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The validity of Tutte's 3-flow conjecture for some Cayley graphs*

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Abstract

Tutte's 3-flow conjecture claims that every bridgeless graph with no 3-edge-cut admits a nowhere-zero 3-flow. In this paper we verify the validity of Tutte's 3-flow conjecture on Cayley graphs of certain classes of finite groups. In particular, we show that every Cayley graph of valency at least 4 on a generalized dicyclic group has a nowhere-zero 3-flow. We also show that if G is a solvable group with a cyclic Sylow 2-subgroup and the connection sequence S with $|S| \geq 4$ contains a central generator element, then the corresponding Cayley graph $\mathrm{Cay}(G, S)$ admits a nowhere-zero 3-flow.

Keywords: Nowhere-zero flow, Cayley graph, Tutte's 3-flow conjecture, connection sequence, solvable group, nilpotent group.

Math. Subj. Class.: 05C25, 05C21, 20D10

1 Introduction

Let D be an orientation of a graph Γ and let k be a positive integer. A *k-flow* on a graph Γ is a pair (D, f) where f is an integer valued function

$$f\colon E(\Gamma) \to \mathbb{Z}$$

such that $|f(e)| < k$ for every $e \in E(\Gamma)$, and for every $v \in V(\Gamma)$,

$$\sum_{e\in E(v)^+} f(e) = \sum_{e\in E(v)^-} f(e),$$

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where $E(v)^+$ and $E(v)^-$ are the all edges with tails at v and heads at v, respectively. A *nowhere-zero k-flow* (abbreviated a k-NZF) is a pair (D, f) such that for every $e \in E(\Gamma)$, $f(e) \neq 0$.

The following conjecture is due to Tutte and is known as Tutte's 3-flow conjecture:

Conjecture 1.1 (Tutte's 3-flow conjecture [8, 9])**.** *Every bridgeless graph with no* 3*-edge-cut has a* 3*-NZF.*

Although Tutte's 3-flow conjecture has been studied by many authors, it is still widely open.

Let G be a finite group with identity 1 and $S = (s_1, s_2, \dots, s_n)$ be a sequence of elements of $G \setminus \{1\}$ such that the mapping $s_i \to s_i^{-1}$ permutes the entries of S. We call S a *connection sequence* (note that all entries of S are distinct unless stated otherwise). A *Cayley graph*, denoted by $\mathrm{Cay}(G, S)$, is a graph whose vertex set is G with adjacency defined by

$$g \sim h \quad \text{if and only if} \quad g^{-1}h \in S,$$

for every $g, h \in G$. We see at once that if S generates G, then $\mathrm{Cay}(G, S)$ is connected.

Alspach et al. [1] conjectured that every Cayley graph of valency at least 3 has a nowhere-zero 4-flow. They also showed their conjecture to be true for solvable groups. Their result was significantly strengthened and extended by Nedela and Škoviera to a much wider class of groups [5].

By combining the fact that a k-valent Cayley graph is k-edge-connected graph with the fact that every 4-edge-connected graph has a 4-NZF [2], we deduce that every Cayley graph of valency at least 4 has a 4-NZF. Thus the question about the existence of a nowhere-zero 4-flow is interesting only for cubic Cayley graphs. Since 4-regular graphs admit a nowhere-zero 2-flow, the important question about flows on Cayley graphs of valency greater than 3 is whether every Cayley graph of valency at least 5 has a nowhere-zero 3-flow. In other words, it is interesting to verify whether Tutte's 3-flow conjecture holds on such Cayley graphs.

In [6], it has been proved that every abelian Cayley graph of valency k, where $k \geq 4$, admits a 3-NZF. Nánásiová and Škoviera [4] improved the above result to Cayley graphs on a group G whose Sylow 2-subgroup is the direct factor of G, and as a consequence, they showed that every Cayley graph of valency at least 4 on a nilpotent group has a 3-NZF. Recently, Yang and Li [11] showed the same fact for a Cayley graph on a dihedral group, and L. Li and X. Li [3] verified Tutte's 3-flow conjecture for Cayley graphs on generalized dihedral groups and generalized quaternion groups.

In this paper, we investigate Tutte's 3-flow conjecture for Cayley graphs on a solvable group with a suitable normal subgroup (Theorems 3.1 and 3.2 and Remark 3.5) and as a consequence of these theorems, we show that every Cayley graph of valency at least 4 on a generalized dicyclic group satisfies Tutte's 3-flow conjecture. By using Theorem 3.6 we can obtain the results of [3] and [11] by a different method.

In [4], the authors showed that a Cayley graph of valency at least 4 with the connection sequence containing a central involution admits a 3-NZF. In Theorem 3.6, we extend this result to the case when Sylow 2-subgroups of G are cyclic and the connection sequence of G contains a central generator element. As a consequence of this theorem, we show that if a Cayley graph of valency at least 4 on a solvable group G, with a cyclic Sylow 2-subgroup, admits a 3-NZF, then every Cayley graph of valency at least 4 on the direct product of G and a nilpotent group admits a 3-NZF.

2 Notation and preliminaries

The terminology and notation used in this paper are standard both in group theory and graph theory, see for instance [7, 10].

An element g of G is called an *involution* if g has order 2. Let $Z(G)$ be the center of a group G. We say that an element x of G is *central* if $x \in Z(G)$. The group generated by a sequence S is denoted by $\langle S \rangle$ and the element $x \in G$ is named a *generator element* of G in S if $\langle S \setminus \{x\} \rangle \neq \langle S \rangle$. For integers $m, n \geq 2$, a cycle of length n and a path of length $m-1$ are denoted by C_n and P_m, respectively. For an integer $m \geq 3$ and for $n \in \mathbb{Z}_m$, the Cayley graph $\mathrm{Cay}(\mathbb{Z}_m, \{-1, 1, -n, n\})$ will be denoted by $C(m, n)$. Let N be a subgroup of G and x belongs to a left transversal set of N in G. The image of $\mathrm{Cay}(N, S)$ under left translation by x is denoted by $x\,\mathrm{Cay}(N, S)$. The *Cartesian product* $H_1 \square H_2$ of graphs H_1 and H_2 is a graph such that $V(H_1) \times V(H_2)$ is its vertex set and any two vertices (u, u') and (v, v') are adjacent in $H_1 \square H_2$ if and only if either $u = v$ and $u'v' \in E(H_2)$ or $u' = v'$ and $uv \in E(H_1)$. Set $L = P_n \square K_2$, where $V(P_n) = \{1, 2, \ldots, n\}$ and $V(K_2) = \{1, 2\}$. The *Möbius ladder* ML_n is a graph obtained by adding the edges $(12)(n1)$ and $(11)(n2)$ to L. Also, by adding the edges $(11)(n1)$ and $(12)(n2)$ to L, we obtain a graph is called the *circular ladder* CL_n. In fact $CL_n \cong C_n \square K_2$. Any graph isomorphic to either CL_n or ML_n for some n will be referred to as a *closed ladder*. It is easy to check that the circular ladder is bipartite if and only if n is even while the Möbius ladder is bipartite if and only if n is odd.

Lemma 2.1 ([4, Theorems 3.3 and 4.3]). *Let* $\mathrm{Cay}(G, S)$ *be a Cayley graph of valency* k*, where* $k \geq 4$*. If* S *contains a central involution, then* $\mathrm{Cay}(G, S)$ *has a* 3*-NZF. In particular, if* G *is nilpotent, then* $\mathrm{Cay}(G, S)$ *has a* 3*-NZF.*

Lemma 2.2 ([4, Proposition 4.1]). *Let* G *be a group,* H *be a normal subgroup of* G *and let* S *be a connection sequence with no intersection with* H*. If* $\mathrm{Cay}(G/H, S/H)$ *has a* 3*-NZF, then so does* $\mathrm{Cay}(G, S)$*.*

Note that in Lemma 2.2, according to the paragraph before Proposition 4.1 in [4], for distinct elements $s, t \in S$, we regard sH and tH as distinct elements of S/H. So, $\mathrm{Cay}(G/H, S/H)$ may have parallel edges even when $\mathrm{Cay}(G, S)$ is simple and $|S/H| = |S|$.

Lemma 2.3 ([6, Theorem 1.1]). *Every abelian Cayley graph of valency* k*, where* $k \geq 4$*, admits a* 3*-NZF.*

Lemma 2.4 ([6, Proposition 2.5]). *Let* $m, n \geq 3$ *be integers. Then the graph* $C_n \square C_m \square K_2$ *admits a* 3*-NZF.*

Lemma 2.5 ([6, Proposition 2.6]). *Let* $m, n \geq 3$ *be two integers such that* $m > n \geq 1$ *and* $m \geq 3$*. Then the graph* $C(m, n) \square K_2$ *admits a* 3*-NZF.*

Lemma 2.6 ([6, Corollary 2.2]). *A regular bipartite graph of valency at least* 2 *admits a* 3*-NZF.*

Lemma 2.7 ([10, page 308]). *A cubic graph has a* 3*-NZF if and only if it is bipartite.*

Lemma 2.8. *Let* G *be a group and* N *be a subgroup of* G *of index* 2*. Then* $\mathrm{Cay}(G, S \setminus (S \cap N))$ *is bipartite.*

Proof. Since the index of N in G is 2, there exists $d \in G \setminus N$ such that $G = N \cup dN$. So, we can consider the vertices of $\mathrm{Cay}(G, S)$ as two partitions N and dN. Since for every $m, n \in N$, m and n are adjacent, and dm and dn are adjacent if and only if $m^{-1}n \in S \cap N$, we obtain that $\mathrm{Cay}(G, S \setminus S \cap N)$ is a bipartite graph with partite sets N and dN. □

Lemma 2.9 ([10, page 308])**.** *A graph has a 2-NZF if and only if it is an even graph.*

Remark 2.10. According to the above lemma, for discussion about a nowhere-zero 3-flow in a Cayley graph with a connection sequence S, it is enough to investigate the case when $|S|$ is odd.

Remark 2.11. Let G be a group and N be a subgroup of G. Let $T = \{x_1, \dots, x_t\}$, where $t \in \mathbb{N}$, be a left transversal set of N in G. If S is a connection sequence of N such that $\mathrm{Cay}(N, S)$ is connected, then

$$\{x_i \, \mathrm{Cay}(N, S) : 1 \le i \le t\}$$

is the set of connected components of $\mathrm{Cay}(G, S)$. For every x_i where $i \in \{1, \dots, t\}$, $\mathrm{Cay}(N, S)$ and $x_i \, \mathrm{Cay}(N, S)$ are isomorphic, because for every $m, n \in N$,

$$\begin{aligned} x_i m \sim x_i n \quad (\text{in } x_i \, \mathrm{Cay}(N, S \cap N)) \quad &\text{if and only if} \\ (x_i m)^{-1}(x_i n) \in S \cap N \quad &\text{if and only if} \quad m^{-1}n \in S \cap N \\ &\text{if and only if} \quad m \sim n \quad (\text{in } \mathrm{Cay}(N, S \cap N)). \end{aligned}$$

Thus if $\mathrm{Cay}(N, S)$ has a 3-NZF, then $\mathrm{Cay}(G, S)$ has a 3-NZF. Hence for finding a 3-NZF in $\mathrm{Cay}(G, S)$, we reduce to find a 3-NZF in $\mathrm{Cay}(N, S)$.

3 Main results

In this section we show the validity of Tutte's 3-flow conjecture for a solvable group with a suitable normal subgroup. As examples, we show the same result for Cayley graphs on generalized dicyclic groups, generalized dihedral groups and quaternion groups. We also prove that every Cayley graph $\mathrm{Cay}(G, S)$ on a solvable group G with a cyclic Sylow 2-subgroup such that the connection sequence S contains a central generator element, admits a 3-NZF.

Theorem 3.1. *Let G be a solvable group, N be a subgroup of G of index 2 and let S be a connection sequence of G such that $|S| \ge 5$ is odd and $S \cap Z(N) \neq \emptyset$. If*

(1) $\mathrm{Cay}(N, S \cap N)$ *admits a 3-NZF and*

(2) for every $d \in S \setminus N$, $\quad d^{-1}(S \cap N)d = S \cap N$,

then $\mathrm{Cay}(G, S)$ *has a 3-NZF.*

Proof. Without loss of generality, we can assume that there exists an element $d \in S \setminus N$, because otherwise $S \subset N$ and by Condition (1), we could conclude that $\mathrm{Cay}(G, S)$ has a 3-NZF. Thus, there is $d \in S \setminus N$. Note that $|S|$ is odd.

We continue the proof in the following two cases:

Case 1. If $|S \cap N|$ is odd, then since $|S \setminus (S \cap N)| = |S| \setminus |S \cap N|$ is even, Lemma 2.9 shows that $\mathrm{Cay}(G, S \setminus (S \cap N))$ admits a 3-NZF. Also by Condition (1), $\mathrm{Cay}(N, S \cap N)$ admits a 3-NZF, and so does $\mathrm{Cay}(G, S) = \mathrm{Cay}(G, S \setminus (S \cap N)) \cup \mathrm{Cay}(G, S \cap N)$.

Case 2. If $|S \cap N|$ is even, then the proof will be divided into two subcases:

Subcase 1. Assume that $|S\backslash(S\cap N)| \geq 2$. By Lemma 2.8, $\mathrm{Cay}(G, S\backslash(S\cap N))$ is bipartite. So Lemma 2.6 shows that $\mathrm{Cay}(G, S \setminus (S \cap N))$ admits a 3-NZF. Since $\mathrm{Cay}(G, S \cap N)$ admits a 3-NZF, we deduce that $\mathrm{Cay}(G, S)$ has a 3-NZF.

Subcase 2. Assume that $|S \setminus (S \cap N)| = 1$. Thus $\{S \setminus (S \cap N)\} = \{d\}$, so $O(d) = 2$ and it is not hard to check that G is the semidirect product of N and $\langle d \rangle$. We want to show that $\mathrm{Cay}(N, S \cap N) \,\square\, \mathrm{Cay}(\langle d \rangle, \{d\}) \cong \mathrm{Cay}(G, S)$. For this purpose, we define ϕ: $\mathrm{Cay}(N, S \cap N) \,\square\, \mathrm{Cay}(\langle d \rangle, \{d\}) \to \mathrm{Cay}(G, S)$ such that $\phi(m, x) = mx$ for every $m \in N$ and $x \in \langle d \rangle$. Since G is the semidirect product of N and $\langle d \rangle$, it is obvious that ϕ is a bijective function. Now we will show that ϕ is homomorphism. For every $m, n \in N$ and $x, y \in \langle d \rangle$, we have:

$$(m, x) \sim (n, y) \quad (\text{in } \mathrm{Cay}(N, S \cap N) \,\square\, \mathrm{Cay}(\langle d \rangle, \{d\})) \quad \text{if and only if} \quad m = n,\ x \sim y \quad \text{or} \quad n \sim m,\ x = y.$$

We should check the following cases:

(1) If $m = n$, $x = 1$ and $y = d$, then $(\phi(m, x))^{-1}\phi(n, y) = m^{-1}nd = d \in S$. Thus $\phi(m, x) \sim \phi(n, y)$ in $\mathrm{Cay}(G, S)$.

(2) If $m = n$, $x = d$ and $y = 1$, then $(\phi(m, x))^{-1}\phi(n, y) = d^{-1}m^{-1}n = d \in S$. Thus $\phi(m, x) \sim \phi(n, y)$ in $\mathrm{Cay}(G, S)$.

(3) If $m \sim n$ and $x = y = 1$, then $m^{-1}n \in S \cap N$. Thus $(\phi(m, x))^{-1}\phi(n, y) = (mx)^{-1}(ny) = m^{-1}n \in N \cap S$. So $\phi(m, x) \sim \phi(n, y)$ in $\mathrm{Cay}(G, S)$.

(4) If $m \sim n$ and $x = y = d$, then $m^{-1}n \in S \cap N$. Thus $(\phi(m, x))^{-1}\phi(n, y) = d^{-1}(m^{-1}n)d \in d^{-1}(S \cap N)d = N \cap S \subset S$. So $\phi(m, x) \sim \phi(n, y)$ in $\mathrm{Cay}(G, S)$.

Now, let $t_1 \sim t_2$ in $\mathrm{Cay}(G, S)$. Since G is the semidirect product of N and $\langle d \rangle$, there exist $m, n \in N$ and $x, y \in \langle d \rangle$ such that $t_1 = mx$ and $t_2 = ny$. We continue the proof in the following cases:

(i) If $x = 1$ and $y = d$, then $m^{-1}nd = t_1^{-1}t_2 \in S\backslash(S\cap N) = \{d\}$. Therefore, $m^{-1}n = 1$ and so $m = n$. From this, we have $\phi^{-1}(t_1) = (m, x) \sim (n, y) = \phi^{-1}(t_2)$.

(ii) If $x = d$ and $y = 1$, the above reason shows that $\phi^{-1}(t_1) = (m, x) \sim (n, y) = \phi^{-1}(t_2)$.

(iii) If $x = y = 1$, then $m^{-1}n = t_1^{-1}t_2 \in S \cap N$. Therefore $m \sim n$ in $\mathrm{Cay}(N, S \cap N)$ and hence $\phi^{-1}(t_1) = (m, x) \sim (n, y) = \phi^{-1}(t_2)$.

(iv) If $x = y = d$, then $d^{-1}m^{-1}nd = t_1^{-1}t_2 \in d(S \cap N)d^{-1} = S \cap N$. Therefore $m^{-1}n \in d(S \cap N)d^{-1} = S \cap N$ and hence, $\phi^{-1}(t_1) = (m, x) \sim (n, y) = \phi^{-1}(t_2)$.

These show that $\mathrm{Cay}(N, S\cap N) \,\square\, \mathrm{Cay}(\langle d \rangle, \{d\}) \cong \mathrm{Cay}(G, S)$. Now, suppose that the theorem is false, and let G be the smallest group satisfying the hypothesis and $\mathrm{Cay}(G, S)$ does not admit a 3-NZF. Note that $|S| \geq 5$. We examine the following possibilities:

Subcase 2.1. If there is $y \in S \cap Z(N)$ of order $n > 2$ such that $d^{-1}yd \notin \{y, y^{-1}\}$, then since $Z(N)$ is normal in G, the assumption guarantees the existence of an element $z \in S \cap Z(N)$ such that $d^{-1}yd = z$. Since $O(d) = 2$, we see that $d^{-1}zd = y$.

Thus $\langle y, y^{-1}, z, z^{-1}\rangle \trianglelefteq \langle y, y^{-1}, z, z^{-1}, d\rangle$. If $G \neq \langle y, y^{-1}, z, z^{-1}, d\rangle$, then by our assumption on G, $\mathrm{Cay}(\langle y, y^{-1}, z, z^{-1}, d\rangle, \{y, y^{-1}, z, z^{-1}, d\})$ admits a 3-NZF. Thus since $|S \setminus \{y, y^{-1}, z, z^{-1}, d\}|$ is even, we get that $\mathrm{Cay}(G, S)$ admits a 3-NZF. This is a contradiction. Therefore, we can assume that $G = \langle y, y^{-1}, z, z^{-1}, d\rangle$, $N = \langle y, y^{-1}, z, z^{-1}\rangle$, $S = \{y, y^{-1}, z, z^{-1}, d\}$ and $S \cap N = \{y, y^{-1}, z, z^{-1}\}$. Let K be a minimal normal subgroup of G such that $K \leq Z(N)$. If $K \cap S = \emptyset$, then $N/K \trianglelefteq G/K$ with $[G/K : N/K] = 2$ and $Z(N/K) \cap S/K \neq \emptyset$. Note that $|S/K| = 5$ and $|(S \cap N)/K| = 4$. So $\mathrm{Cay}(N/K, (S \cap N)/K)$ admits a 3-NZF. Also $|G/K| < |G|$. Thus our assumption on G leads us to see that $\mathrm{Cay}(G/K, S/K)$ admits a 3-NZF, and so does $\mathrm{Cay}(G, S)$ by Lemma 2.2. This is a contradiction. Thus $K \cap S \neq \emptyset$. Without loss of generality, we can suppose that $y \in K$, so $d^{-1}yd = z \in K$. Therefore, $K = N$. This forces N to be cyclic or elementary abelian. Thus either $N = \langle y\rangle$ or $N = \langle S \cap N\rangle = \langle y\rangle \times \langle z\rangle$ and hence, either $z = y^i$ and

$$\mathrm{Cay}(N, N \cap S) = \mathrm{Cay}(\langle y\rangle, \{y, y^{-1}, y^i, y^{-i}\}) \cong C(n, i) \quad \text{or}$$
$$\mathrm{Cay}(N, N \cap S) = \mathrm{Cay}(\langle y\rangle, \{y, y^{-1}\}) \,\Box\, \mathrm{Cay}(\langle z\rangle, \{z, z^{-1}\}) \cong C_n \,\Box\, C_n.$$

Note that $\mathrm{Cay}(G, S) = \mathrm{Cay}(N, S \cap N) \,\Box\, K_2$. Thus $\mathrm{Cay}(G, S)$ is isomorphic to either $C(n, i) \,\Box\, K_2$ or $(C_n \,\Box\, C_n) \,\Box\, K_2$. So Lemmas 2.5 and 2.4 guarantee that $\mathrm{Cay}(G, S)$ admits a 3-NZF. This is a contradiction.

Subcase 2.2. If $S \cap Z(N)$ contains an involution y such that $d^{-1}yd \neq y$, then there exists an element $z \in S \cap Z(N)$ such that $d^{-1}yd = z$. Therefore, $\langle y, z\rangle$ is an elementary abelian 2-group of order 4. So $\mathrm{Cay}(\langle y, z, d\rangle, \{y, z, d\})$ is the circular ladder CL_4 (see Figure 1) which is bipartite and hence, it admits a 3-NZF. Also, $\mathrm{Cay}(G, S \setminus \{y, z, d\})$ admits a 3-NZF, and so does $\mathrm{Cay}(G, S)$. This is a contradiction.

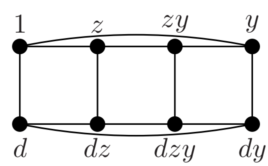

Figure 1: The circular ladder CL_4.

Subcase 2.3. Suppose that for every $y \in Z(N) \cap S$, $d^{-1}yd \in \{y, y^{-1}\}$. Applying the above argument shows that there exists an element $y \in Z(N) \cap S$ such that $\langle y\rangle$ is a minimal normal subgroup of G. If the order of y is 2, then y is a central involution and hence, $\mathrm{Cay}(G, S)$ admits a 3-NZF. This is a contradiction. Thus the order of y is an odd prime number. Now if $N \cap S$ contains an element z such that $O(z) \geq 3$ and $d^{-1}zd \in \{z, z^{-1}\}$, then applying the same argument as that of used in Subcase 2.1 leads us to get a contradiction. Now suppose that there exists an element $z \in (S \cap N) \setminus \{y, y^{-1}\}$ such that $O(z) \geq 3$ and $d^{-1}zd \notin \{z, z^{-1}\}$. So our assumption on G allows us to assume that $S = \{y, y^{-1}, z, z^{-1}, d^{-1}zd, d^{-1}z^{-1}d, d\}$. Let K be a normal subgroup of G containing y such that $K \leq N$ and it is maximal with the property $K \cap (S \setminus \{y, y^{-1}\}) = \emptyset$. If M/K is a minimal normal subgroup of G/K such that $M/K \leq N/K$, then our assumption on K shows that $M \cap (S \setminus \{y, y^{-1}\}) \neq \emptyset$. Without loss of generality, we can assume that $z \in M$. Since M is normal in G, we deduce that $d^{-1}zd \in M$ and hence, $S - \{d\} \subseteq M$. Thus $M = N$. Set $S_1 = \{z, z^{-1}, d^{-1}zd, d^{-1}z^{-1}d, d\}$. Moreover $M/K = N/K$ is abelian and normal in G/K of index 2 such that $S_1/K \setminus (S_1/K \cap M/K) = \{dK\}$ and $dK(S_1/K \cap M/K)dK = (S_1/K \cap M/K)$. By our assumption on G, $\mathrm{Cay}(G/K, S_1/K)$ admits a 3-NZF. But $S_1 \cap K = \emptyset$, so Lemma 2.2 shows that $\mathrm{Cay}(G, S_1)$ admits a 3-NZF. In addition, since $|S \setminus S_1| = 2$, $\mathrm{Cay}(G, S \setminus S_1)$ admits a 3-NZF and hence, $\mathrm{Cay}(G, S)$ admits a 3-NZF. This is a contradiction. Finally, let $N \cap S$ contain an element z of order 2. Since $|S \cap N|$ is even, our assumption on G allows us to assume that there exists an involution $w \in (S \cap N) \setminus \{z\}$ such that $G = \langle y, y^{-1}, z, w, d\rangle$. Since z, w are distinct involutions, we have that either $\langle z, w\rangle$ is an elementary abelian 2-group of order 4 or a dihedral group. We can see at once that $\mathrm{Cay}(\langle w, z, d\rangle, \{w, z, d\})$ is a circular ladder CL_k, for some even number k, which is bipartite. Therefore, $\mathrm{Cay}(\langle w, z, d\rangle, \{w, z, d\})$ admits a 3-NZF, and so does $\mathrm{Cay}(G, S)$. This is a contradiction.

This shows that $\mathrm{Cay}(G, S)$ admits a 3-NZF, as desired. □

Theorem 3.2. *Let G be a group, N be an abelian subgroup of G of index 2 and let S be a connection sequence of G such that $|S| \geq 4$. If there exists $d \in S \setminus (S \cap N)$ such that $d^{-1}(S \cap N)d = S \cap N$, then $\mathrm{Cay}(G, S)$ admits a 3-NZF.*

Proof. First, assume that $|S \cap N| \geq 4$. By Lemma 2.3, $\mathrm{Cay}(N, S \cap N)$ has a 3-NZF. Since $|G/N| = 2$, we can assume that $G/N = \langle dN\rangle$, and hence for every $y \in S \setminus (S \cap N)$, $yN \in \langle dN\rangle$. Thus there exists $t \in N$ such that $y = td$ and

$$\text{for every } s \in S \cap N \text{ and } y \in S \setminus (S \cap N), \quad y^{-1}sy \in S \cap N. \tag{3.1}$$

So the Conditions (1) and (2) of Theorem 3.1 are fulfilled and hence $\mathrm{Cay}(G, S)$ admits a 3-NZF. Now, we assume that $|S \cap N| \leq 3$. The proof falls naturally into several parts. If $|S \cap N| = 0$, then by Lemma 2.8, $\mathrm{Cay}(G, S)$ is bipartite, and hence Lemma 2.6 shows that $\mathrm{Cay}(G, S)$ admits a 3-NZF. Moreover, if $|S \cap N| = 2$, then Lemma 2.9 forces $\mathrm{Cay}(N, S \cap N)$ to admit a 3-NZF. Also by (3.1), for every $s \in S \cap N$, $y^{-1}sy = d^{-1}sd \in S \cap N$. So Theorem 3.1 completes the proof. Therefore, $|S \cap N| \in \{1, 3\}$. We consider these possibilities in the following cases:

Case 1. Assume that $|S \cap N| = 1$. So $S \cap N = \{x\}$. Clearly, $O(x) = 2$ and $d^{-1}xd = x^{-1} = x$. Also, for every $y \in S \setminus (S \cap N)$, we have $yN \in \langle dN\rangle$ and hence, $y = md$ for some $m \in N$. Therefore, we can see $y^{-1}xy = x$. Thus $x \in Z(\langle S\rangle)$ is of order 2. Hence by Lemma 2.1, we have $\mathrm{Cay}(\langle S\rangle, S)$ admits a 3-NZF, and so does $\mathrm{Cay}(G, S)$.

Case 2. Assume that $|S \cap N| = 3$. We continue the proof in two subcases:

Subcase 1. Let $S \cap N = \{x, y, y^{-1}\}$, where $O(x) = 2$ and $O(y) \geq 3$. Since $d^{-1}xd \in S \cap N$ and $O(d^{-1}xd) = 2$, the same argument as that of used in Case 1 completes the proof.

Subcase 2. Let $S \cap N = \{x, y, z\}$, where $O(x) = O(y) = O(z) = 2$. First, assume that none of the elements in $S \cap N$ generates by the other ones. Since x, y, z are of order 2 and N is abelian, we have

$$\langle N \cap S\rangle = \{x^i y^j z^l \mid 1 \leq i, j, k \leq 2\} = \langle x\rangle \times \langle y\rangle \times \langle z\rangle \leq N.$$

It is easy to check that $\mathrm{Cay}(\langle N \cap S\rangle, S \cap N)$ is bipartite (similar to Figure 1) and hence by Lemma 2.6, $\mathrm{Cay}(N, S \cap N)$ admits a 3-NZF. The rest of the proof runs as the case when $|S \cap N| \geq 4$.

Otherwise, without loss of generality, assume that $S \cap N = \{x, y, xy\}$. Set $S_1 = \{d, d^{-1}\}$. Note that $|S|$ is odd. Thus $|S \setminus ((S \cap N) \cup S_1)| = 0$ or $2k$ where $k \in \mathbb{N}$. Set $S_2 = S \setminus ((S \cap N) \cup S_1)$ and $H = \langle (S \cap N) \cup S_1 \rangle$. In fact,

$$\mathrm{Cay}(G, S_2) \cup \mathrm{Cay}(G, (S \cap N) \cup S_1) = \mathrm{Cay}(G, S)$$

and $\mathrm{Cay}(G, S_2)$ admits a 3-NZF. So it is sufficient to find a 3-NZF in $\mathrm{Cay}(G, (S \cap N) \cup S_1)$.

We know that $d^{-1}xd \in S \cap N$. If $d^{-1}xd = x$, then since N is abelian, we have $x \in Z(H)$ and its order is 2, so the proof is complete by Lemma 2.1. Now, assume that $d^{-1}xd = y$. Since $N \neq dN \in G/N$ and $|G/N| = 2$, we have $O(dN) = 2$, and hence $d^2 \in N$. It follows that $x = d^2xd^{-2} = dyd^{-1}$. Therefore,

$$d^{-1}xyd = d^{-1}xdd^{-1}yd = yx = xy.$$

Thus $xy \in Z(H)$ and $O(xy) = 2$. Lemma 2.1 shows that $\mathrm{Cay}(H, (S \cap N) \cup S_1)$ admits a 3-NZF, and so does $\mathrm{Cay}(G, (S \cap N) \cup S_1)$, as desired. The same reasoning can be applied to the case $d^{-1}xd = xy$. □

In the following we show that Theorem 3.2 guarantees the existence of a 3-NZF in a Cayley graph on a generalized dicyclic group.

Example 3.3. Let H be an abelian group, having a specific element $y \in H$ of order 2. A group G is called a *generalized dicyclic group*, $\mathrm{Dic}(H, y)$, if it is generated by H and an additional element x. Moreover, we have $[G : H] = 2$, $x^2 = y$ and $x^{-1}ax = a^{-1}$ for every $a \in H$. It is easy to see that every Cayley graph of valency at least 4 on $\mathrm{Dic}(H, y)$ has a 3-NZF by applying Theorem 3.2.

Note that in [3, 11], as the main theorems, it is showed that the graphs mentioned in Example 3.4 admit nowhere-zero 3-flows.

Example 3.4.

(1) Let H be an abelian group. The *generalized dihedral group* D_H is a group of order $2|H|$ generated by H and an element p where $p \notin H$, $p^2 = 1$ and $p^{-1}hp = h^{-1}$ for all $h \in H$. We see at once that every Cayley graph of valency at least 4 on D_H satisfies the conditions of Theorem 3.2, and hence it admits a 3-NZF. In particular, $G = \langle x, a \mid a^n = x^2 = 1, x^{-1}ax = a^{-1} \rangle$ is a special case of D_H, where $H = \langle a \rangle$, $p = x$ and it is called a *dihedral group* and denoted by D_{2n}.

(2) Let $G = \langle z, a \mid a^n = z^2, a^n = 1, z^{-1}az = a^{-1} \rangle$ which is called a *generalized quaternion group*, denoted by Q_{4n}. Note that G is a special case of a generalized dicyclic group where $\langle a \rangle$ and z play the roles of H and x, respectively. Thus every Cayley graph of valency at least 4 on Q_{4n} admits a 3-NZF.

Remark 3.5. Let G be a group, N be a normal subgroup of G of an odd index at least 3 and S be a connection sequence of G such that $|S| \geq 4$. Assume that $T = \{x_1, \dots, x_{2k+1}\}$ is a left transversal set of N in G and $\mathrm{Cay}(N, S \cap N))$ has a 3-NZF. Note that by Remark 2.11,

$$\mathrm{Cay}(G, S) = \left(\bigcup_{i=1}^{2k+1} x_i \,\mathrm{Cay}(N, S \cap N) \right) \cup \mathrm{Cay}(G, S \setminus (S \cap N)).$$

By the assumption, for every $i \in \{1, \ldots, 2k+1\}$, $x_i \operatorname{Cay}(N, S \cap N)$ admits a 3-NZF. For finding a 3-NZF in $\operatorname{Cay}(G, S)$, it is enough to find a 3-NZF in $\operatorname{Cay}(G, S \setminus (S \cap N))$. If $|S \setminus (S \cap N)|$ is odd, then there exists $y \in S \setminus (S \cap N)$ such that $O(y) = 2$ and hence $yN \in G/N$ and $O(yN) = 2$. So we have $2 \mid |G/N|$. This is impossible. Thus $|S \setminus (S \cap N)|$ is even and hence $\operatorname{Cay}(G, S \setminus (S \cap N))$ admits a 3-NZF by Lemma 2.9. Therefore if $\operatorname{Cay}(N, S \cap N)$ has a 3-NZF, then so does $\operatorname{Cay}(G, S)$.

Theorem 3.6. *Let G be a solvable group with a cyclic Sylow 2-subgroup and let S be a connection sequence of G with $|S| \geq 4$. If there exists an element $x \in Z(G) \cap S$ such that x is a generator element of G in S, then $\operatorname{Cay}(G, S)$ admits a 3-NZF.*

Proof. Suppose that G is the smallest counterexample satisfies the above conditions, but $\operatorname{Cay}(G, S)$ does not admit a 3-NZF. Without loss of generality, we can assume that $|S| = 5$ and $x \in Z(G) \cap S$. Thus $O(x) \geq 3$ by Lemma 2.1. If there exists $u \in Z(G)$ such that $\langle u \rangle \cap S = \emptyset$, then $|S/\langle u \rangle| = |S|$, $x\langle u \rangle \in Z(G/\langle u \rangle) \cap S/\langle u \rangle$ and $|G/\langle u \rangle| < |G|$. If $x\langle u \rangle$ is a generator element of $G/\langle u \rangle$ in $S/\langle u \rangle$, then by our assumption, $\operatorname{Cay}(G/\langle u \rangle, S/\langle u \rangle)$ admits a 3-NZF. Lemma 2.2 forces $\operatorname{Cay}(G, S)$ to admit a 3-NZF, a contradiction. Thus $x\langle u \rangle$ is not a generator element. Therefore, there exist an element $t \in \langle S \setminus \{x, x^{-1}\} \rangle$ and $i \in \mathbb{N}$ such that $xu^i = t$ and hence $t \in Z(G)$. If there exists $t_1 \in \langle t \rangle \cap S$, then as stated above, we can see that $O(t_1) \geq 3$. Thus $Z(G) \cap S = \{x, x^{-1}, t_1, t_1^{-1}\}$. Therefore $|G/Z(G)| \in \{1, 2\}$ and hence, $G/Z(G)$ is cyclic. So G is an abelian group. This forces $\operatorname{Cay}(G, S)$ to admit a 3-NZF, a contradiction. Thus $\langle t \rangle \cap S = \emptyset$. Moreover, we can see at once that $x\langle t \rangle$ is a generator element of $G/\langle t \rangle$ in $S/\langle t \rangle$, $|S/\langle t \rangle| = |S|$ and $|G/\langle t \rangle| < |G|$. Therefore, our assumption forces $\operatorname{Cay}(G/\langle t \rangle, S/\langle t \rangle)$ to admit a 3-NZF, and so does $\operatorname{Cay}(G, S)$ by Lemma 2.2. This is a contradiction. So for every $u \in Z(G)$, we have $\langle u \rangle \cap S \neq \emptyset$. We continue the proof in two cases:

Case 1. Suppose that $|Z(G)|$ is even. So there exists $w \in Z(G)$ of order 2. By our assumption, $\langle w \rangle \cap S \neq \emptyset$, and hence S contains a central involution. Lemma 2.1 shows that $\operatorname{Cay}(\langle S \rangle, S)$ admits a 3-NZF, and so does $\operatorname{Cay}(G, S)$. This is a contradiction.

Case 2. Let $|Z(G)|$ be odd. Since $|S| = 5$, S contains an involution y. We continue the proof in three subcases:

Subcase 1. Suppose that $|S \cap Z(G)|$ is odd, so $Z(G)$ contains an involution. This is a contradiction, because $|Z(G)|$ is odd.

Subcase 2. Suppose that $|Z(G) \cap S| = 2$. So we have $Z(G) \cap S = \{x, x^{-1}\}$, where $O(x)$ is an odd prime number p. Therefore, $\langle x \rangle$ is a cyclic subgroup of order p. By the assumption, $x \notin \langle S \setminus \{x, x^{-1}\} \rangle$ and hence, we deduce that $G = \langle x \rangle \times M$, where $M = \langle S \setminus \{x, x^{-1}\} \rangle$ is a maximal subgroup of G. Let N be a minimal normal subgroup of G such that $N \leq M$. So N is an elementary abelian q-group, where q is a prime number. If $N \cap S = \emptyset$, then $x\langle N \rangle \in Z(G/N) \cap S/N$ is a generator element of G/N in S/N, $|G/N| < |G|$ and $|S/N| = 5$. Thus by our assumption on G, $\operatorname{Cay}(G/N, S/N)$ admits a 3-NZF, and so does $\operatorname{Cay}(G, S)$. This contradicts our assumption. If $N \cap S \neq \emptyset$, then the proof falls naturally into several parts:

- **(a)** If $y \in N \cap S$ such that $O(y) = 2$, then $2 \mid |N|$. Since N is elementary abelian, we get that N is an elementary abelian 2-group. Thus $|N| = 2$ by the assumption. Therefore $y \in N \leq Z(G)$, and hence $|Z(G)|$ is even. This is a contradiction.

(b) If $N \cap S = \{z, z^{-1}\}$, where $O(z) \geq 3$, then $S \setminus \{x, x^{-1}\} = \{z, z^{-1}, y\}$. Since N is an elementary abelian q-group where q is a prime number, we get $O(z) = q \neq 2$. So $y \notin N$. If $yz = zy$, then G is an abelian group and hence, Lemma 2.3 forces $\mathrm{Cay}(G, S)$ to admit a 3-NZF, a contradiction. If $yz \neq zy$ and $O(yz) = 2$, then we have $yzy = z^{-1}$. Thus $L = \langle x, x^{-1}, z, z^{-1}\rangle \lhd G = \langle x, x^{-1}, z, z^{-1}, y\rangle$. Therefore, $[G : L] = 2$ and $L \lhd G$. We thus get that $\mathrm{Cay}(G, S)$ admits a 3-NZF by Theorem 3.1. This is a contradiction. Now, suppose that $yz \neq zy$ and $O(yz) \geq 3$. Since $O(z) = q$, $z \in N$ and $|M/N| = |\langle yN\rangle| = 2$, we have $|M| = 2q^t$, where $t \in \mathbb{N}$. If $O(yz) = q^n$, where $n \leq t$, then $yz \in N$. So $y \in N$, a contradiction. Suppose that $O(yz) = 2q^n$ where $n \leq t$. Since $\gcd(2, q^n) = 1$, there exist $k, s \in \mathbb{Z}$ such that $2s + kq^n = 1$. So, $O((yz)^{2s}) = q^n$ and $O((yz)^{kq^n}) = 2$. Thus we have $(yz)^{2s} \in N$. Since $z \in N$ and N is abelian, we can see that $(yz)^{2s}y = y(yz)^{2s}$. Therefore $(yz)^{2s} \in Z(M) \leq Z(G)$. Thus $\langle (yz)^{2s}\rangle$ is a normal subgroup of G and $\langle (yz)^{2s}\rangle \leq N$. So $z \in N = \langle (yz)^{2s}\rangle \leq Z(M) \leq Z(G)$ and hence $yz = zy$. This is a contradiction with the above statements.

Subcase 3. Suppose that $|S \cap Z(G)| = 4$. Since $|S| = 5$, we can see $|S| \setminus |S \cap Z(G)| = 1$. It follows that $[\langle S\rangle : \langle S \cap Z(G)\rangle] = 2$. So $\langle S\rangle/(\langle S \cap Z(G)\rangle)$ is a cyclic group. On the other hand, $\langle S \cap Z(G)\rangle \leq Z(\langle S\rangle)$. Therefore $\langle S\rangle$ is abelian, and hence Lemma 2.3 yields that $\mathrm{Cay}(\langle S\rangle, S)$ admits a 3-NZF, and so does $\mathrm{Cay}(G, S)$, a contradiction. □

Corollary 3.7. *Let G be a solvable group such that the Sylow 2-subgroups of G are cyclic and every Cayley graph of valency at least 4 on G admits a 3-NZF. If H is a nilpotent group, then every Cayley graph of valency at least 4 on $G \times H$ admits a 3-NZF.*

Proof. Suppose that H is the smallest nilpotent group such that $\mathrm{Cay}(G \times H, S)$ does not admit a 3-NZF. Note that by the assumption on G, we have $H \neq 1$. If there exists $1 \neq t \in Z(H)$ such that $\langle t\rangle \cap S = \emptyset$, then since $\langle t\rangle \lhd G \times H$, our assumption on H shows that $\mathrm{Cay}((G \times H)/\langle t\rangle, S/\langle t\rangle)$ admits a 3-NZF. So Lemma 2.2 forces $\mathrm{Cay}(G \times H, S)$ to admit a 3-NZF. This is a contradiction. Thus for every $t \in Z(H)$, $\langle t\rangle \cap S \neq \emptyset$. If $|H|$ is even, then S contains a central involution and hence, Lemma 2.1 shows that $\mathrm{Cay}(G \times H, S)$ admits a 3-NZF, a contradiction. Thus $|H|$ is odd. Let the order of $t \in Z(H) \cap S$ be odd. If $|H \cap S|$ is odd, then $2 \mid |H|$. This is a contradiction. If $|H \cap S| = 2$, then $H \cap S = \{x, x^{-1}\}$ and hence, $Z(H) \cap S = \{x, x^{-1}\}$ and $O(x)$ is a prime number. Since G is solvable, we can assume that K is a normal subgroup of $G \times H$ such that $K \leq G$ and K is maximal with the property that $S \cap K = \emptyset$. If $G = K$, then $(G \times H)/G$ is nilpotent and $|S/G| = |S|$, and hence, $\mathrm{Cay}((G \times H)/G, S/G)$ admits a 3-NZF, and so does $\mathrm{Cay}(G \times H, S)$. This is a contradiction. Thus $G \neq K$ and for a minimal normal subgroup M/K of $(G \times H)/K$ such that $M/K \leq G/K$, we have $M \cap S \neq \emptyset$. So one of the following possibilities occurs:

(I) Suppose that $M \cap S$ contains an involution z. Then $2 \mid |M/K|$. Since M/K is elementary abelian and the Sylow 2-subgroups of G are cyclic, we have $M/K = \langle zK\rangle$ and hence $\langle zK\rangle \leq Z((G \times H)/K)$. Therefore, Lemma 2.1 shows that $\mathrm{Cay}((G \times H)/K, S/K)$ admits a 3-NZF, and so does $\mathrm{Cay}(G \times H, S)$ by Lemma 2.2, a contradiction.

(II) If $M \cap S$ does not contain any involution, then $|M \cap S|$ is an even number. Since $|S|$ is odd, we get that $S \setminus (M \cap S)$ contains an involution z. But $|H|$ is odd, so $z \in G$. Let $S_1 = (M \cap S) \cup \{z, x, x^{-1}\}$. We have $\langle S_1\rangle = \langle M \cap S, z\rangle \times \langle x\rangle$ and $|S_1| \geq 5$ is an odd number. Thus Theorem 3.6 shows that $\mathrm{Cay}(\langle S_1\rangle, S_1)$ admits a 3-NZF, so does $\mathrm{Cay}(G \times H, S_1)$. Since $|S \setminus S_1|$ is even, $\mathrm{Cay}(G \times H, S)$ admits a 3-NZF, a contradiction.

If $|H \cap S| \geq 4$, then there exists an element $x \in S$ such that $O(x) = 2$. Since $|H|$ is odd, we have $x \notin H \cap S$ and the Sylow 2-subgroups of $G \times H$ are the Sylow 2-subgroups of G and hence, $x \in G$. Therefore $x \in C_{G \times H}(H \cap S)$, the centralizer of $H \cap S$ in $G \times H$, and hence $x \in Z(\langle H \cap S \rangle \times \langle x \rangle)$. So Lemma 2.1 forces $\mathrm{Cay}(\langle H \cap S \rangle \times \langle x \rangle, (H \cap S) \cup \{x\})$ to admit a 3-NZF, so does $\mathrm{Cay}(G \times H, (H \cap S) \cup \{x\})$. But $|S \setminus ((H \cap S) \cup \{x\})|$ is even, So $\mathrm{Cay}(G \times H, S)$ admits a 3-NZF, a contradiction. □

Corollary 3.8. *If L is a nilpotent group, then for every generalized dihedral group D_H, the Cayley graph of valency at least 4 on $D_H \times L$ admits a 3-NZF.*

Proof. Let D_H be the smallest generalized dihedral group such that the Cayley graph of valency at least 4 on $D_H \times L$ does not admit a 3-NZF. If $|H|$ is odd, then the Sylow 2-subgroups of D_H are cyclic, and hence Corollary 3.7 shows that $\mathrm{Cay}(D_H \times L, S)$ admits a 3-NZF, a contradiction. If $|H|$ is even, then H contains a central involution t. If $t \in S$, then Lemma 2.1 shows that the Cayley graph of valency at least 4 on $D_H \times L$ admits a 3-NZF, a contradiction. If $t \notin S$, then by our assumption, $\mathrm{Cay}((D_H \times L)/\langle t \rangle, S/\langle t \rangle)$ admits a 3-NZF. It follows that $\mathrm{Cay}(D_H \times L, S)$ admits a 3-NZF by Lemma 2.2. This is impossible. These contradictions show that every Cayley graph of valency at least 4 on $D_H \times L$ admits a 3-NZF. □

References

[1] B. Alspach, Y. Liu and C. Zhang, Nowhere-zero 4-flows and Cayley graphs on solvable groups, *SIAM J. Discrete Math.* **9** (1996), 151–154, doi:10.1137/s0895480193258017.

[2] F. Jaeger, Flows and generalized coloring theorems in graphs, *J. Comb. Theory Ser. B* **26** (1979), 205–216, doi:10.1016/0095-8956(79)90057-1.

[3] L. Li and X. Li, Nowhere-zero 3-flows in Cayley graphs on generalized dihedral group and generalized quaternion group, *Front. Math. China* **10** (2015), 293–302, doi:0.1007/s11464-014-0378-2.

[4] M. Nánásiová and M. Škoviera, Nowhere-zero 3-flows in Cayley graphs and Sylow 2-subgroups, *J. Algebraic Combin.* **30** (2009), 103–111, doi:10.1007/s10801-008-0153-0.

[5] R. Nedela and M. Škoviera, Cayley snarks and almost simple groups, *Combinatorica* **21** (2001), 583–590, doi:10.1007/s004930100014.

[6] M. Potočnik, P. Škoviera and R. Škrekovski, Nowhere-zero 3-flows in abelian Cayley graphs, *Discrete Math.* **297** (2005), 119–127, doi:10.1016/j.disc.2005.04.013.

[7] J. S. Rose, *A Course on Group Theory*, Cambridge University Press, Cambridge, 1978.

[8] W. T. Tutte, A contribution to the theory of chromatic polynomials, *Canad. J. Math.* **6** (1954), 80–91, doi:10.4153/cjm-1954-010-9.

[9] W. T. Tutte, On the algebraic theory of graph colorings, *J. Comb. Theory* **1** (1966), 15–50, doi:10.1016/s0021-9800(66)80004-2.

[10] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, NJ, 1996.

[11] F. Yang and X. Li, Nowhere-zero 3-flows in dihedral Cayley graphs, *Inform. Process. Lett.* **111** (2011), 416–419, doi:10.1016/j.ipl.2011.01.017.