - n| \leq 1$.

Proof. Let $A = \mathrm{Aut}(\Gamma)$, and let P be a sylow p-subgroup of A such that $R(H) \leq P$. Since Γ is edge-transitive, Lemma 4.3 gives that $|A| = |R(H)| \cdot p \cdot m$, where $(p, m) = 1$. It follows that $|P| = p|R(H)|$, and hence $P \leq N_A(R(H))$. Furthermore, for any $e \in E(\Gamma)$, we have $|A : A_e| = |E(\Gamma)| = p|R(H)|$, and so $|A_e| = m$. It follows that $P_e = P \cap A_e = 1$, and hence $|P : P_e| = |P| = p|R(H)| = |E(\Gamma)|$. Thus, P is transitive on the edges of Γ. Thus, Γ is normal edge-transitive.

Let $N = N_A(R(H))$. Then N is transitive on the edges of Γ. Since $R(H) \trianglelefteq N$, the two orbits H_0, H_1 of $R(H)$ do not contain any edge of Γ, and so $\mathcal{R} = \mathcal{L} = \emptyset$. By Proposition 2.3, we may assume that $1 \in S$. Since N is transitive on the edges of Γ and Γ has valency p, N_{1_0} has an element $\sigma_{\alpha,h}$ of order p for some $\alpha \in \mathrm{Aut}(H)$ and $1 \neq h \in H$. Furthermore, $\sigma_{\alpha,h}$ cyclically permutes the elements in $\Gamma(1_0)$. So we have $\Gamma(1_0) = \{1_1, h_1, (hh^\alpha)_1, \ldots, (hh^\alpha \cdots h^{\alpha^{p-2}})_1\}$ and $hh^\alpha h^{\alpha^2} \cdots h^{\alpha^{p-1}} = 1$. This implies that

$$S = \{1, h, hh^\alpha, \ldots, hh^\alpha \cdots h^{\alpha^{p-2}}\},$$

and $h^{\alpha^p} = h$. Since Γ is connected, one has $H = \langle S \rangle = \langle h^{\alpha^i} \mid 0 \leq i \leq p-1 \rangle$. As $h^{\alpha^p} = h$, α^p is a trivial automorphism of H. Consequently, we have $o(\alpha) = 1$ or p and (1) is proved.

For (2), without loss of generality, assume that $H/K \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ with $m > n$, where K is a characteristic subgroup of H. Let $T = \langle R(x) \in R(H) \mid x^{p^n} \in K \rangle$. Then T is characteristic in $R(H)$ and $R(H)/T \cong \mathbb{Z}_{p^{m-n}}$. Propositions 2.1 and 2.2 implies that the quotient graph Γ_T of Γ relative to T is a graph of valency p with N/T as an edge-transitive group of automorphisms. Clearly, $R(H)/T$ is semiregular on $V(\Gamma_T)$ with two orbits and $R(H)/T \trianglelefteq N/T$, so Γ_T is a normal edge-transitive bi-Cayley graph over $R(H)/T \cong \mathbb{Z}_{p^{m-n}}$ of valency p.

So to complete the proof, it suffices to show that if $H \cong \mathbb{Z}_{p^m}$ then $m \leq 1$. Suppose to the contrary that $H \cong \mathbb{Z}_{p^m}$ with $m \geq 2$. Since $H = \langle h^{\alpha^i} \mid 0 \leq i \leq p-1 \rangle$, we have $H = \langle h \rangle$. Let $h^\alpha = h^\lambda$ for some $\lambda \in \mathbb{Z}_{p^m}^*$. Then

$$1 = hh^\alpha h^{\alpha^2} \cdots h^{\alpha^{p-1}} = h^{1+\lambda+\lambda^2+\cdots+\lambda^{p-1}},$$

and then

$$1 + \lambda + \lambda^2 + \cdots + \lambda^{p-1} \equiv 0 \pmod{p^m}.$$

It follows that $\lambda^p \equiv 1 \pmod{p^m}$, and hence $\lambda \equiv 1 \pmod{p}$. Let $\lambda = kp + 1$ for some integer k. Since $m \geq 2$, we have

$$1 + (kp+1) + (kp+1)^2 + \cdots + (kp+1)^{p-1} \equiv 0 \pmod{p^2}.$$

It follows that

$$1 + (kp+1) + (2kp+1) + \cdots + ((p-1)kp+1) \equiv 0 \pmod{p^2},$$

and hence

$$p + \frac{1}{2}p(p-1)kp \equiv 0 \pmod{p^2}.$$

A contradiction occurs. □

4.2 Proof of Thorem 4.2

Throughout this subsection, we shall always let H be an inner-abelian metacyclic p-group with p an odd prime, and Γ be a connected edge-transitive bi-Cayley graph over H of valency p.

In view of Lemma 4.4(1) and since H is inner abelian, we may make the following assumption throughout this subsection.

Assumption 4.5. *$\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$, where $S = \{1, h, hh^\alpha, \ldots, hh^\alpha \cdots h^{\alpha^{p-2}}\}$ for some $1 \neq h \in H$ and $\alpha \in \mathrm{Aut}(H)$ satisfying $hh^\alpha h^{\alpha^2} \cdots h^{\alpha^{p-1}} = 1$ and $o(\alpha) = p$.*

Proof of Theorem 4.2. Suppose to the contrary that $p > 3$. Since H is an inner-abelian metacyclic p-group, by elementary group theory (see also [20] or [3, Lemma 65.2]), we may assume that

$$H = \left\langle a, b \mid a^{p^{t+1}} = b^{p^s} = 1, b^{-1}ab = a^{p^t+1} \right\rangle,$$

where $t \geq 1, s \geq 1$. Note that $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^s}$. By Lemma 4.4, we have $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^t}$, $\mathbb{Z}_{p^t} \times \mathbb{Z}_{p^{t+1}}$ or $\mathbb{Z}_{p^t} \times \mathbb{Z}_{p^{t-1}}$.

If $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^{t-1}}$, then $s = t - 1$ and

$$H = \left\langle a, b \mid a^{p^{t+1}} = b^{p^{t-1}} = 1, b^{-1}ab = a^{p^t+1} \right\rangle.$$

Let $T = \langle R(x) \mid x \in H, x^{p^{t-1}} = 1 \rangle$. Then T is characteristic in $R(H)$ and $R(H)/T$ is isomorphic to $\mathbb{Z}_{p^2}$. However, by the proof of Lemma 4.4, this is impossible.

If $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^t}$, then $s = t$ and

$$H = \left\langle a, b \mid a^{p^{t+1}} = b^{p^t} = 1, b^{-1}ab = a^{p^t+1} \right\rangle,$$

where $t \geq 1$. We shall show that this is impossible in Lemma 4.6.

If $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^{t+1}}$, then $s = t + 1$ and

$$H = \left\langle a, b \mid a^{p^{t+1}} = b^{p^{t+1}} = 1, b^{-1}ab = a^{p^t+1} \right\rangle,$$

where $t \geq 1$. We shall show that this is impossible in Lemma 4.7. □

Lemma 4.6. *If* $H = \left\langle a, b \mid a^{p^{t+1}} = b^{p^t} = 1, b^{-1}ab = a^{p^t+1} \right\rangle$ $(t > 0)$*, then* $p = 3$.

Proof. Suppose to the contrary that $p > 3$. We first define the following four maps. Let

$$\gamma\colon a \mapsto a^{1+p}, b \mapsto b, \qquad \delta\colon a \mapsto a, b \mapsto b^{1+p},$$
$$\sigma\colon a \mapsto a, b \mapsto ba^p, \qquad \tau\colon a \mapsto ba, b \mapsto b.$$

Let $x_1 = a^{1+p}$, $x_2 = x_3 = a$, $x_4 = ba$, $y_1 = y_4 = b$, $y_2 = b^{1+p}$ and $y_3 = ba^p$. Since H is an inner-abelian metacyclic-p group, by Proposition 3.1 and a direct computation, we have $o(x_{i_1}) = o(a) = p^{t+1}$, $o(y_{i_1}) = o(b) = p^t$ and it is direct to check that x_{i_1} and y_{i_1} have the same relations as do a and b, where $i_1 \in \{1,2,3,4\}$. Moreover, for any $i_1 \in \{1,2,3,4\}$, we have $\langle x_{i_1}, y_{i_1}\rangle = H$. It follows that each of the above four maps induces an automorphism of H.

Set $P = \langle \sigma, \gamma, \delta, \tau\rangle$. By a direct computation, we have $o(\gamma) = p^t, o(\delta) = p^{t-1}$ and $o(\sigma) = o(\tau) = p^t$. Furthermore, $\gamma\delta = \delta\gamma$, $\gamma^{-1}\sigma\gamma = \sigma^{p+1}$ and $\delta^{-1}\sigma\delta = \sigma^\ell$ with $\ell(p+1) \equiv 1 \pmod{p^t}$. As both γ and δ fixes the subgroup $\langle b\rangle$ while σ does not, one has

$$\langle \sigma, \gamma, \delta\rangle = \langle\sigma\rangle \rtimes (\langle\gamma\rangle \times \langle\delta\rangle) \cong \mathbb{Z}_{p^t} \rtimes (\mathbb{Z}_{p^t} \times \mathbb{Z}_{p^{t-1}}).$$

Observing that $\langle \sigma, \gamma, \delta\rangle$ fixes the subgroup $\langle a\rangle$ setwise but τ does not, it follows that $\langle \sigma, \gamma, \delta\rangle \cap \langle\tau\rangle = 1$, and hence $|P| \geq p^{4t-1}$. In view of [13, Theorem 2.8], $\mathrm{Aut}(H)$ has a normal Sylow p-subgroup of order p^{4t-1}. It follows that $P = \langle \sigma, \gamma, \delta, \tau\rangle$ is the unique Sylow p-subgroup of $\mathrm{Aut}(H)$. In particular, we have $P = \langle\gamma\rangle\langle\delta\rangle\langle\sigma\rangle\langle\tau\rangle$.

Recall that $S = \{1, h, hh^\alpha, \ldots, hh^\alpha \cdots h^{\alpha^{p-2}}\}$. Assume that $h = b^u a^v$ for some $u \in \mathbb{Z}_{p^t}$ and $v \in \mathbb{Z}_{p^{t+1}}$. Since $H = \langle S\rangle$, we have $o(h) = \exp(H)$. It follows that $(v, p) = 1$. Then the map $\varphi_1\colon a \mapsto a^v, b \mapsto b$ induces an automorphism of H. Let $\varphi = (\tau^u\varphi_1)^{-1}$. Then $\varphi \in \mathrm{Aut}(H)$ and $h^\varphi = a$. By Proposition 2.4(3), we have that $\Gamma \cong \Gamma' = \mathrm{BiCay}(H, \emptyset, \emptyset, S^\varphi)$. Let $\beta = \varphi^{-1}\alpha\varphi$. Then $\sigma_{\beta,a} \in \mathrm{Aut}(\Gamma')$ cyclically permutates the elements in $\Gamma'(1_0)$. It follows that

$$S^\varphi = \{1, a, aa^\beta, aa^\beta a^{\beta^2}, \ldots, aa^\beta a^{\beta^2} \cdots a^{\beta^{p-2}}\},$$

and $aa^\beta a^{\beta^2} \cdots a^{\beta^{p-1}} = 1$. Clearly, $o(\beta) = o(\alpha) = p$, so $\beta \in P$. We assume that $\beta = \gamma^i\delta^j\sigma^k\tau^l$ for some $i, k, l \in \mathbb{Z}_{p^t}$ and $j \in \mathbb{Z}_{p^{t-1}}$.

By Lemma 3.2(2)–(3) and Proposition 3.1(1), we have

$$\beta\colon \begin{cases} a \mapsto (b^l a)^{(1+p)^i} = b^{(1+p)^i l} a^{(1+p)^i} \\ b \mapsto (b \cdot (b^l a)^{pk})^{(1+p)^j} = b^{(1+p)^j(1+pkl)} a^{(1+p)^j pk} \end{cases} \tag{4.1}$$

Let $\mho_1(H) = \{x^p \mid x \in H\}$. Then $\mho_1(H) \leq Z(H)$ and

$$\beta\colon \begin{cases} a \mapsto b^l a \cdot w \\ b \mapsto b \cdot w' \end{cases} \tag{4.2}$$

for some w, $w' \in \mho_1(H)$. Since Γ' is connected, by Proposition 2.3, we have $H = \langle S^\varphi\rangle$. By Proposition 3.1(1), it follows that $(l, p) = 1$.

We shall finish the proof by the following steps.

Step 1: $t > 1$.

Suppose to the contrary that $t = 1$. Then $H = \langle a, b \mid a^{p^2} = b^p = 1, b^{-1}ab = a^{1+p} \rangle$. We shall first show that for any $r \geq 1$,

$$a^{\beta^r} = b^{rl} a^{1+\frac{1}{2}r(r-1)klp+irp} \tag{4.3}$$

By Equation (4.1) we have

$$\beta\colon \begin{cases} a \mapsto b^l a^{1+ip} \\ b \mapsto ba^{kp} \end{cases}$$

So Equation (4.3) holds when $r = 1$. Now assume that $r > 1$ and

$$a^{\beta^{r-1}} = b^{(r-1)l} a^{1+\frac{1}{2}(r-1)(r-2)klp+i(r-1)p}.$$

By a direct computation, we have

$$\begin{aligned} a^{\beta^r} &= (b^{(r-1)l} a^{1+\frac{1}{2}(r-1)(r-2)klp+i(r-1)p})^{\beta} \\ &= (ba^{kp})^{(r-1)l} (b^l a^{1+ip})^{1+\frac{1}{2}(r-1)(r-2)klp+i(r-1)p} \\ &= b^{(r-1)l} a^{(r-1)lkp} b^l a^{1+\frac{1}{2}[(r-1)^2-(r-1)]klp+irp} \\ &= b^{(r-1)l+l} a^{1+[\frac{1}{2}(r-1)^2-\frac{1}{2}(r-1)+(r-1)]klp+irp} \\ &= b^{rl} a^{1+\frac{1}{2}r(r-1)klp+irp} \end{aligned}$$

By induction, we have Equation (4.3).

Now we show that for any $r \geq 1$,

$$a \cdot a^{\beta} \cdots a^{\beta^r} = b^{\frac{1}{2}r(r+1)l} a^{(r+1)+[\frac{1}{6}r(r+1)(2r+1)l+\frac{1}{2}r(r+1)i+\frac{1}{6}(r-1)r(r+1)kl]p}. \tag{4.4}$$

By Equation (4.3) and Lemma 3.2(1)&(3), we have

$$a \cdot a^{\beta} = a \cdot b^l a^{1+ip} = b^l a^{(1+p)^l} a^{1+ip} = b^l a^{1+lp} a^{1+ip} = b^l a^{2+(l+i)p}.$$

So Equation 4.4 holds when $r = 1$. Now assume that $r > 1$ and

$$a \cdot a^{\beta} \cdots a^{\beta^{r-1}} = b^{\frac{1}{2}(r-1)rl} a^{r+[\frac{1}{6}(r-1)r(2r-1)l+\frac{1}{2}(r-1)ri+\frac{1}{6}(r-2)(r-1)rkl]p}.$$

By a direct computation, we have

$$\begin{aligned} &aa^{\beta}a^{\beta^2} \cdots a^{\beta^r} \\ &= b^{\frac{1}{2}(r-1)rl} a^{r+[\frac{1}{6}(r-1)r(2r-1)l+\frac{1}{2}(r-1)ri+\frac{1}{6}(r-2)(r-1)rkl]p} \cdot b^{rl} a^{1+\frac{1}{2}r(r-1)klp+irp} \\ &= b^{\frac{1}{2}r(r+1)l} a^{\{r+[\frac{1}{6}(r-1)r(2r-1)l+\frac{1}{2}(r-1)ri+\frac{1}{6}(r-2)(r-1)rkl]p\}\cdot(1+rlp)+1+\frac{1}{2}r(r-1)klp+irp} \\ &= b^{\frac{1}{2}r(r+1)l} a^{r(1+rlp)+[\frac{1}{6}(r-1)r(2r-1)l+\frac{1}{2}(r-1)ri+\frac{1}{6}(r-2)(r-1)rkl]p+1+\frac{1}{2}r(r-1)klp+irp} \\ &= b^{\frac{1}{2}r(r+1)l} a^{(r+1)+[\frac{1}{6}(r-1)r(2r-1)+r^2]lp+[\frac{1}{2}(r-1)r+r]ip+[\frac{1}{6}(r-2)(r-1)+\frac{1}{2}r(r-1)]rklp} \\ &= b^{\frac{1}{2}r(r+1)l} a^{(r+1)+[\frac{1}{6}r(r+1)(2r+1)l+\frac{1}{2}r(r+1)i+\frac{1}{6}(r-1)r(r+1)kl]p}. \end{aligned}$$

By induction, we have Equation (4.4).

Since p is a prime and $p > 3$, by Equation (4.4), we have

$$aa^\beta a^{\beta^2}\cdots a^{\beta^{p-1}} = b^{\frac{1}{2}(p-1)pl}a^{p+[\frac{1}{6}(p-1)p(2p-1)l+\frac{1}{2}(p-1)pi+\frac{1}{6}(p-2)(p-1)pkl]p} = a^p \neq 1,$$

a contradiction.

Step 2: A final contradiction

Let $\mho_2(H) = \{x^{p^2} \mid x \in H\}$. Then $\mho_2(H) \leq Z(H)$. By Equation (4.1), we have

$$\begin{aligned} a^\beta &= b^{(1+ip)l}a^{1+ip}\cdot\varpi,\\ b^\beta &= b^{1+jp+pkl}a^{pk}\cdot\varpi', \end{aligned}$$

for some $\varpi, \varpi' \in \mho_2(H)$. Let $m \equiv il \pmod p$, $n \equiv i \pmod p$, $f \equiv j + kl \pmod p$ for some $m, n, f \in \mathbb{Z}_p$. Then

$$\beta\colon \begin{cases} a \mapsto b^{mp+l}a^{np+1}\cdot\varpi_1 \\ b \mapsto b^{fp+1}a^{kp}\cdot\varpi_1' \end{cases} \tag{4.5}$$

for some $\varpi_1, \varpi_1' \in \mho_2(H)$.

We shall first prove the following claim.

Claim. *For any $r \geq 2$, $a^{\beta^r} = b^{c_rp+2l}a^{d_rp}\varpi_r$ for some $c_r, d_r \in \mathbb{Z}_p$ and $\varpi_r \in \mho_2(H)$.*

Since $t > 1$, for any positive integer i_0, by Lemma 3.2(1)&(3), we have

$$ab^{i_0} = b^{i_0}a^{(1+p^t)^{i_0}} = b^{i_0}a^{1+i_0p^t} = b^{i_0}a\cdot\varpi_0, \tag{4.6}$$

for some $\varpi_0 \in \mho_2(H)$. Then by Equations (4.5) and (4.6), we have

$$\begin{aligned} a^{\beta^2} &= (b^{fp+1}a^{kp}\cdot\varpi_1')^{mp+l}(b^{mp+l}a^{np+1}\cdot\varpi_1)^{np+1}\cdot\varpi_1^\beta\\ &= b^{(2m+fl+nl)p+2l}a^{(2n+kl)p}\cdot\varpi_2, \end{aligned}$$

for some $\varpi_2 \in \mho_2(H)$. Take $c_2, d_2 \in \mathbb{Z}_p$ such that $2m + fl + nl \equiv c_2 \pmod p$ and $2n + kl \equiv d_2 \pmod p$. If $r = 2$, then Claim is clearly true. Now assume that $r > 2$ and Claim holds for any positive integer less than r. Then

$$a^{\beta^{r-1}} = b^{c_{r-1}p+2l}a^{d_{r-1}p}\cdot\varpi_{r-1},$$

for some $c_{r-1}, d_{r-1} \in \mathbb{Z}_p$ and $\varpi_{r-1} \in \mho_2(H)$, and then

$$\begin{aligned} a^{\beta^r} &= (b^{fp+1}a^{kp}\cdot\varpi_1')^{c_{r-1}p+2l}(b^{mp+l}a^{np+1}\cdot\varpi_1)^{d_{r-1}p}\cdot\varpi_{r-1}^\beta\\ &= b^{(c_{r-1}+2fl+ld_{r-1})p+2l}a^{(2kl+d_{r-1})p}\cdot\varpi_r, \end{aligned}$$

for some $\varpi_r \in \mho_2(H)$. Take $c_r, d_r \in \mathbb{Z}_p$ such that $c_{r-1} + 2fl + ld_{r-1} \equiv c_r \pmod p$ and $2kl + d_{r-1} \equiv d_r \pmod p$. By induction, we complete the proof of Claim.

Now by our Claim, we have

$$a^{\beta^p} = b^{c_pp+2l}a^{d_pp}\cdot\varpi_p = a,$$

for some $c_p, d_p \in \mathbb{Z}_p$ and $\varpi_p \in \mho_2(H)$. It follows that $c_pp + 2l \equiv 0 \pmod{p^2}$, a contradiction. This completes the proof of our lemma. □

Lemma 4.7. *If* $H = \left\langle a, b \mid a^{p^{t+1}} = b^{p^{t+1}} = 1, b^{-1}ab = a^{p^t+1} \right\rangle$ $(t > 0)$*, then* $p = 3$*.*

Proof. Suppose to the contrary that $p > 3$. We first define the following four maps. Let

$$\gamma\colon a \mapsto a^{1+p}, b \mapsto b, \qquad \delta\colon a \mapsto a, b \mapsto b^{1+p},$$
$$\sigma\colon a \mapsto b^p a, b \mapsto b, \qquad \tau\colon a \mapsto a, b \mapsto ba.$$

Let $x_1 = a^{1+p}$, $x_2 = x_4 = a$, $x_3 = b^p a$, $y_1 = y_3 = b$, $y_2 = b^{1+p}$ and $y_4 = ba$. Since H is an inner-abelian metacyclic-p group, by Proposition 3.1 and a direct computation, we have $o(x_{i_1}) = o(a) = p^{t+1}$, $o(y_{i_1}) = o(b) = p^t$ and it is direct to check that x_{i_1} and y_{i_1} have the same relations as do a and b, where $i_1 \in \{1, 2, 3, 4\}$. Moreover, for any $i_1 \in \{1, 2, 3, 4\}$, we have $\langle x_{i_1}, y_{i_1} \rangle = H$. It follows that each of the above four maps induces an automorphism of H.

Set $P = \langle \sigma, \gamma, \delta, \tau \rangle$. By a direct computation, we have $o(\gamma) = o(\delta) = p^t$, $o(\sigma) = p^t$ and $o(\tau) = p^{t+1}$. Moreover, we have $\gamma\delta = \delta\gamma$, $\delta^{-1}\sigma\delta = \sigma^{p+1}$ and $\gamma^{-1}\sigma\gamma = \sigma^{\ell}$ with $\ell(p+1) \equiv 1 \pmod{p^{t+1}}$. As both γ and δ fixes the subgroup $\langle a \rangle$ while σ does not, one has

$$\langle \sigma, \gamma, \delta \rangle = \langle \sigma \rangle \rtimes (\langle \gamma \rangle \times \langle \delta \rangle) \cong \mathbb{Z}_{p^t} \rtimes (\mathbb{Z}_{p^t} \times \mathbb{Z}_{p^t}).$$

Observing that $\langle \sigma, \gamma, \delta \rangle$ fixes the subgroup $\langle b \rangle$ setwise but τ does not, it follows that $\langle \sigma, \gamma, \delta \rangle \cap \langle \tau \rangle = 1$, and hence $|P| \geq p^{4t+1}$. In view of [13, Theorem 2.8], $\mathrm{Aut}(H)$ has a normal Sylow p-subgroup of order p^{4t+1}. It follows that $P = \langle \sigma, \gamma, \delta, \tau \rangle$ is the unique Sylow p-subgroup of $\mathrm{Aut}(H)$. In particular, we have $P = \langle \gamma \rangle \langle \delta \rangle \langle \sigma \rangle \langle \tau \rangle$.

Recall that $S = \{1, h, hh^{\alpha}, \ldots, hh^{\alpha} \cdots h^{\alpha^{p-2}}\}$ and $o(\alpha) = p$. Assume that $h = b^u a^v$ for some $u \in \mathbb{Z}_{p^{t+1}}$ and $v \in \mathbb{Z}_{p^{t+1}}$. Since $H = \langle S \rangle$, we obtain that $o(h) = \exp(H)$. It follows that $(u, p) = 1$. Then there exists $u' \in \mathbb{Z}^*_{p^{t+1}}$ such that $u \equiv u'v \pmod{p^{t+1}}$. Let $\varphi = \sigma^{u'}(\delta^u)^{-1}(\tau^v)^{-1}$. Then $\varphi \in \mathrm{Aut}(H)$ and $h^{\varphi} = b$. By Proposition 2.4(3), we have $\Gamma \cong \mathrm{BiCay}(H, \emptyset, \emptyset, S^{\varphi})$. Let $\Gamma' = \mathrm{BiCay}(H, \emptyset, \emptyset, S^{\varphi})$ and $\beta = \varphi^{-1}\alpha\varphi$. Then $\sigma_{\beta,b} \in \mathrm{Aut}(\Gamma')$ cyclically permutates the elements in $\Gamma'(1_0)$. It follows that $bb^{\beta}b^{\beta^2} \cdots b^{\beta^{p-1}} = 1$ and

$$S^{\varphi} = \{1, b, bb^{\beta}, bb^{\beta}b^{\beta^2}, \ldots, bb^{\beta}b^{\beta^2} \cdots b^{\beta^{p-2}}\}.$$

Since $o(\beta) = o(\alpha) = p$, we have $\beta \in P$. Assume that $\beta = \gamma^i\delta^j\sigma^k\tau^l$ for some $i, j, k \in \mathbb{Z}_{p^t}$ and $l \in \mathbb{Z}_{p^{t+1}}$. Then by Lemma 3.2(2)–(3) and Proposition 3.1(1), we have

$$\beta\colon \begin{cases} a \mapsto (ba^l)^{(1+p)^i kp} a^{(1+p)^i} = b^{(1+p)^i kp} a^{(1+p)^i(1+klp)} \\ b \mapsto (ba^l)^{(1+p)^j} = b^{(1+p)^j} a^{(1+p)^j l} \end{cases} \tag{4.7}$$

and then

$$\beta\colon \begin{cases} a \mapsto a \cdot w \\ b \mapsto ba^l \cdot w' \end{cases} \tag{4.8}$$

for some $w, w' \in \mho_1(H)$. Since $\Gamma' \cong \Gamma$ is connected, we derive from Proposition 2.3 that $H = \langle S^{\varphi} \rangle$. By Proposition 3.1(1), it follows that $(l, p) = 1$. We shall finish the proof by the following steps.

Step 1: $t > 1$.

Suppose to the contrary that $t = 1$. Then $H = \langle a, b \mid a^{p^2} = b^{p^2} = 1, b^{-1}ab = a^{1+p}\rangle$. We shall first show that for any $r \geq 1$,

$$b^{\beta^r} = b^{1+(rj+\frac{1}{2}r(r-1)kl)p}a^{rl+\frac{1}{2}r(r+1)jlp+\frac{1}{2}r(r-1)(i+kl)lp+\frac{1}{6}r(r-1)(r-2)kl^2p}. \tag{4.9}$$

By Equation (4.7), we have

$$\beta\colon \begin{cases} a \mapsto b^{kp}a^{1+(i+kl)p} \\ b \mapsto b^{1+jp}a^{l+jlp}. \end{cases}$$

Thus Equation (4.9) holds when $r = 1$. Now assume that $r > 1$ and

$$b^{\beta^{r-1}} = b^{1+((r-1)j+\frac{1}{2}(r-1)(r-2)kl)p} \cdot a^{(r-1)l+\frac{1}{2}(r-1)rjlp+\frac{1}{2}(r-1)(r-2)(i+kl)lp+\frac{1}{6}(r-1)(r-2)(r-3)kl^2p}.$$

By a direct computation, we have

$$\begin{aligned} b^{\beta^r} &= (b^{1+jp}a^{l+jlp})^{1+((r-1)j+\frac{1}{2}(r-1)(r-2)kl)p} \\ &\quad \cdot (b^{kp}a^{1+(i+kl)p})^{(r-1)l+\frac{1}{2}(r-1)rjlp+\frac{1}{2}(r-1)(r-2)(i+kl)lp+\frac{1}{6}(r-1)(r-2)(r-3)kl^2p} \\ &= b^{1+(rj+\frac{1}{2}(r-1)(r-2)kl+(r-1)kl)p} \cdot a^{l+jlp+[(r-1)lj+\frac{1}{2}(r-1)(r-2)kl^2]p} \\ &\quad \cdot a^{(r-1)l(1+(i+kl)p)+\frac{1}{2}(r-1)rjlp+\frac{1}{2}(r-1)(r-2)(i+kl)lp+\frac{1}{6}(r-1)(r-2)(r-3)kl^2p} \\ &= b^{1+rjp+[\frac{1}{2}(r-1)(r-2)+(r-1)]klp} \\ &\quad \cdot a^{[l+(r-1)l]+[1+(r-1)+\frac{1}{2}(r-1)r]jlp+\frac{1}{2}r(r-1)(i+kl)lp+[\frac{1}{2}+\frac{1}{6}(r-3)](r-1)(r-2)kl^2p} \\ &= b^{1+(rj+\frac{1}{2}r(r-1)kl)p}a^{rl+\frac{1}{2}r(r+1)jlp+\frac{1}{2}r(r-1)(i+kl)lp+\frac{1}{6}r(r-1)(r-2)kl^2p}. \end{aligned}$$

By induction, we have Equation (4.9). Then by Equation (4.9), we have

$$b^{\beta^p} = b^{1+(pj+\frac{1}{2}p(p-1)kl)p}a^{pl+\frac{1}{2}p(p+1)jlp+\frac{1}{2}p(p-1)(i+kl)lp+\frac{1}{6}p(p-1)(p-2)kl^2p} = ba^{pl} \neq b,$$

a contradiction.

Step 2: A final contradiction.

Let $\mho_2(H) = \{x^{p^2} \mid x \in H\}$. Then $\mho_2(H) \leq Z(H)$. By Equation (4.7), we have

$$\begin{aligned} a^\beta &= b^{kp}a^{(i+kl)p+1} \cdot \varpi' \\ b^\beta &= b^{jp+1}a^{jlp+l} \cdot \varpi \end{aligned}$$

for some $\varpi, \varpi' \in \mho_2(H)$. Let $f \equiv i + kl \pmod p$, $n \equiv j \pmod p$, $m \equiv jl \pmod p$ for some $m, n, f \in \mathbb{Z}_p$. Then

$$\beta\colon \begin{cases} a \mapsto b^{kp}a^{fp+1} \cdot \varpi_1' \\ b \mapsto b^{np+1}a^{mp+l} \cdot \varpi_1 \end{cases} \tag{4.10}$$

for some $\varpi_1, \varpi_1' \in \mho_2(H)$.

We shall first prove the following claim.

Claim. *For any* $r \geq 1$, $b^{\beta^r} = b^{rnp+\frac{r(r-1)}{2}klp+1}a^{rmp+\frac{r(r-1)}{2}(n+f)lp+\frac{r(r-1)(r-2)}{6}kl^2p+rl}\cdot\varpi_r$ *with* $\varpi_r \in \mho_2(H)$.

If $r = 1$, then by Equation (4.10), Claim is clearly true. Now assume that $r > 1$ and Claim holds for any positive integer less than r. Then

$$b^{\beta^{r-1}} = b^{(r-1)np+\frac{(r-1)(r-2)}{2}klp+1} \cdot a^{(r-1)mp+\frac{(r-1)(r-2)}{2}(n+f)lp+\frac{(r-1)(r-2)(r-3)}{6}kl^2p+(r-1)l}\cdot\varpi_{r-1},$$

for some $\varpi_{r-1} \in \mho_2(H)$. Since $t > 1$, for any positive integer i_0, by Lemma 3.2(1)&(3), we have

$$ab^{i_0} = b^{i_0}a^{(1+p^t)^{i_0}} = b^{i_0}a^{1+i_0p^t} = b^{i_0}a\cdot\varpi_0, \tag{4.11}$$

for some $\varpi_0 \in \mho_2(H)$. Then by Equations (4.10) and (4.11), we have

$$\begin{aligned} b^{\beta^r} &= (b^{np+1}a^{mp+l}\cdot\varpi_1)^{(r-1)np+\frac{(r-1)(r-2)}{2}klp+1} \\ &\quad\cdot(b^{kp}a^{fp+1}\cdot\varpi_1')^{(r-1)mp+\frac{(r-1)(r-2)}{2}(n+f)lp+\frac{(r-1)(r-2)(r-3)}{6}kl^2p+(r-1)l}\cdot\varpi_{r-1}^{\beta} \\ &= b^{(r-1)np+\frac{(r-1)(r-2)}{2}klp+np+1+k(r-1)lp}\cdot\varpi_r\cdot a^{(r-1)nlp+\frac{(r-1)(r-2)}{2}kl^2p} \\ &\quad\cdot a^{mp+l+(r-1)mp+\frac{(r-1)(r-2)}{2}(n+f)lp+\frac{(r-1)(r-2)(r-3)}{6}kl^2p+(r-1)l+(r-1)flp} \\ &= b^{rnp+\frac{r(r-1)}{2}klp+1}\cdot a^{rmp+\frac{r(r-1)}{2}(n+f)lp+\frac{r(r-1)(r-2)}{6}kl^2p+rl}\cdot\varpi_r. \end{aligned}$$

for some $\varpi_r \in \mho_2(H)$. By induction, we complete the proof of Claim.

Now by our Claim and $o(\beta) = p$, we have

$$b^{\beta^p} = b^{np^2+\frac{(p-1)}{2}klp^2+1}\cdot a^{mp^2+\frac{(p-1)}{2}(n+f)lp^2+\frac{(p-1)(p-2)}{6}kl^2p^2+pl}\cdot\varpi_p = b$$

for some $\varpi_p \in \mho_2(H)$. It follows that $pl \equiv 0 \pmod{p^2}$, a contradiction. This completes the proof of our lemma. □

5 Proof of Theorem 1.3

We first prove a lemma.

Lemma 5.1. *Let p be an odd prime, and let H be a metacyclic p-group. If Γ is a connected edge-transitive bi-Cayley graph over H of valency p, then H is either abelian or inner-abelian.*

Proof. We may assume that H is non-abelian. By Proposition 3.1, the group H has the following presentation:

$$H = \left\langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, a^b = a^{1+p^r}\right\rangle,$$

where r, s, t, u are non-negative integers with $u \leq r$ and $r \geq 1$.

Let $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ be a connected edge-transitive bi-p-Cayley graph over H of valency p. Let $A = \mathrm{Aut}(\Gamma)$, and let P be a Sylow p-subgroup of A such that

$R(H) \leq P$. From the proof of Lemma 4.4(1), we see that P is transitive on the edges of Γ. Since $H' = \langle a^{p^r} \rangle \cong \mathbb{Z}_{p^{s+u}}$, we have

$$H/H' = \left\langle \overline{a}, \overline{b} \mid \overline{a}^{p^r} = \overline{b}^{p^{r+s+t}} = 1, \overline{a}^{\overline{b}} = \overline{a} \right\rangle \cong \mathbb{Z}_{p^r} \times \mathbb{Z}_{p^{r+s+t}},$$

where $\overline{a} = aH'$ and $\overline{b} = bH'$. By Lemma 4.4(2), we have $s + t = 0$ or 1, and so $(s,t) = (0,0), (1,0)$ or $(0,1)$.

Let $n = 2r+2s+u+t$. We use induction on n. If $n = 1$ or 2, then H is clearly abelian, as desired. Assume $n \geq 3$. Let N be a minimal normal subgroup of P and $N \leq R(H)$. Since H is metacyclic, we have $N \cong \mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. Suppose that $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Note that $R(H)' \cong \mathbb{Z}_{p^{s+u}}$. Let Q be the subgroup of $R(H)'$ of order p. Since Q is characteristic in $R(H)'$ and $R(H)'$ is characteristic in $R(H)$, $R(H) \trianglelefteq P$ gives that $Q \trianglelefteq P$. By Lemma 3.2(4), each subgroup of $R(H)$ of order p is contained in N. It follows that $Q < N$, contrary to the minimality of N. Thus $N \cong \mathbb{Z}_p$.

Consider the quotient graph Γ_N of Γ corresponding to the orbits of N. Clearly, N is intransitive on both H_0 and H_1, the two orbits of $R(H)$ on $V(\Gamma)$, and by Propositions 2.1 and 2.2, N is semiregular and Γ_N is a graph of valency p with P/N as an edge-transitive group of automorphisms. Clearly, Γ_N is a bi-Cayley graph over the group $R(H)/N$ of order $2 \cdot p^{n_1}$ with $n_1 < n$. By induction, we have $R(H)/N$ is either abelian or inner-abelian. If $R(H)/N$ is abelian, then $R(H)' \leq N \cong \mathbb{Z}_p$. It follows that $R(H)' = 1$ or $R(H)' \cong \mathbb{Z}_p$, implying that $H \cong R(H)$ is abelian or inner-abelian, as required.

In what follows, we always assume that $R(H)/N$ is inner-abelian, and for any element $h \in H$, we use $\overline{h}$ to denote hN.

By Theorem 4.2, we have $p = 3$. Recall that $(s,t) = (0,0), (1,0)$ or $(0,1)$.

Case 1: $(s,t) = (0,0)$.

In this case, we have

$$H = \left\langle a, b \mid a^{3^{r+u}} = 1, b^{3^r} = a^{3^r}, a^b = a^{1+3^r} \right\rangle.$$

Let $x = a$ and $y = ba^{-1}$. Since $b^{3^r} = a^{3^r}$, by Proposition 3.1(2), we conclude that $y^{3^r} = (ba^{-1})^{3^r} = 1$ and

$$x^y = a^{ba^{-1}} = (a^b)^{a^{-1}} = (a^{1+3^r})^{a^{-1}} = a^{1+3^r} = x^{1+3^r}.$$

Then

$$R(H) \cong H = \left\langle x, y \mid x^{3^{r+u}} = y^{3^r} = 1, x^y = x^{1+3^r} \right\rangle.$$

Recall that $N \cong \mathbb{Z}_3$ and $N \leq R(H)$. By Lemma 3.2(4), N is one of the following four groups: $\langle x^{3^{r+u-1}} \rangle$, $\langle y^{3^{r-1}} \rangle$, $\langle y^{3^{r-1}} x^{3^{r+u-1}} \rangle$, $\langle y^{3^{r-1}} x^{2 \cdot 3^{r+u-1}} \rangle$.

First suppose that $N \neq \langle x^{3^{r+u-1}} \rangle$. Then $\overline{x}$ has order 3^{r+u}. We shall show that H/N has the following presentation:

$$H/N = \left\langle \overline{x}, \overline{h} \mid \overline{x}^{3^{r+u}} = \overline{h}^{3^{r-1}} = \overline{1}, \overline{x}^{\overline{h}} = \overline{x}^{1+3^r} \right\rangle.$$

Actually, if $N = \langle y^{3^{r-1}} \rangle$, then we may take $h = y$. If $N = \langle y^{3^{r-1}} x^{3^{r+u-1}} \rangle$, then take $h = yx^{3^u}$, and then by Lemma 3.2(2) – (3), we have

$$\begin{aligned}(yx^{3^u})^{3^{r-1}} &= y^{3^{r-1}} x^{3^u[1+(1+3^r)+(1+3^r)^2+\cdots+(1+3^r)^{3^{r-1}-1}]} \\ &= y^{3^{r-1}} x^{3^u[1+(1+3^r)+(1+2\cdot 3^r)+\cdots+(1+(3^{r-1}-1)\cdot 3^r)]} \\ &= y^{3^{r-1}} x^{3^u \cdot 3^{r-1}} \\ &= y^{3^{r-1}} x^{3^{u+r-1}} \in N.\end{aligned}$$

If $N = \langle y^{3^{r-1}} x^{2\cdot 3^{r+u-1}} \rangle$, then take $h = yx^{2\cdot 3^u}$, and then by Lemma 3.2(2) – (3), we have

$$\begin{aligned}(yx^{2\cdot 3^u})^{3^{r-1}} &= y^{3^{r-1}} x^{2\cdot 3^u[1+(1+3^r)+(1+3^r)^2+\cdots+(1+3^r)^{3^{r-1}-1}]} \\ &= y^{3^{r-1}} x^{2\cdot 3^u[1+(1+3^r)+(1+2\cdot 3^r)+\cdots+(1+(3^{r-1}-1)\cdot 3^r)]} \\ &= y^{3^{r-1}} x^{2\cdot 3^u \cdot 3^{r-1}} \\ &= y^{3^{r-1}} x^{2\cdot 3^{u+r-1}} \in N.\end{aligned}$$

Clearly, in each case, we have $\overline{x}^{\overline{h}} = \overline{x}^{1+3^r}$. So H/N always has the above presentation. Since $R(H)/N$ is inner-abelian, by [20] or [3, Lemma 65.2], we have $u = 1$. However, by Proposition 4.1, there is no cubic edge-transitive bi-Cayley graph over $R(H)/N$, a contradiction.

Suppose now that $N = \langle x^{3^{r+u-1}} \rangle$. Then

$$H/N = \left\langle \overline{x}, \overline{y} \mid \overline{x}^{3^{r+u-1}} = \overline{y}^{3^r} = \overline{1}, \overline{x}^{\overline{y}} = \overline{x}^{1+3^r} \right\rangle,$$

Since $R(H)/N$ is inner-abelian, by [20] or [3, Lemma 65.2], we have $u = 2$. Then

$$H = \left\langle x, y \mid x^{3^{r+2}} = y^{3^r} = 1, x^y = x^{1+3^r} \right\rangle,$$

where $r \geq 1$.

If $r = 1$, then by MAGMA [5], there is no cubic edge-transitive bi-Cayley graph over H, a contradiction. If $r \geq 2$, then by Lemma 4.4(1), we have $\mathcal{R} = \mathcal{L} = \emptyset$. Assume that $S = \{1, g, h\}$. Since Γ is connected, by Proposition 2.3(1), we have $H = \langle S \rangle = \langle g, h \rangle$. It follows that $o(g) = o(h) = \exp(H) = 3^{r+2}$, and so $H' = \langle x^{3^r} \rangle = \langle g^{3^r} \rangle = \langle h^{3^r} \rangle$. Moreover, by Lemma 4.4(1), there exists $\alpha \in \mathrm{Aut}(H)$ such that $g^\alpha = g^{-1}h$, $h^\alpha = g^{-1}$ and $o(\alpha) \mid 3$. Suppose that α is trivial. Then $h = g^{-1}$, and then $H = \langle g \rangle$, a contradiction. Thus, α has order 3. Assume that $(g^{3^r})^\alpha = g^{\lambda\cdot 3^r}$ for some $\lambda \in \mathbb{Z}_9^*$. Then $(h^{3^r})^\alpha = h^{\lambda\cdot 3^r}$. Since $g^\alpha = g^{-1}h$ and $h^\alpha = g^{-1}$, we have $g^{\lambda\cdot 3^r} = g^{-3^r}h^{3^r}$ and $h^{\lambda\cdot 3^r} = g^{-3^r}$. Then

$$g^{\lambda^2\cdot 3^r} = (g^{\lambda\cdot 3^r})^\lambda = (g^{-3^r}h^{3^r})^\lambda = g^{-\lambda\cdot 3^r}h^{\lambda\cdot 3^r} = g^{-\lambda\cdot 3^r}g^{-3^r} = g^{(-\lambda-1)\cdot 3^r}.$$

It follows that $g^{(\lambda^2+\lambda+1)\cdot 3^r} = 1$, and so $9 \mid \lambda^2 + \lambda + 1$, a contradiction.

Case 2: $(s, t) = (1, 0)$.

In this case, we have

$$H = \left\langle a, b \mid a^{3^{r+u+1}} = 1, b^{3^{r+1}} = a^{3^{r+1}}, a^b = a^{1+3^r} \right\rangle.$$

Let $x = a$ and $y = ba^{-1}$. Since $b^{3^{r+1}} = a^{3^{r+1}}$, by Proposition 3.1(2), we obtain that $y^{3^{r+1}} = (ba^{-1})^{3^{r+1}} = 1$ and

$$x^y = a^{ba^{-1}} = (a^b)^{a^{-1}} = (a^{1+3^r})^{a^{-1}} = a^{1+3^r} = x^{1+3^r}.$$

Then

$$R(H) \cong H = \left\langle x, y \mid x^{3^{r+u+1}} = y^{3^{r+1}} = 1, x^y = x^{1+3^r} \right\rangle,$$

Recall that $N \cong \mathbb{Z}_3$ and $N \leq R(H)$. By Lemma 3.2(4), N is one of the following four groups: $\langle x^{3^{r+u}} \rangle$, $\langle y^{3^r} \rangle$, $\langle y^{3^r} x^{3^{r+u}} \rangle$, $\langle y^{3^r} x^{2\cdot 3^{r+u}} \rangle$.

Suppose first that $N \neq \langle x^{3^{r+u}} \rangle$. Then $\overline{x}$ has order 3^{r+u+1}. We shall show that H/N has the following presentation:

$$H/N = \left\langle \overline{x}, \overline{h} \mid \overline{x}^{3^{r+u+1}} = \overline{h}^{3^r} = \overline{1}, \overline{x}^{\overline{h}} = \overline{x}^{1+3^r} \right\rangle.$$

Actually, if $N = \langle y^{3^r} \rangle$, then we may take $h = y$. If $N = \langle y^{3^r} x^{3^{r+u}} \rangle$, then take $h = yx^{3^u}$, and then by Lemma 3.2(2) – (3), we have

$$\begin{aligned}
(yx^{3^u})^{3^r} &= y^{3^r} x^{3^u[1+(1+3^r)+(1+3^r)^2+\cdots+(1+3^r)^{3^r-1}]} \\
&= y^{3^r} x^{3^u[1+(1+3^r)+(1+2\cdot 3^r)+\cdots+(1+(3^r-1)\cdot 3^r)]} \\
&= y^{3^r} x^{3^u[3^r+\frac{3^r\cdot(3^r-1)}{2}\cdot 3^r]} \\
&= y^{3^r} x^{3^{u+r}} \in N.
\end{aligned}$$

If $N = \langle y^{3^r} x^{2\cdot 3^{r+u}} \rangle$, then take $h = yx^{2\cdot 3^u}$, and then by Lemma 3.2(2) – (3), we have

$$\begin{aligned}
(yx^{2\cdot 3^u})^{3^r} &= y^{3^r} x^{2\cdot 3^u[1+(1+3^r)+(1+3^r)^2+\cdots+(1+3^r)^{3^r-1}]} \\
&= y^{3^r} x^{2\cdot 3^u[1+(1+3^r)+(1+2\cdot 3^r)+\cdots+(1+(3^r-1)\cdot 3^r)]} \\
&= y^{3^r} x^{2\cdot 3^u[3^r+\frac{3^r\cdot(3^r-1)}{2}\cdot 3^r]} \\
&= y^{3^r} x^{2\cdot 3^{u+r}} \in N.
\end{aligned}$$

Clearly, in each case, we have $\overline{x}^{\overline{h}} = \overline{x}^{1+3^r}$. So H/N always has the above presentation. Since $R(H)/N$ is inner-abelian, by [20] or [3, Lemma 65.2], we have $u = 0$. Then

$$H = \left\langle x, y \mid x^{3^{r+1}} = y^{3^{r+1}} = 1, x^y = x^{1+3^r} \right\rangle,$$

where $r \geq 1$. By [20] or [3, Lemma 65.2], H is inner-abelian, as required.

Suppose now $N = \langle x^{3^{r+u}} \rangle$. Then

$$R(H)/N = \left\langle \overline{x}, \overline{y} \mid \overline{x}^{3^{r+u}} = \overline{y}^{3^{r+1}} = \overline{1}, \overline{x}^{\overline{y}} = \overline{x}^{1+3^r} \right\rangle.$$

Since $R(H)/N$ is inner-abelian, by [20] or [3, Lemma 65.2], we have $u = 1$. Then

$$H = \left\langle x, y \mid x^{3^{r+2}} = y^{3^{r+1}} = 1, x^y = x^{1+3^r} \right\rangle,$$

where $r \geq 1$.

If $r = 1$, then by MAGMA [5], there is no cubic edge-transitive bi-Cayley graph over H, a contradiction. If $r \geq 2$, then by Lemma 4.4(1), we have $\mathcal{R} = \mathcal{L} = \emptyset$. Assume that $S = \{1, g, h\}$. Since Γ is connected, by Proposition 2.3(1), we have $H = \langle S \rangle = \langle g, h \rangle$. It follows that $o(g) = o(h) = \exp(H) = 3^{r+2}$. By Lemma 4.4(1), there exists $\alpha \in \mathrm{Aut}(H)$ such that $g^\alpha = g^{-1}h$, $h^\alpha = g^{-1}$ and $o(\alpha) \mid 3$. Suppose that α is trivial. Then $h = g^{-1}$, and then $H = \langle g \rangle$, a contradiction. Thus, α has order 3. Note that

$$\Omega_r(H) = \left\langle z^{3^r} \mid z \in H \right\rangle = \left\langle x^{3^r} \right\rangle \times \left\langle y^{3^r} \right\rangle \cong \mathbb{Z}_9 \times \mathbb{Z}_3$$

and $g^{3^r}, h^{3^r} \in \Omega_r(H)$.

If $\langle g^{3^r} \rangle = \langle h^{3^r} \rangle$, then we may assume that $(g^{3^r})^\alpha = g^{\lambda \cdot 3^r}$ for some $\lambda \in \mathbb{Z}_9^*$. Then $(h^{3^r})^\alpha = h^{\lambda \cdot 3^r}$. Since $g^\alpha = g^{-1}h$ and $h^\alpha = g^{-1}$, we have $g^{\lambda \cdot 3^r} = g^{-3^r} h^{3^r}$ and $h^{\lambda \cdot 3^r} = g^{-3^r}$. Then

$$g^{\lambda^2 \cdot 3^r} = (g^{\lambda \cdot 3^r})^\lambda = (g^{-3^r} h^{3^r})^\lambda = g^{-\lambda \cdot 3^r} h^{\lambda \cdot 3^r} = g^{-\lambda \cdot 3^r} g^{-3^r} = g^{(-\lambda - 1) \cdot 3^r}.$$

It follows that $g^{(\lambda^2 + \lambda + 1) \cdot 3^r} = 1$, and so $9 \mid \lambda^2 + \lambda + 1$, a contradiction.

Suppose $\langle g^{3^r} \rangle \neq \langle h^{3^r} \rangle$. Then $\Omega_r(H) = \langle g^{3^r}, h^{3^r} \rangle$ and $H' = \langle x^{3^r} \rangle \cong \mathbb{Z}_9$. Assume that $x^{3^r} = g^{i \cdot 3^r} h^{j \cdot 3^r}$ for some $i, j \in \mathbb{Z}_9$. Then either $(i, 3) = 1$ or $(j, 3) = 1$. Since $H' = \langle x^{3^r} \rangle$, we have $\langle x^{3^r} \rangle^\alpha = \langle x^{3^r} \rangle$. So $(g^{i \cdot 3^r} h^{j \cdot 3^r})^\alpha = (g^{i \cdot 3^r} h^{j \cdot 3^r})^k$ for some $k \in \mathbb{Z}_9$. Then

$$g^{ik \cdot 3^r} h^{jk \cdot 3^r} = (g^{i \cdot 3^r} h^{j \cdot 3^r})^\alpha = (g^\alpha)^{i \cdot 3^r} (h^\alpha)^{j \cdot 3^r} = g^{-i \cdot 3^r} h^{i \cdot 3^r} g^{-j \cdot 3^r} = g^{-(i+j) \cdot 3^r} h^{i \cdot 3^r}.$$

It follows that $-(i + j) \equiv ik \pmod 9$ and $i \equiv jk \pmod 9$. Then $-(jk + j) \equiv jk^2 \pmod 9$, and so $j(1 + k + k^2) \equiv 0 \pmod 9$, forcing that $3 \mid j$. Furthermore, since $i \equiv jk \pmod 9$, we have $3 \mid i$, a contradiction.

Case 3: $(s, t) = (0, 1)$.

In this case, we have

$$H = \left\langle a, b \mid a^{3^{r+u}} = 1, b^{3^{r+1}} = a^{3^r}, a^b = a^{1+3^r} \right\rangle.$$

Let $x = b, y = b^3 a^{-1}$. Since $a^b = a^{1+3^r}$, we have $b^{-1}aba^{-1} = a^{3^r}$, and then

$$aba^{-1} = ba^{3^r} = bb^{3^{r+1}} = b^{1+3^{r+1}}.$$

Since $b^{3^{r+1}} = a^{3^r}$, by Proposition 3.1(2), we have

$$x^{3^{r+u+1}} = b^{3^{r+u+1}} = a^{3^{r+u}} = 1, \qquad y^{3^r} = (b^3 a^{-1})^{3^r} = 1,$$
$$x^y = b^{b^3 a^{-1}} = (b)^{a^{-1}} = aba^{-1} = b^{1+3^{r+1}} = x^{1+3^{r+1}}.$$

Then

$$R(H) \cong H = \left\langle x, y \mid x^{3^{r+u+1}} = y^{3^r} = 1, x^y = x^{1+3^{r+1}} \right\rangle.$$

Recall that $N \cong \mathbb{Z}_3$ and $N \leq R(H)$. By Lemma 3.2(4), N is one of the following four groups: $\langle x^{3^{r+u}} \rangle$, $\langle y^{3^{r-1}} \rangle$, $\langle y^{3^{r-1}} x^{3^{r+u}} \rangle$, $\langle y^{3^{r-1}} x^{2 \cdot 3^{r+u}} \rangle$.

Suppose first that $N \neq \langle x^{3^{r+u}} \rangle$. Then $\overline{x}$ has order 3^{r+u+1}. We shall show that H/N has the following presentation:

$$H/N = \left\langle \overline{x}, \overline{h} \mid \overline{x}^{3^{r+u+1}} = \overline{h}^{3^{r-1}} = \overline{1}, \overline{x}^{\overline{h}} = \overline{x}^{1+3^{r+1}} \right\rangle.$$

Actually, if $N = \langle y^{3^{r-1}} \rangle$, then we may take $h = y$. If $N = \langle y^{3^{r-1}} x^{3^{r+u}} \rangle$, then take $h = yx^{3^{u+1}}$, and then by Lemma 3.2(2)–(3), we have

$$\begin{aligned}
(yx^{3^{u+1}})^{3^{r-1}} &= y^{3^{r-1}} x^{3^{u+1}[1+(1+3^{r+1})+(1+3^{r+1})^2+\cdots+(1+3^{r+1})^{3^{r-1}-1}]} \\
&= y^{3^{r-1}} x^{3^{u+1}[1+(1+3^{r+1})+(1+2\cdot 3^{r+1})+\cdots+(1+(3^{r-1}-1)\cdot 3^{r+1})]} \\
&= y^{3^{r-1}} x^{3^{u+1}[3^{r-1}+\frac{3^{r-1}\cdot(3^{r-1}-1)}{2}\cdot 3^{r+1}]} \\
&= y^{3^{r-1}} x^{3^{u+r}} \in N.
\end{aligned}$$

If $N = \langle y^{3^{r-1}} x^{2\cdot 3^{r+u}} \rangle$, then take $h = yx^{2\cdot 3^{u+1}}$, and then by Lemma 3.2(2)–(3), we have

$$\begin{aligned}
(yx^{2\cdot 3^{u+1}})^{3^{r-1}} &= y^{3^{r-1}} x^{2\cdot 3^{u+1}[1+(1+3^{r+1})+(1+3^{r+1})^2+\cdots+(1+3^{r+1})^{3^{r-1}-1}]} \\
&= y^{3^{r-1}} x^{2\cdot 3^{u+1}[1+(1+3^{r+1})+(1+2\cdot 3^{r+1})+\cdots+(1+(3^{r-1}-1)\cdot 3^{r+1})]} \\
&= y^{3^{r-1}} x^{2\cdot 3^{u+1}[3^{r-1}+\frac{3^{r-1}\cdot(3^{r-1}-1)}{2}\cdot 3^{r+1}]} \\
&= y^{3^{r-1}} x^{2\cdot 3^{u+r}} \in N.
\end{aligned}$$

Clearly, in each case, we have $\overline{x}^{\overline{h}} = \overline{x}^{1+3^r}$. So H/N always has the above presentation. Since $R(H)/N$ is inner-abelian, by [20] or [3, Lemma 65.2], we have $u = 1$. However, by Proposition 4.1, there is no cubic edge-transitive bi-Cayley graph over $R(H)/N$, a contradiction.

Suppose now that $N = \langle x^{3^{r+u}} \rangle$. Then

$$R(H)/N = \left\langle \overline{x}, \overline{y} \mid \overline{x}^{3^{r+u}} = \overline{y}^{3^r} = \overline{1}, \overline{x}^{\overline{y}} = \overline{x}^{1+3^{r+1}} \right\rangle.$$

Since $R(H)/N$ is inner-abelian, by [20] or [3, Lemma 65.2], we have $u = 2$. However, by Proposition 4.1, there is no cubic edge-transitive bi-Cayley graph over $R(H)/N$, a contradiction. □

Now we are ready to finish the proof of Theorem 1.3.

Proof of Theorem 1.3. By Lemma 5.1, if H is non-abelian, then H is inner-abelian. By Theorem 4.2, we have $p = 3$, and then by Proposition 4.1, Γ is isomorphic to either Γ_r or Σ_r, as desired. □

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