ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 15 (2018) 81–95
https://doi.org/10.26493/1855-3974.1382.dee
(Also available at http://amc-journal.eu)

On 2-distance-balanced graphs*

Boštjan Frelih †, Štefko Miklavič ‡

University of Primorska, FAMNIT and IAM, Muzejski trg 2, 6000 Koper, Slovenia

Received 12 April 2017, accepted 11 September 2017, published online 26 January 2018

Abstract

Let n denote a positive integer. A graph Γ of diameter at least n is said to be n*-distance-balanced* whenever for any pair of vertices u, v of Γ at distance n, the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u. In this article we consider $n = 2$ (e.g. we consider 2*-distance-balanced graphs*). We show that there exist 2-distance-balanced graphs that are not 1-distance-balanced (e.g. distance-balanced). We characterize all connected 2-distance-balanced graphs that are not 2-connected. We also characterize 2-distance-balanced graphs that can be obtained as cartesian product or lexicographic product of two graphs.

Keywords: n*-distance-balanced graph, cartesian product, lexicographic product.*

Math. Subj. Class.: 05C12, 05C76

1 Introductory remarks

A graph Γ is *distance-balanced* if for each pair u, v of adjacent vertices of Γ the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u. Although these graphs are interesting from the purely graph-theoretical point of view, they also have applications in other areas of research, such as mathematical chemistry and communication networks. It is for that reason that they have been studied from various different points of view in the literature.

Distance-balanced graphs were first studied by Handa [9] in 1999. The name *distance-balanced*, however, was introduced nine years later by Jerebic, Klavžar and Rall [12]. The

*We thank the two anonymous referees for many useful comments and suggestions that have greatly improved our initial manuscript, especially for pointing out the mistake in Theorem 5.4.

†This work is partially supported by Erasmus+ Programme of the European Union (project ECCUM – Establishment of Computing Centers and Curriculum Development in Mathematical Engineering Master Programme).

‡The author acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1-0285 and research projects N1-0032, N1-0038, J1-5433, J1-6720, J1-7051).

E-mail addresses: bostjan.frelih@upr.si (Boštjan Frelih), stefko.miklavic@upr.si (Štefko Miklavič)

family of distance-balanced graphs is very rich (for instance, every distance-regular graph as well as every vertex-transitive graph has this property [13]). In the literature these graphs were studied from various purely graph-theoretic aspects such as symmetry [13], connectivity [9, 16] or complexity aspects of algorithms related to such graphs [6], to name just a few. However, it turns out that these graphs have applications in other areas, such as mathematical chemistry (see for instance [3, 11, 12]) and communication networks (see for instance [3]).

Another interesting fact is that these graphs can be characterized by properties that do not seem to have much in common with the original definition from [12]. For example, the distance-balanced graphs coincide with *self-median* graphs, that is graphs for which the sum of the distances from a given vertex to all other vertices is independent of the chosen vertex (see [4]). In [1], distance-balanced graphs are called *transmission regular*. Finally, even order distance-balanced graphs possess yet another nice property, making them what are called *equal opportunity graphs* (see [3] for the definition).

In distance-balanced graphs one only considers pairs of adjacent vertices. However, it is very natural to extend the definition to the pairs of nonadjacent vertices. This generalized concept of n-*distance-balanced* graphs (see Section 2 for the definition) was first introduced by Boštjan Frelih in 2014 [8] (we point out that certain other generalizations of this concept, where one still focuses just on pairs of adjacent vertices, have also been considered in the recent years [10, 14, 15]). The n-distance-balanced graphs and their properties were extensively studied in [17]. They are also the main topic in the paper [7], but in this paper some of the stated results do not hold. We comment on one of these problems later (see Remark 5.1).

In this article we consider 2-*distance-balanced graphs*. We now summarize our results. After some preliminaries in Section 2, we show in Section 3 that there exist 2-distance-balanced graphs that are not 1-distance-balanced (e.g. distance-balanced). It was shown in [9] that every distance-balanced graph is 2-connected. It turns out that not all 2-distance-balanced graphs are 2-connected. However, we characterize all connected 2-distance-balanced graphs that are not 2-connected.

In [12] distance-balanced cartesian products and distance-balanced lexicographic products of two graphs were characterized. We characterize 2-distance-balanced cartesian products and 2-distance-balanced lexicographic products of two graphs in Section 4 and 5, respectively.

2 Preliminaries

In this section we review some basic definitions that we will need later. Throughout this paper, all graphs are assumed to be finite, undirected, without loops and multiple edges. Given a graph Γ let $V(\Gamma)$ and $E(\Gamma)$ denote its vertex set and edge set, respectively.

For $v \in V(\Gamma)$ we denote the set of vertices adjacent to v by $N_\Gamma(v)$. If the number $|N_\Gamma(v)|$ is independent of the choice of $v \in V(\Gamma)$, then we call this number the *valency of* Γ and we denote it by k_Γ (or simply by k if the graph Γ is clear from the context). In this case we say that Γ is *regular with valency* k or k-*regular*.

For $u, v \in V(\Gamma)$ we denote the distance between u and v by $\partial_\Gamma(u, v)$ (or simply by $\partial(u, v)$ if the graph Γ is clear from the context). The *diameter* $\max\{\partial_\Gamma(u, v) \mid u, v \in V(\Gamma)\}$ of Γ will be denoted by D_Γ (or simply by D if the graph Γ is clear from the context). For any pair of vertices $u, v \in V(\Gamma)$ we let W^Γ_{uv} be the set of vertices of Γ that are closer to u than to v, that is

$$W_{uv}^{\Gamma} = \{w \in V(\Gamma) \mid \partial_\Gamma(u,w) < \partial_\Gamma(v,w)\}.$$

Let n denote a positive integer. A graph Γ of diameter at least n is said to be *n-distance-balanced*, if $|W_{uv}^{\Gamma}| = |W_{vu}^{\Gamma}|$ for any $u, v \in V(\Gamma)$ at distance n. The distance-balanced graphs are n-distance-balanced graphs for $n = 1$.

For $W \subseteq V(\Gamma)$ the subgraph of Γ induced by W is denoted by $\langle W \rangle$ (we abbreviate $\Gamma - W = \langle V(\Gamma) \setminus W \rangle$). A *vertex cut* of a connected graph Γ is a set $W \subseteq V(\Gamma)$ such that $\Gamma - W$ is disconnected. A vertex cut of size k is called a *k-cut*. A graph is said to be *k-connected* if it has at least $k+1$ vertices and the size of the smallest vertex cut is at least k. If a vertex cut consists of a single vertex v, then v is called the *cut vertex*.

We complete this section by defining the cartesian product and the lexicographic product of graphs G and H. In both cases, the vertex set of the product is $V(G) \times V(H)$. Pick $(g_1, h_1), (g_2, h_2) \in V(G) \times V(H)$.

In the *cartesian product* of G and H, denoted by $G \Box H$, (g_1, h_1) and (g_2, h_2) are adjacent if and only if $g_1 = g_2$ and h_1, h_2 are adjacent in H, or $h_1 = h_2$ and g_1, g_2 are adjacent in G. Note that the cartesian product is commutative.

In the *lexicographic product* of G and H, denoted by $G[H]$, (g_1, h_1) and (g_2, h_2) are adjacent if and only if $g_1 = g_2$ and h_1, h_2 are adjacent in H, or g_1, g_2 are adjacent in G.

3 On the connectivity of 2-distance-balanced graphs

In this section we characterize connected 2-distance-balanced graphs that are not 2-connected (Corollary 3.4). As a consequence, using the well known fact that an arbitrary connected distance-balanced graph is at least 2-connected (see [9]), we construct an infinite family of 2-distance-balanced graphs that are not distance-balanced.

Let G be an arbitrary (not necessary connected) graph, and let c be a vertex that does not belong to the set of vertices of G. We construct a graph, denoted by $\Gamma(G, c)$, with the set of vertices

$$V(\Gamma(G, c)) = V(G) \cup \{c\}$$

and the set of edges

$$E(\Gamma(G, c)) = E(G) \cup \{cv \mid v \in V(G)\}.$$

This graph is obviously connected. Next theorem follows directly from the construction of $\Gamma(G, c)$.

Theorem 3.1. *G is not connected if and only if $\Gamma(G, c)$ is not 2-connected.* □

We show that regularity of G is a sufficient condition for $\Gamma(G, c)$ to be 2-distance-balanced.

Theorem 3.2. *If G is a regular graph that is not a complete graph, then $\Gamma = \Gamma(G, c)$ is 2-distance-balanced.*

Proof. Assume that G is a k-regular graph that is not a complete graph. Let $G_1, G_2, \ldots, G_n$ be its connected components for some positive integer n. If G is connected, then $n = 1$, otherwise G has at least two connected components. Since G is not a complete graph, it is clear that the diameter of Γ equals 2, which means that two arbitrary vertices of Γ are either adjacent or they are at distance 2.

There are two different types of vertices at distance 2 in Γ. The first type is when both vertices at distance 2 belong to the same connected component of G. The second type is when vertices at distance 2 belong to different connected components of G.

Let G_i be an arbitrary connected component of G. Let $v_1, v_2 \in V(G_i)$ be arbitrary vertices at distance 2 in Γ. We count vertices that are closer to v_1 than to v_2 in Γ and vertices that are closer to v_2 than to v_1 in Γ. We get

$$W^{\Gamma}_{v_1v_2} = \{v_1\} \cup (N_{G_i}(v_1) \setminus (N_{G_i}(v_1) \cap N_{G_i}(v_2))).$$

It follows that

$$\left|W^{\Gamma}_{v_1v_2}\right| = 1 + |N_{G_i}(v_1)| - |N_{G_i}(v_1) \cap N_{G_i}(v_2)|.$$

Changing the roles of vertices v_1 and v_2, we get

$$\left|W^{\Gamma}_{v_2v_1}\right| = 1 + |N_{G_i}(v_2)| - |N_{G_i}(v_2) \cap N_{G_i}(v_1)|.$$

Since G is regular, the number of vertices that are closer to v_1 than to v_2 in Γ equals the number of vertices that are closer to v_2 than to v_1 in Γ.

Let now G_i and G_j be arbitrary different connected components of a disconnected graph G. Pick arbitrary $v_1 \in V(G_i)$ and $v_2 \in V(G_j)$. Obviously these two vertices are at distance 2 in Γ. Observe that

$$W^{\Gamma}_{v_1v_2} = \{v_1\} \cup N_{G_i}(v_1) \quad \text{and} \quad W^{\Gamma}_{v_2v_1} = \{v_2\} \cup N_{G_j}(v_2).$$

Since every connected component of a k-regular graph is also a k-regular (induced) subgraph, it follows that

$$\left|W^{\Gamma}_{v_1v_2}\right| = 1 + k \quad \text{and} \quad \left|W^{\Gamma}_{v_2v_1}\right| = 1 + k,$$

where k is the valency of G. So the number of vertices that are closer to v_1 than to v_2 in Γ equals the number of vertices that are closer to v_2 than to v_1 in Γ. Since this is true for an arbitrary pair of vertices at distance 2 in Γ, this graph is 2-distance-balanced. □

Next we prove that every connected 2-distance-balanced graph, that is not 2-connected, is isomorphic to $\Gamma(G, c)$ for some regular graph G that is not connected.

Theorem 3.3. *Let Γ be a connected 2-distance-balanced graph that is not 2-connected. Then Γ is isomorphic to $\Gamma(G, c)$ for some disconnected regular graph G.*

Proof. Since Γ is not 2-connected, there exists a cut vertex $c \in V(\Gamma)$. Let $G_1, G_2, \ldots, G_n$ be connected components of $G = \Gamma - \{c\}$, $n \geq 2$. We want to prove that G is regular and that the cut vertex c is adjacent to every other vertex in Γ. To do this we will first prove some partial results.

First we claim that the cut vertex c is adjacent to every vertex in a connected component G_ℓ of G for at least one integer ℓ, $1 \leq \ell \leq n$. Suppose that this is not true. Let G_i and G_j be two different connected components of G. Then there exist $v_2 \in V(G_i)$ and $u_2 \in V(G_j)$, both at distance 2 from c in Γ. This means that there exists $v_1 \in V(G_i)$ that is adjacent to c and v_2 in Γ, and there exists $u_1 \in V(G_j)$ that is adjacent to c and u_2 in Γ. If we compare the set of vertices that are closer to c than to v_2 in Γ and the set of vertices that are closer to v_2 than to c in Γ, we get

$$W^{\Gamma}_{cv_2} \supseteq \{c\} \cup V(G_j) \quad \text{and} \quad W^{\Gamma}_{v_2c} \subseteq V(G_i) \setminus \{v_1\}.$$

It follows that

$$1 + |V(G_j)| \le \left|W^{\Gamma}_{cv_2}\right| \quad \text{and} \quad \left|W^{\Gamma}_{v_2c}\right| \le |V(G_i)| - 1.$$

Since Γ is 2-distance-balanced, we get

$$|V(G_j)| \le |V(G_i)| - 2. \tag{3.1}$$

Similarly as above (changing vertex v_2 with u_2) we get

$$|V(G_i)| + 2 \le |V(G_j)|. \tag{3.2}$$

However, inequalities (3.1) and (3.2) imply

$$|V(G_i)| + 2 \le |V(G_j)| \le |V(G_i)| - 2,$$

a contradiction. It follows that the cut vertex $c \in V(\Gamma)$ is adjacent to every vertex in $V(G_\ell)$ for at least one integer ℓ, $1 \le \ell \le n$. Without loss of generality we may assume that $\ell = 1$.

Next we claim that the induced subgraph G_1 of Γ is regular. Pick some $u \in V(G) \setminus V(G_1)$ that is adjacent to the cut vertex c in Γ. Since c is adjacent to every vertex in $V(G_1)$, the distance between u and an arbitrary $v \in V(G_1)$ equals 2 in Γ. Pick $v \in V(G_1)$. Notice that

$$W^{\Gamma}_{vu} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\}).$$

It follows that

$$\left|W^{\Gamma}_{vu}\right| = 1 + |N_{\Gamma}(v)| - 1 = |N_{G_1}(v)| + 1.$$

Pick $w \in V(G_1)$. Since Γ is 2-distance-balanced and c is adjacent to every vertex of $V(G_1)$, we get the following sequence of equalities

$$\begin{aligned} |N_{G_1}(v)| + 1 = |N_{\Gamma}(v)| &= \left|W^{\Gamma}_{vu}\right| = \left|W^{\Gamma}_{uv}\right| = \left|W^{\Gamma}_{uw}\right| \\ &= \left|W^{\Gamma}_{wu}\right| = |N_{\Gamma}(w)| = |N_{G_1}(w)| + 1. \end{aligned}$$

So

$$|N_{G_1}(v)| = |N_{G_1}(w)|$$

for arbitrary $v, w \in V(G_1)$. From now on we may assume that the induced subgraph G_1 of Γ is k-regular. This also means that every vertex in $V(G_1)$ has valency $k+1$ in Γ.

Our next step is to show that the cut vertex $c \in V(\Gamma)$ is adjacent to every vertex in $V(G) \setminus V(G_1)$. Suppose that this is not true. Then there exists some vertex u_2 in a connected component G_ℓ of G, $2 \le \ell \le n$, that is at distance 2 from c in Γ. Without loss of generality we can take $\ell = 2$. Consequently there exists some $u_1 \in V(G_2)$ that is adjacent to both c and u_2 in Γ. Pick an arbitrary $v \in V(G_1)$. We have already proved that the valency of an arbitrary vertex in $V(G_1)$ is $k+1$ in Γ. Now we count vertices that are closer to v than to u_1 in Γ. Since

$$W^{\Gamma}_{vu_1} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\}),$$

we get

$$\left|W^{\Gamma}_{vu_1}\right| = 1 + k + 1 - 1 = k + 1.$$

In addition, for vertices that are closer to c than to u_2 in Γ, we have

$$W^{\Gamma}_{cu_2} \supseteq V(G_1) \cup \{c\}.$$

It follows that

$$\left|W^{\Gamma}_{cu_2}\right| \geq |V(G_1)| + 1 \geq k + 2. \tag{3.3}$$

Consider the distance partition of Γ according to adjacent vertices c and u_1 that is shown in Figure 1. The symbol D^i_j denotes the set of vertices that are at distance i from u_1 and at distance j from c in Γ. Define a set

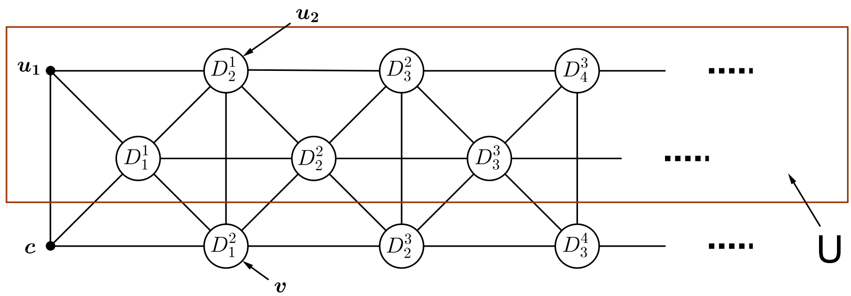

Figure 1: The distance partition of Γ according to adjacent vertices c and u_1.

$$U = \bigcup_{i=1}^{D} = \left(D^{i-1}_i \cup D^i_i\right),$$

where D denotes the diameter of Γ. First we show that $W^{\Gamma}_{u_2c} \subseteq U$. Recall that for $u, v \in V(\Gamma)$, $\partial(u, v)$ denotes the distance between vertices u and v. Pick an arbitrary $w \in W^{\Gamma}_{u_2c}$. Since u_1, u_2 are, by the assumption, adjacent vertices in Γ, the triangle inequality tells us that

$$\partial(u_1, w) \in \{\partial(u_2, w) - 1, \partial(u_2, w), \partial(u_2, w) + 1\}.$$

If we consider all three cases, we get

$$\begin{aligned}
\partial(u_1, w) &= \partial(u_2, w) - 1 < \partial(c, w),\\
\partial(u_1, w) &= \partial(u_2, w) < \partial(c, w),\\
\partial(u_1, w) &= \partial(u_2, w) + 1 \leq \partial(c, w).
\end{aligned}$$

Each considered case gives us that $w \in U$ and so $W^{\Gamma}_{u_2c} \subseteq U$. Note also that $U \subseteq V(G_2)$. Now we show that $U \subseteq W^{\Gamma}_{u_1v}$ (recall that v is an abitrary vertex in $V(G_1)$). Let w be an arbitrary vertex in U, which means that w is also in $V(G_2)$. We get that

$$\partial(u_1, w) \leq \partial(c, w) < \partial(v, w),$$

since vertices v and c are adjacent in Γ, and v is not in $V(G_2)$. It follows that $w \in W^{\Gamma}_{u_1v}$, and so $U \subseteq W^{\Gamma}_{u_1v}$. From relations $W^{\Gamma}_{u_2c} \subseteq U \subseteq W^{\Gamma}_{u_1v}$, we get that $W^{\Gamma}_{u_2c} \subseteq W^{\Gamma}_{u_1v}$ and so

$$\left|W^{\Gamma}_{u_2c}\right| \leq \left|W^{\Gamma}_{u_1v}\right| = \left|W^{\Gamma}_{vu_1}\right| = k + 1. \tag{3.4}$$

By taking into account inequalities (3.3) and (3.4), and since Γ is 2-distance-balanced, we get

$$k+2 \leq |W^{\Gamma}_{cu_2}| = |W^{\Gamma}_{u_2c}| \leq k+1,$$

which is a contradiction. This shows that the cut vertex $c \in V(\Gamma)$ is adjacent to all vertices in $V(G)$.

It remains to prove that the induced subgraph G_ℓ $(2 \leq \ell \leq n)$ of Γ is k-regular. Without loss of generality assume $\ell = 2$. Since we already know that the cut vertex c is adjacent to every vertex in Γ, an arbitrary vertex u in $V(G_2)$ is at distance 2 from an arbitrary vertex v in $V(G_1)$ in Γ. Observe that

$$W^{\Gamma}_{uv} = \{u\} \cup (N_\Gamma(u) \setminus \{c\}) = \{u\} \cup N_{G_2}(u)$$

and

$$W^{\Gamma}_{vu} = \{v\} \cup (N_\Gamma(v) \setminus \{c\}) = \{v\} \cup N_{G_1}(v).$$

This means that

$$\left|W^{\Gamma}_{uv}\right| = 1 + |N_{G_2}(u)| \quad \text{and} \quad \left|W^{\Gamma}_{vu}\right| = 1 + k.$$

Since Γ is 2-distance balanced, it follows that $\left|W^{\Gamma}_{uv}\right| = \left|W^{\Gamma}_{vu}\right|$ and so $|N_{G_2}(u)| = k$ for an arbitrary vertex $u \in V(G_2)$. Therefore, G_2 is regular and has the same valency k as the induced subgraph G_1. It follows that G is regular and this completes the proof. □

The characterization of all connected 2-distance-balanced graphs that are not 2-connected follows immediately from Theorems 3.1, 3.2 and 3.3.

Corollary 3.4. *Let Γ be a connected graph. Then Γ is 2-distance-balanced and not 2-connected if and only if it is isomorphic to $\Gamma(G,c)$ for some disconnected regular graph G.* □

4 2-distance-balanced cartesian product

Throughout this section let G and H be graphs and let $\Gamma = G \Box H$ be the cartesian product of G and H. We characterize connected 2-distance-balanced cartesian products of graphs G and H (see Theorem 4.4). It follows from the definition that the cartesian product Γ is connected if and only if G and H are both connected. In order to avoid trivialities we assume that $|V(G)| \geq 2$ and $|V(H)| \geq 2$.

Recall that

$$\partial_\Gamma((g_1,h_1),(g_2,h_2)) = \partial_G(g_1,g_2) + \partial_H(h_1,h_2) \tag{4.1}$$

for arbitrary $(g_1,h_1),(g_2,h_2) \in V(\Gamma)$. Since we are dealing with 2-distance-balanced cartesian products of graphs, we are interested in vertices at distance 2. It follows from equality (4.1), that there exist three different types of vertices at distance 2 in Γ. We now state these three types and we will refer to them later. Let (g_1,h_1), $(g_2,h_2) \in V(\Gamma)$ be vertices at distance 2 in Γ. We say that these two vertices are of type

- $G2$, if $h_1 = h_2$ and $\partial_G(g_1,g_2) = 2$,
- $H2$, if $g_1 = g_2$ and $\partial_H(h_1,h_2) = 2$,
- $GH2$, if $\partial_G(g_1,g_2) = \partial_H(h_1,h_2) = 1$.

Note that vertices of type $G2$ ($H2$, respectively) do not exist if G (H, respectively) is a complete graph. Denote the set of vertices that are at equal distance from g_1 and g_2 in G by $E^G_{g_1g_2}$, and the set of vertices that are at equal distance from h_1 and h_2 in H by $E^H_{h_1h_2}$.

We first prove three lemmas that we will need later in the proof of the main theorem of this section.

Lemma 4.1. *Let (g_1, h) and (g_2, h) be arbitrary vertices of type $G2$ in $\Gamma = G\square H$. Then*

$$\left|W^\Gamma_{(g_1,h)(g_2,h)}\right| = |H|\left|W^G_{g_1g_2}\right| \quad \text{and} \quad \left|W^\Gamma_{(g_2,h)(g_1,h)}\right| = |H|\left|W^G_{g_2g_1}\right|.$$

Proof. Let (a, x) be an arbitrary vertex of Γ. It follows from the equality (4.1) that

$$\partial_\Gamma((g_1, h),(a, x)) = \partial_G(g_1, a) + \partial_H(h, x)$$

and

$$\partial_\Gamma((g_2, h),(a, x)) = \partial_G(g_2, a) + \partial_H(h, x).$$

So (a, x) is closer to (g_1, h) than to (g_2, h) in Γ if and only if a is closer to g_1 than to g_2 in G. Since $(a, x) \in V(\Gamma)$ was an arbitrary vertex, this means that

$$\left|W^\Gamma_{(g_1,h)(g_2,h)}\right| = |H|\left|W^G_{g_1g_2}\right|.$$

Similarly we get that

$$\left|W^\Gamma_{(g_2,h)(g_1,h)}\right| = |H|\left|W^G_{g_2g_1}\right|.$$

□

Lemma 4.2. *Let (g, h_1) and (g, h_2) be arbitrary vertices of type $H2$ in $\Gamma = G\square H$. Then*

$$\left|W^\Gamma_{(g,h_1)(g,h_2)}\right| = |G|\left|W^H_{h_1h_2}\right| \quad \text{and} \quad \left|W^\Gamma_{(g,h_2)(g,h_1)}\right| = |G|\left|W^H_{h_2h_1}\right|.$$

Proof. Similar to the proof of Lemma 4.1. □

Lemma 4.3. *Let (g_1, h_1) and (g_2, h_2) be arbitrary vertices of type $GH2$ in $\Gamma = G\square H$. Then*

$$\left|W^\Gamma_{(g_1,h_1)(g_2,h_2)}\right| = \left|E^H_{h_1h_2}\right|\left|W^G_{g_1g_2}\right| + \left|W^H_{h_1h_2}\right|\left|W^G_{g_1g_2} \cup E^G_{g_1g_2}\right|$$

and

$$\left|W^\Gamma_{(g_2,h_2)(g_1,h_1)}\right| = \left|E^H_{h_1h_2}\right|\left|W^G_{g_2g_1}\right| + \left|W^H_{h_2h_1}\right|\left|W^G_{g_2g_1} \cup E^G_{g_1g_2}\right|.$$

Proof. Let (a, x) be an arbitrary vertex of Γ. It follows from the equality (4.1) that

$$\partial_\Gamma((g_1, h_1),(a, x)) = \partial_G(g_1, a) + \partial_H(h_1, x) \tag{4.2}$$

and

$$\partial_\Gamma((g_2, h_2),(a, x)) = \partial_G(g_2, a) + \partial_H(h_2, x). \tag{4.3}$$

There are three different cases according to the distance of h_1 and h_2 from x in H.

In the first case let $\partial_H(h_1, x) = \partial_H(h_2, x)$. From equalities (4.2) and (4.3) we get that

$$\partial_\Gamma((g_1, h_1),(a, x)) < \partial_\Gamma((g_2, h_2),(a, x)) \iff \partial_G(g_1, a) < \partial_G(g_2, a).$$

This is true for exactly those $(a, x) \in V(\Gamma)$, for which $a \in W^G_{g_1g_2}$. Similarly we get that

$$\partial_\Gamma((g_2, h_2), (a, x)) < \partial_\Gamma((g_1, h_1), (a, x)) \iff \partial_G(g_2, a) < \partial_G(g_1, a).$$

And this is true for exactly those $(a, x) \in V(\Gamma)$, for which $a \in W^G_{g_2g_1}$.

In the second case let $\partial_H(h_1, x) < \partial_H(h_2, x)$. Since h_1 and h_2 are adjacent in H, it is obvious that $\partial_H(h_2, x) = \partial_H(h_1, x) + 1$. From equalities (4.2) and (4.3) we get that

$$\partial_\Gamma((g_1, h_1), (a, x)) < \partial_\Gamma((g_2, h_2), (a, x)) \iff \partial_G(g_1, a) < \partial_G(g_2, a) + 1.$$

This is true for exactly those $(a, x) \in V(\Gamma)$, for which $a \in W^G_{g_1g_2} \cup E^G_{g_1g_2}$. Similarly we get that

$$\partial_\Gamma((g_2, h_2), (a, x)) < \partial_\Gamma((g_1, h_1), (a, x)) \iff \partial_G(g_2, a) + 1 < \partial_G(g_1, a).$$

But such vertices do not exist, since $\partial_G(g_1, a) \leq \partial_G(g_2, a) + 1$ by the triangle inequality.

In the third case let $\partial_H(h_2, x) < \partial_H(h_1, x)$. Similarly as above we get that (a, x) is closer to (g_2, h_2) that to (g_1, h_1) if and only if $a \in W^G_{g_2g_1} \cup E^G_{g_1g_2}$, and that (a, x) is never closer to (g_1, h_1) that to (g_2, h_2).

It follows from the above comments that

$$W^\Gamma_{(g_1,h_1)(g_2,h_2)} = \left(E^H_{h_1h_2} \times W^G_{g_1g_2}\right) \bigcup \left(W^H_{h_1h_2} \times \left(W^G_{g_1g_2} \cup E^G_{g_1g_2}\right)\right)$$

and

$$W^\Gamma_{(g_2,h_2)(g_1,h_1)} = \left(E^H_{h_1h_2} \times W^G_{g_2g_1}\right) \bigcup \left(W^H_{h_2h_1} \times \left(W^G_{g_2g_1} \cup E^G_{g_1g_2}\right)\right).$$

The result follows. □

Next theorem gives the characterization of connected 2-distance-balanced cartesian products of graphs G and H.

Theorem 4.4. *The cartesian product $\Gamma = G \square H$ is a connected 2-distance-balanced graph if and only if each of G, H is either a connected 2-distance-balanced and 1-distance-balanced graph, or a complete graph.*

Proof. We first prove that if each of G, H is either a connected 2-distance-balanced and 1-distance-balanced graph or a complete graph, then Γ is a connected 2-distance-balanced graph.

Let us assume that G and H are connected 2-distance-balanced and 1-distance-balanced graphs. The connectivity of Γ follows from the connectivity of G and H. In this case all three types of vertices at distance 2 are present in Γ.

Let (g_1, h) and (g_2, h) be arbitrary vertices of type $G2$ in Γ. Since G is, by the assumption, 2-distance-balanced and since vertices g_1, g_2 are at distance 2 in G, it follows from Lemma 4.1 that

$$\left|W^\Gamma_{(g_1,h)(g_2,h)}\right| = |H|\left|W^G_{g_1g_2}\right| = |H|\left|W^G_{g_2g_1}\right| = \left|W^\Gamma_{(g_2,h)(g_1,h)}\right|.$$

So for arbitrary vertices (g_1, h), $(g_2, h) \in V(\Gamma)$ of type $G2$, the number of vertices that are closer to (g_1, h) than to (g_2, h) in Γ equals the number of vertices that are closer to (g_2, h) than to (g_1, h) in Γ.

If (g, h_1) and (g, h_2) are arbitrary vertices of type $H2$ in Γ, then similarly as above (using Lemma 4.2 instead of Lemma 4.1) we find that the number of vertices that are closer to (g, h_1) than to (g, h_2) in Γ equals the number of vertices that are closer to (g, h_2) than to (g, h_1) in Γ.

Let $(g_1, h_1), (g_2, h_2) \in V(\Gamma)$ be arbitrary vertices of type $GH2$ in Γ. Since G and H are both, by the assumption, 1-distance-balanced, and since g_1, g_2 are adjacent in G and h_1, h_2 are adjacent in H, we have

$$\left|W^G_{g_1g_2}\right| = \left|W^G_{g_2g_1}\right| \quad \text{and} \quad \left|W^H_{h_1h_2}\right| = \left|W^H_{h_2h_1}\right|.$$

It follows from Lemma 4.3 that

$$\left|W^\Gamma_{(g_1,h_1)(g_2,h_2)}\right| = \left|W^\Gamma_{(g_2,h_2)(g_1,h_1)}\right|$$

for arbitrary vertices of type $GH2$ in Γ. So we proved that if G and H are both connected 2-distance-balanced and 1-distance-balanced graphs, then the cartesian product $\Gamma = G \Box H$ is a connected 2-distance-balanced graph. Note that since G and H are 1-distance-balanced graphs, it follows that the cartesian product $\Gamma = G \Box H$ is also 1-distance-balanced (see [12, Proposition 4.1]).

If one (or both) of G, H is a complete graph, then the proof that $\Gamma = G \Box H$ is a connected 2-distance balanced graph is similar to the proof above. The only diference is that we do not have to consider vertices of type $G2$ ($H2$, respectively).

Assume now that $\Gamma = G \Box H$ is a connected 2-distance-balanced graph. The connectivity of G and H follows from the connectivity of Γ. If G and H are complete graphs, then we are done. Therefore we assume that at least one of G or H is not a complete graph. First we show that in this case G and H are 2-distance-balanced graphs provided they are not complete.

Assume that G is not a complete graph. For an arbitrary $h \in V(H)$ and arbitrary $g_1, g_2 \in V(G)$ that are at distance 2 in G, consider $(g_1, h), (g_2, h) \in V(\Gamma)$. Note that $\partial_\Gamma((g_1, h), (g_2, h)) = 2$ by (4.1) and that

$$\left|W^\Gamma_{(g_1,h)(g_2,h)}\right| = |H| \left|W^G_{g_1g_2}\right| \quad \text{and} \quad \left|W^\Gamma_{(g_2,h)(g_1,h)}\right| = |H| \left|W^G_{g_2g_1}\right|$$

by Lemma 4.1. Since Γ is 2-distance-balanced, it follows that $\left|W^G_{g_1g_2}\right| = \left|W^G_{g_2g_1}\right|$, so also G is a 2-distance-balanced graph. Due to commutativity of the cartesian product, if H is not a complete graph, we can similarly show that H is a 2-distance-balanced graph.

Finally we show that G and H are also 1-distance-balanced graphs. Pick arbitrary adjacent vertices g_1, g_2 of G and arbitrary adjacent vertices h_1, h_2 of H, and note that (g_1, h_1), $(g_2, h_2) \in V(\Gamma)$ are at distance 2. Since Γ is 2-distance-balanced, it follows that

$$\left|W^\Gamma_{(g_1,h_1)(g_2,h_2)}\right| = \left|W^\Gamma_{(g_2,h_2)(g_1,h_1)}\right|.$$

From Lemma 4.3 we get that

$$\begin{aligned}\left|E^H_{h_1h_2}\right| \left(\left|W^G_{g_1g_2}\right| - \left|W^G_{g_2g_1}\right|\right) = \left|W^H_{h_2h_1}\right| \left|W^G_{g_2g_1} \cup E^G_{g_1g_2}\right| \\ - \left|W^H_{h_1h_2}\right| \left|W^G_{g_1g_2} \cup E^G_{g_1g_2}\right|. \quad (4.4)\end{aligned}$$

Assume that G is not a 1-distance-balanced graph. Then we could choose g_1, g_2 in such a way that $\left|W^G_{g_1g_2}\right| > \left|W^G_{g_2g_1}\right|$. As a consequence we also have that

$$\left|W^G_{g_1g_2} \cup E^G_{g_1g_2}\right| > \left|W^G_{g_2g_1} \cup E^G_{g_1g_2}\right|.$$

It follows from (4.4) that $\left|W^H_{h_2h_1}\right| > \left|W^H_{h_1h_2}\right|$. Consider now vertices $(g_1, h_2), (g_2, h_1)$, which are also at distance 2 in Γ. Similar argument as above shows that $\left|W^H_{h_2h_1}\right| < \left|W^H_{h_1h_2}\right|$, which is a contradiction. So G is a 1-distance balanced graph. Since the cartesian product is commutative, the proof that H is a 1-distance balanced graph is analogous to the proof for G. □

5 2-distance-balanced lexicographic product

Throughout this section let G and H be graphs and let $\Gamma = G[H]$ be the lexicographic product of G and H. It follows from the definition that the lexicographic product Γ is connected if and only if G is connected. In order to avoid trivialities we assume that $|V(G)| \geq 2$ and $|V(H)| \geq 2$. We characterize connected 2-distance-balanced lexicographic products of G and H (see Theorem 5.4).

Remark 5.1. A more general result about the characterization of connected n-distance-balanced lexicographic products of G and H as in Theorem 5.4 is stated in [7, Theorem 3.4]. But the result is not correct for at least $n = 2$. As a counterexample, let both G and H be paths on 3 vertices, which are connected graphs. Observe that G is 2-distance-balanced, and that H is locally regular (in a sense that any non-adjacent vertices in H have the same number of neighbours). By [7, Theorem 3.4], $G[H]$ is 2-distance-balanced. However, one can easily check that the $G[H]$ is not 2-distance-balanced.

Notice that there exist two different types of vertices at distance 2 in Γ. We now state these two types and we will refer to them in the proof of the Theorem 5.4. Let (g_1, h_1), $(g_2, h_2) \in V(\Gamma)$ be vertices at distance 2 in Γ. We say that this two vertices are of type

- $G2$, if $\partial_G(g_1, g_2) = 2$,
- $H2$, if $g_1 = g_2$ and $\partial_H(h_1, h_2) \geq 2$.

It follows from the definition that there exist vertices of type $G2$ in Γ if and only if G is connected non-complete graph. Similarly, there exist vertices of type $H2$ in Γ if and only if H is non-complete graph.

The following two lemmas will be used in the proof of the main theorem of this section.

Lemma 5.2. *Let (g_1, h_1) and (g_2, h_2) be arbitrary vertices of type $G2$ in $\Gamma = G[H]$. Then*

$$\left|W^\Gamma_{(g_1,h_1)(g_2,h_2)}\right| = 1 + |N_H(h_1)| + \left(\left|W^G_{g_1g_2}\right| - 1\right)|V(H)|$$

and

$$\left|W^\Gamma_{(g_2,h_2)(g_1,h_1)}\right| = 1 + |N_H(h_2)| + \left(\left|W^G_{g_2g_1}\right| - 1\right)\left|V(H)\right|.$$

Proof. Let (g_1, h_1) and (g_2, h_2) be arbitrary vertices of type $G2$ in Γ. Clearly, (g_1, h_1) is closer to itself than to (g_2, h_2). Now consider vertices of Γ of type (g_1, h), where $h \neq h_1$. Note that $\partial_\Gamma((g_1, h), (g_2, h_2)) = 2$, and so $(g_1, h) \in W^\Gamma_{(g_1,h_1)(g_2,h_2)}$ if and only if $h \in N_H(h_1)$. Finally, consider vertices of Γ of type (g, h), where $g \neq g_1$. Then $\partial_\Gamma((g_1, h_1), (g, h)) = \partial_G(g_1, g)$, and so $(g, h) \in W^\Gamma_{(g_1,h_1)(g_2,h_2)}$ if and only if $g \in W^G_{g_1 g_2} \setminus \{g_1\}$. It follows that

$$W^\Gamma_{(g_1,h_1)(g_2,h_2)} = \{(g_1, h_1)\} \cup (\{g_1\} \times N_H(h_1)) \cup ((W^G_{g_1 g_2} \setminus \{g_1\}) \times V(H)).$$

Similarly we get

$$W^\Gamma_{(g_2,h_2)(g_1,h_1)} = \{(g_2, h_2)\} \cup (\{g_2\} \times N_H(h_2)) \cup ((W^G_{g_2 g_1} \setminus \{g_2\}) \times V(H)).$$

The result follows. □

Lemma 5.3. *Let (g, h_1) and (g, h_2) be arbitrary vertices of type $H2$ in $\Gamma = G[H]$. Then*

$$\left|W^\Gamma_{(g,h_1)(g,h_2)}\right| = 1 + |N_H(h_1)| - |N_H(h_1) \cap N_H(h_2)|$$

and

$$\left|W^\Gamma_{(g,h_2)(g,h_1)}\right| = 1 + |N_H(h_2)| - |N_H(h_1) \cap N_H(h_2)|.$$

Proof. Let (g, h_1) and (g, h_2) be arbitrary vertices of type $H2$ in Γ, and let (g', h') be an arbitrary vertex of Γ. Note that if $g \neq g'$ then $\partial_\Gamma((g, h_1), (g', h')) = \partial_\Gamma((g, h_2), (g', h'))$. Assume therefore that $g' = g$. But it is clear that in this case $(g, h') \in W^\Gamma_{(g,h_1)(g,h_2)}$ if and only if $\partial_H(h_1, h') \leq 1 < \partial_H(h_2, h')$. It follows that

$$W^\Gamma_{(g,h_1)(g,h_2)} = \{(g, h_1)\} \cup (\{g\} \times (N_H(h_1) \setminus (N_H(h_1) \cap N_H(h_2)))).$$

Similarly we get

$$W^\Gamma_{(g,h_2)(g,h_1)} = \{(g, h_2)\} \cup (\{g\} \times (N_H(h_2) \setminus (N_H(h_1) \cap N_H(h_2)))).$$

The result follows. □

Next theorem gives the characterization of connected 2-distance-balanced lexicographic products of graphs G and H.

Theorem 5.4. *The lexicographic product $\Gamma = G[H]$ is a connected 2-distance-balanced graph if and only if one of the following (i), (ii) holds:*

(i) G is a connected 2-distance-balanced graph and H is a regular graph.

(ii) G is a complete graph, H is not a complete graph, and each connected component of the complement of H induces a regular subgraph of the complement of H.

Proof. We first prove that if one of (i), (ii) holds, then Γ is a connected 2-distance-balanced graph. The connectivity of Γ follows from the connectivity of G.

Assume that (i) holds. Take arbitrary (g_1, h_1), $(g_2, h_2) \in V(\Gamma)$ of type $G2$. Since G is a 2-distance-balanced graph and H is a regular graph, we have that $\left|W^G_{g_1 g_2}\right| = \left|W^G_{g_2 g_1}\right|$ and $|N_H(h_1)| = |N_H(h_2)|$. It follows from Lemma 5.2 that

$$\left|W^\Gamma_{(g_1,h_1)(g_2,h_2)}\right| = \left|W^\Gamma_{(g_2,h_2)(g_1,h_1)}\right|$$

for arbitrary vertices of type $G2$ in Γ.

Take now arbitrary (g, h_1), $(g, h_2) \in V(\Gamma)$ of type $H2$. Since, by the assumption, H is a regular graph, we have that $|N_H(h_1)| = |N_H(h_2)|$. It follows from Lemma 5.3 that

$$\left|W^{\Gamma}_{(g,h_1)(g,h_2)}\right| = \left|W^{\Gamma}_{(g,h_2)(g,h_1)}\right|$$

for arbitrary vertices of type $H2$ in Γ. So, if (i) holds then Γ is a connected 2-distance-balanced graph.

Assume that (ii) holds. Then G is a complete graph and H is not a complete graph, so we only have vertices of type $H2$ in Γ. Let us denote the complement of H by $\overline{H}$. Let (g, h_1), $(g, h_2) \in V(\Gamma)$ be arbitrary vertices of type $H2$. Note that this implies that h_1, h_2 are not adjacent in H, and so h_1, h_2 are adjacent in $\overline{H}$. As a consequence, h_1, h_2 are contained in the same connected component of $\overline{H}$. It follows that $|N_{\overline{H}}(h_1)| = |N_{\overline{H}}(h_2)|$, and consequently also $|N_H(h_1)| = |N_H(h_2)|$. It follows from Lemma 5.3 that

$$\left|W^{\Gamma}_{(g,h_1)(g,h_2)}\right| = \left|W^{\Gamma}_{(g,h_2)(g,h_1)}\right|.$$

So, if (ii) holds then Γ is a connected 2-distance-balanced graph.

Assume now that the lexicographic product $\Gamma = G[H]$ is a connected 2-distance-balanced graph. The connectivity of G follows from the connectivity of Γ. In what follows we first treat the case where G is not a complete graph, and then the case when G is a complete graph.

Suppose that G is not a complete graph. Take arbitrary $g_1, g_2 \in V(G)$ at distance 2 in G. Then $(g_1, h), (g_2, h) \in V(\Gamma)$ are of type $G2$ in Γ for an arbitrary $h \in V(H)$. Since Γ is, by the assumption, a 2-distance-balanced graph, it follows from Lemma 5.2 that $\left|W^G_{g_1g_2}\right| = \left|W^G_{g_2g_1}\right|$ for arbitrary vertices at distance 2 in G. So, G is a connected 2-distance-balanced graph. For arbitrary $h_1, h_2 \in V(H)$ and arbitrary $g_1, g_2 \in V(G)$ at distance 2 in G, consider $(g_1, h_1), (g_2, h_2) \in V(\Gamma)$. These two vertices are of type $G2$ in Γ. Since Γ is, by the assumption, a 2-distance-balanced graph and we already know that G is also 2-distance-balanced graph, it follows from Lemma 5.2 that $|N_H(h_1)| = |N_H(h_2)|$ for arbitrary two vertices in H. So, H is a regular graph and (i) holds.

From now on let G be a complete graph. Since Γ is not a complete graph, it follows that also H is not a complete graph. This means that all vertices at distance 2 in Γ are of type $H2$. We want to show that in this case each connected component of the complement of H induces a regular subgraph of the complement of H.

Let $h_1, h_2 \in V(H)$ be arbitrary vertices at distance greater or equal than 2 in H (that is, vertices h_1, h_2 are not adjacent in H). Observe that $(g, h_1), (g, h_2) \in V(\Gamma)$ are of type $H2$ for an arbitrary $g \in V(G)$. From Lemma 5.3 we get that $|N_H(h_1)| = |N_H(h_2)|$, and consequently also $|N_{\overline{H}}(h_1)| = |N_{\overline{H}}(h_2)|$. This shows that any adjacent vertices of $\overline{H}$ have the same valency in $\overline{H}$, and therefore each connected component of $\overline{H}$ induces a regular subgraph of $\overline{H}$. □

We finish our paper with a suggestion for further research. A fullerene is a cubic planar graph having all faces 5- or 6-cycles. Examples include the dodecahedron and generalized Petersen graph $\mathrm{GP}(12, 2)$. Dodecahedron is distance-regular, and so it is n-distance-balanced for every $1 \leq n \leq 5$ (recall that the diameter of dodecahedron is 5). On the other hand, the diameter of $\mathrm{GP}(12, 2)$ is also 5, but $\mathrm{GP}(12, 2)$ is n-distance-balanced only for $n = 5$, see [17]. Therefore, it would be interesting to know, which fullerenes are n-distance-balanced at least for some values of n (for example, for $n \in \{1, 2, D\}$, where D is the diameter of a fullerene in question). For more on fullerenes see [2, 5, 18].

References

[1] A. Abiad, B. Brimkov, A. Erey, L. Leshock, X. Martinez-Rivera, S. O, S. Y. Song and J. Williford, On the Wiener index, distance cospectrality and transmission-regular graphs, *Discrete Appl. Math.* **230** (2017), 1–10, doi:10.1016/j.dam.2017.07.010.

[2] V. Andova, F. Kardoš and R. Škrekovski, Mathematical aspects of fullerenes, *Ars Math. Contemp.* **11** (2016), 353–379, https://amc-journal.eu/index.php/amc/article/view/834.

[3] K. Balakrishnan, B. Brešar, M. Changat, S. Klavžar, A. Vesel and P. Žigert Pleteršek, Equal opportunity networks, distance-balanced graphs, and Wiener game, *Discrete Optim.* **12** (2014), 150–154, doi:10.1016/j.disopt.2014.01.002.

[4] K. Balakrishnan, M. Changat, I. Peterin, S. Špacapan, P. Šparl and A. R. Subhamathi, Strongly distance-balanced graphs and graph products, *European J. Combin.* **30** (2009), 1048–1053, doi:10.1016/j.ejc.2008.09.018.

[5] A. Behmaram, T. Došlić and S. Friedland, Matchings in m-generalized fullerene graphs, *Ars Math. Contemp.* **11** (2016), 301–313, https://amc-journal.eu/index.php/amc/article/view/882.

[6] S. Cabello and P. Lukšič, The complexity of obtaining a distance-balanced graph, *Electron. J. Combin.* **18** (2011), #P49, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v18i1p49.

[7] M. Faghani, E. Pourhadi and H. Kharazi, On the new extension of distance-balanced graphs, *Trans. Comb.* **5** (2016), 21–34, doi:10.22108/toc.2016.15048.

[8] B. Frelih, *Različni vidiki povezave regularnosti v grafih*, Ph.D. thesis, University of Primorska, 2014, in Slovene.

[9] K. Handa, Bipartite graphs with balanced (a, b)-partitions, *Ars Combin.* **51** (1999), 113–119, http://www.combinatorialmath.ca/arscombinatoria/vol51.html.

[10] T. Hilado and K. Nomura, Distance degree regular graphs, *J. Comb. Theory Ser. B* **37** (1984), 96–100, doi:10.1016/0095-8956(84)90050-9.

[11] A. Ilić, S. Klavžar and M. Milanović, On distance-balanced graphs, *European J. Combin.* **31** (2010), 733–737, doi:10.1016/j.ejc.2009.10.006.

[12] J. Jerebic, S. Klavžar and D. F. Rall, Distance-balanced graphs, *Ann. Comb.* **12** (2008), 71–79, doi:10.1007/s00026-008-0337-2.

[13] K. Kutnar, A. Malnič, D. Marušič and Š. Miklavič, Distance-balanced graphs: Symmetry conditions, *Discrete Math.* **306** (2006), 1881–1894, doi:10.1016/j.disc.2006.03.066.

[14] K. Kutnar, A. Malnič, D. Marušič and Š. Miklavič, Matchings in m-generalized fullerene graphs, *Ars Math. Contemp.* **2** (2009), 41–47, https://amc-journal.eu/index.php/amc/article/view/75.

[15] K. Kutnar and Š. Miklavič, Nicely distance-balanced graphs, *European J. Combin.* **39** (2014), 57–67, doi:10.1016/j.ejc.2013.12.002.

[16] Š. Miklavič and P. Šparl, On the connectivity of bipartite distance-balanced graphs, *European J. Combin.* **33** (2012), 237–247, doi:10.1016/j.ejc.2011.10.002.

[17] Š. Miklavič and P. Šparl, ℓ-distance-balanced graphs, 2017, arXiv:1702.05257 [math.CO].

[18] N. Tratnik and P. Žigert Pleteršek, Resonance graphs of fullerenes, *Ars Math. Contemp.* **11** (2016), 425–435, https://amc-journal.eu/index.php/amc/article/view/1000.