

Vertex transitive maps on the Klein bottle

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Abstract

We examine properties of groups associated with vertex transitive maps on the Klein bottle. As an application, we prove that all 4-line vertex transitive maps on the Klein bottle introduced by Babai admit a free vertex transitive action, but there are others that do not.

Keywords: Klein bottle, wallpaper group, vertex-transitive map.

Math. Subj. Class.: 57M60, 05C10; 20H15

1 Introduction

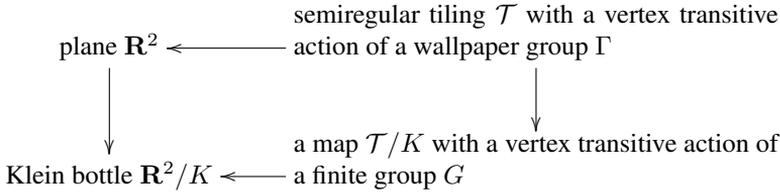
This paper is motivated by a question raised in [8], namely whether all vertex transitive maps on the Klein bottle admit a *free* vertex-transitive action as is the case on torus. Vertex transitive maps on the Klein bottle and groups acting on them have been considered in the works of Babai [1] and Thomassen [9]. In this work we take a different approach, working exclusively on the universal covering space of the Klein bottle, the plane.

Suppose a map M on the Klein bottle KB is given with a vertex-transitive action, by a group G . The map lifts to the universal covering space [1, p. 610], where one obtains a vertex transitive action [3, 4, 5] of a wallpaper group [12] Γ on a semiregular tiling [6, 11] \mathcal{T} , together with a normal subgroup K (isomorphic to pg) of transformations preserving the canonical map $\mathbf{R}^2 \rightarrow KB$. One can recover G as the quotient Γ/K . The action of G is free if and only if the action of Γ is free.

In this paper we aim to provide an insight into the groups involved (Proposition 2.1, Table 1 and 2) and resolve the question of free action (Proposition 5.1).

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2 KBU groups

The group pg is central in our work. It can be introduced in several equivalent ways.

- geometrically by giving two parallel glides P, Q of equal length,
- geometrically by giving a glide g and an orthogonal translation t ,
- as an abstract group $\langle P, Q; P^2 = Q^2 \rangle$,
- or as an abstract group $\langle g, t; gtg^{-1} = t^{-1} \rangle$.

By a *wallpaper* or *plane crystallographic* group one means a group having a discrete action on the plane, and containing two linearly independent translations. There are 17 isomorphism types of wallpaper groups [2]. Not every wallpaper group gives rise to an action on the Klein bottle, because some have no glides, and hence no subgroup isomorphic to pg . We shall call a *KBU group* a wallpaper group that contains a normal subgroup isomorphic to pg .

Proposition 2.1. *A wallpaper group Γ is a KBU group if and only if Γ is isomorphic to one of the following groups: $pm, pg, cm, pmm, pmg, pgg$.*

Proof. Suppose Γ is a KBU group. Clearly, Γ cannot be $p1$ or $p2$, because these groups do not contain any glides. Suppose Γ contains a rotation r of degree > 2 . If g is a glide from the normal subgroup K isomorphic to pg , then g fixes a line L . Using the elementary Lemma 2.2 we see that rgr^{-1} fixes $r \cdot L$. It follows that K contains non-parallel glides, and hence a rotation. This contradicts the fact that pg is torsion-free.

It remains to show that the group listed have normal subgroups isomorphic to pg . Such subgroups for all of them are listed in Table 2. □

Lemma 2.2. Suppose group G acts on a set X . Suppose a subgroup H leaves a subset Y fixed. If g is any element of G then the subgroup gHg^{-1} leaves a subset gY fixed.

Let us recall presentations of KBU groups from [2]:

$$\begin{aligned}
 pm &:= \langle X, Y, R; XY = YX, RY = YR, (RX)^2 = R^2 = 1 \rangle \\
 pg &:= \langle P, Q; P^2 = Q^2 \rangle \\
 cm &:= \langle R, S; (RS)^2 = (SR)^2, R^2 = 1 \rangle \\
 pmm &:= \langle R_1, R_2, R_3, R_4; R_1^2 = R_2^2 = R_3^2 = R_4^2 = 1, \\
 &\quad (R_1R_2)^2 = (R_2R_3)^2 = (R_3R_4)^2 = (R_4R_1)^2 = 1 \rangle \\
 pmg &:= \langle R, T_1, T_2; R^2 = T_1^2 = T_2^2 = 1, T_1RT_1 = T_2RT_2 \rangle \\
 pgg &:= \langle P, O; (PO)^2 = (P^{-1}O)^2 = 1 \rangle \\
 cmm &:= \langle T, R_1, R_2; T^2 = R_1^2 = R_2^2 = 1, (R_1R_2)^2 = (R_1TR_2T)^2 = 1 \rangle
 \end{aligned}$$

Even though the groups are given in terms of presentations, their elements can be classified in geometric terms. Let g be an element of Γ . If the centralizer $Z(g)$ of g is a free abelian group of rank 2, then g is a *translation*. The translations form a free abelian subgroup $T(\Gamma)$ of rank two, of finite index (≤ 4) in Γ . Suppose next that the intersection $Z(g) \cap T(g)$ is a free abelian group of rank 1. Then if g is an involution, then g is a *mirror*, otherwise g is a *glide* (and g is of infinite order). In the remaining cases g is a *rotation*.

2.1 Translations and other elements in KBU groups

Geometric representations of KBU groups can be seen in Figure 1 with indicated generators. Each picture shows the basis of the translation subgroup (the bounding parallelogram), glides (dotted lines), mirrors (thick lines) and rotations by angle π (diamonds). We will now summarize the structure of the translation subgroups of the above seven classes of wallpaper groups.

2.1.1 Elements of pm

The group is generated by three elements: a pair of orthogonal translations X and Y , and a mirror in the direction of Y . The translation subgroup is generated by X and Y and is of index 2 in pm . There is a coset decomposition $pm = T(pm) \cup R \cdot T(pm)$. Elements of form RX^kY^l are mirrors and glides according to whether $l = 0$ or not. From defining relations it follows that

$$RX = X^{-1}R \qquad RY = YR$$

2.1.2 Elements of pg

The group is generated by a pair of parallel glides P and Q satisfying $P^2 = Q^2$. The translation subgroup is generated by P^2 and $P^{-1}Q$ and is of index 2 in pg . There is a coset decomposition $pg = T(pg) \cup P \cdot T(pg)$. Elements of form $P \cdot P^{2l}(P^{-1}Q)^k$ are all glides. We have

$$P \cdot P^2 = P^2 \cdot P \qquad P(P^{-1}Q) = (P^{-1}Q)^{-1}P$$

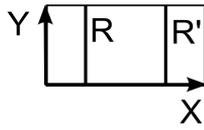
2.1.3 Elements of cm

The group is generated by a translation S and a mirror R , which are neither parallel, nor orthogonal. The translation subgroup is generated by S and RSR and is of index 2 in cm . There is a coset decomposition $cm = T(cm) \cup R \cdot T(cm)$. Elements of form $RS^k(RSR)^l$ are mirrors and glides according to whether $k = -l$ or not. From defining relations it follows

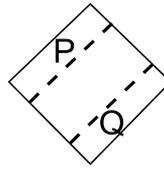
$$R \cdot S = RSR \cdot R \qquad R \cdot RSR = S \cdot R$$

2.1.4 Elements of pmm

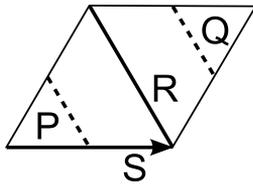
This group is generated by four mirrors R_1-R_4 forming the sides of a square. The translation subgroup is generated by R_1R_3 and R_2R_4 , and is of index 4 in pmm . There is a coset decomposition $pmm = T(pmm) \cup R_1 \cdot T(pmm) \cup R_2 \cdot T(pmm) \cup R_1R_2 \cdot T(pmm)$. All



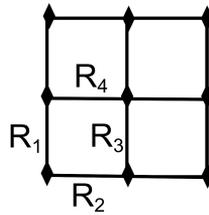
(a) Group pm



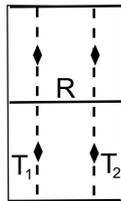
(b) Group pg



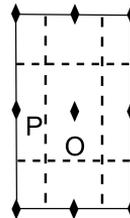
(c) Group cm



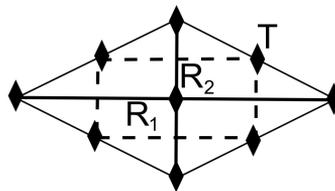
(d) Group pmm



(e) Group pmg



(f) Group pgg



(g) Group cmm

Figure 1: KBU groups.

elements of the last coset are rotations. Elements of the form $R_1(R_1R_3)^k$ and $R_2(R_2R_4)^k$ are mirrors, and the remaining non-translations are glides. From the defining relations we have:

$$\begin{aligned} R_1(R_2R_4) &= (R_2R_4)R_1 & R_1(R_1R_3) &= (R_1R_3)^{-1}R_1 \\ R_2(R_2R_4) &= (R_2R_4)^{-1}R_2 & R_2(R_1R_3) &= (R_1R_3)R_2 \\ (R_1R_2)(R_2R_4) &= (R_2R_4)^{-1}(R_1R_2) & (R_1R_2)(R_1R_3) &= (R_1R_3)^{-1}(R_1R_2) \end{aligned}$$

2.1.5 Elements of pmg

This group is generated by a mirror R and two translations T_1, T_2 lying on a line parallel to it. The translation subgroup is generated by $(T_1R)^2$ and T_1T_2 , and is of index 4 in pmg . There is a coset decomposition $pmg = T(pmg) \cup R \cdot T(pmg) \cup T_1 \cdot T(pmg) \cup RT_1 \cdot T(pmg)$. We have

$$\begin{aligned} R(T_1T_2) &= (T_1T_2)R & (T_1R)^2 \cdot R &= R \cdot (T_1R)^{-2} \\ T_1(T_1T_2) &= (T_1T_2)^{-1}T_1 & T_1(T_1R)^2 &= (T_1R)^{-2}T_1 \\ (RT_1)(T_1T_2) &= (T_1T_2)^{-1}(RT_1) & (RT_1)(T_1R)^2 &= (T_1R)^2(RT_1) \end{aligned}$$

2.1.6 Elements of pgg

This group is generated by a pair of orthogonal glides O and P . The translation subgroup is generated by P^2 and O^2 . There is a coset decomposition $pgg = T(pgg) \cup O \cdot T(pgg) \cup P \cdot T(pgg) \cup (OP) \cdot T(pgg)$. We have

$$\begin{aligned} P \cdot P^2 &= P^2 \cdot P & P \cdot O^2 &= O^{-2} \cdot P \\ O \cdot P^2 &= P^{-2} \cdot O & O \cdot O^2 &= O^2 \cdot O \\ OP \cdot P^2 &= P^{-2} \cdot OP & OP \cdot O^2 &= O^{-2} \cdot OP \end{aligned}$$

2.1.7 Elements of $cm m$

This group is generated by a pair of orthogonal mirrors R_1, R_2 and a rotation not lying on any of them. The translation subgroup is generated by R_1R_2T and R_2TR_1 . We have a coset decomposition $cm m = T(cm m) \cup R_1 \cdot T(cm m) \cup R_2 \cdot T(cm m) \cup T \cdot T(cm m)$. We have

$$R_1(R_1R_2T) = (R_2TR_1)R_1 \qquad R_1(R_2TR_1) = (R_1R_2T)R_1 \qquad (2.1)$$

$$R_2(R_1R_2T) = (R_2TR_1)^{-1}R_2 \qquad R_2(R_2TR_1) = (R_1R_2T)^{-1}R_2 \qquad (2.2)$$

$$T(R_1R_2T)T = (R_1R_2T)^{-1}T \qquad T(R_2TR_1) = (R_2TR_1)^{-1}T \qquad (2.3)$$

3 Ten families of glides of KBU groups

In order to analyze normal subgroups of KBU groups isomorphic with pg it is first necessary to understand the set of glides in these wallpaper groups.

In Table 1 we list ten families of glides parametrized by positive integers $n > 0$. Using the facts from the previous section it is straightforward to verify that the elements g

listed there are glides, that is they are not involutions, and centralize a rank 1 subgroup of translations. The subgroup of translations is in fact generated by the corresponding parallel translation.

Family	Group G	glide g	orthogonal translation	parallel translation
M1	pm	RY^n	X	Y
M2	pmm	$R_1(R_2R_4)^n$	R_1R_3	R_2R_4
M3	pmg	$R(T_1T_2)^n$	$(T_1R)^2$	T_1T_2
M4	cm	$R(RS)^{2n}$	$RSRS^{-1}$	$RSRS$
M5	cmm	$R_1(TR_2)^{2n}$	$(TR_1)^2$	$(TR_2)^2$
G1	pg	P^{2n-1}	$(PQ^{-1})^2$	P^2
G2	pmg	$(T_1R)^{2n-1}$	T_1T_2	$(T_1R)^2$
G3	pgg	O^{2n-1}	P^2	O^2
G4	cm	$(RS)^{2n-1}$	$RSRS^{-1}$	$RSRS$
G5	cmm	$(R_1T)^{2n-1}$	$(R_2T)^2$	$(R_1T)^2$

Table 1: Glides in KBU groups.

Take for instance the glide $g = (T_1R)^{2n-1}$ in family G2. The translation subgroup of pmg is generated by $(T_1R)^2$ and T_1T_2 . It is clear that g commutes with $(T_1R)^2$, and using equations from 2.1.5 we check that it anticommutes with T_1T_2 :

$$\begin{aligned}
 g(T_1T_2)g^{-1} &= (T_1R)^{2n-1}(T_1T_2)(RT_1)^{2n-1} \\
 &= (T_1R)^{2n-1}(RT_1)^{2n-1}(T_1T_2)^{(-1)^{2n-1}} \\
 &= (T_1T_2)^{-1}
 \end{aligned}$$

Since $g^2 = ((T_1R)^2)^{2n-1}$ we see that g is not an involution, and thus is not a mirror.

The list provided in Table 1 is in a way exhaustive.

Proposition 3.1. *If γ is a glide in KBU group Γ , then there exists $n \geq 1$ and an isomorphism $\Gamma \rightarrow G$, such that γ maps to one of the glides g listed in Table 1.*

Proof. Suppose γ is a glide. Clearly, γ cannot belong to $T(\Gamma)$. If γ belonged to $\alpha \cdot T(\Gamma)$ for some rotation α , then $\gamma^2 = 1$, which contradicts the fact that all glides are of infinite order. Let us now distinguish several cases.

Suppose next Γ admits an orthogonal basis of $T(\Gamma)$. This is true for groups pm, pg, pmm, pmg, pgg . As shown in 2.1 we need to distinguish two cases.

Firstly suppose $\gamma = RX^kY^l$, where X and Y is an orthogonal basis of $T(\Gamma)$ such that

$$\begin{aligned}
 RX &= X^{-1}R & RY &= YR
 \end{aligned}$$

Then $\gamma^2 = Y^{2l}$ and thus γ is a glide if and only if $l \neq 0$. The next step is to show that there exists an automorphism of Γ such that RY^l is mapped to RY^{-l} . In the case of groups pmm, pmg, pgg we can take it to be conjugation by any rotation. For groups pm and pg , any rotation of plane of order 2 accomplishes the same, but in these cases it is an outer automorphism. Conjugating RY^l by X^s we obtain

$$\begin{aligned}
 X^sRX^kY^lX^{-s} &= RX^{k-2s}Y^l & X^{-s}RX^kY^lX^s &= RX^{k+2s}Y^l
 \end{aligned}$$

Thus modulo inner automorphisms there are at most two classes of glides in the form RX^kY^l . These are exchanged by conjugation with $\frac{1}{2}X$, which is an outer automorphism. In the case of group pmm we remark that the map

$$\phi(R_1) := R_2, \quad \phi(R_2) := R_3, \quad \phi(R_3) := R_4, \quad \phi(R_4) := R_1$$

is an outer automorphism exchanging glides $R_1(R_2R_4)^k$ with $R_2(R_3R_1)^k$.

Secondly, we need to consider the case $\gamma = P \cdot P^{2k}Y^l$, where P is a glide and Y is an orthogonal translation satisfying

$$PY = Y^{-1}P$$

Then $\gamma^2 = P^{4k+2}$, which shows that γ is a glide for all choices of k and l . Arguing as in the previous case we obtain that modulo automorphisms of the group two glides $P \cdot P^{2k}Y^l$ and $P \cdot P^{2k'}Y^{l'}$ are equivalent if and only if $k = \pm k'$.

Suppose now that Γ is isomorphic to cm or $cm m$. We can assume that the γ belongs to a coset $RT(\Gamma)$, where R is a mirror and $T(\Gamma)$ is generated by (equal length) translations $S, RS R$. Then we can explicitly write

$$\begin{aligned} \gamma &= RS^k(RSR)^l \\ \gamma^2 &= S^{k+l}(RSR)^{k+l} \end{aligned}$$

and thus γ is a glide if and only if $k + l \neq 0$. In case of $cm m$ any rotation in Γ conjugates $RS^k(RSR)^l$ with $RS^{-k}(RSR)^{-l}$. In case of cm any plane rotation does the same, but it is now an outer automorphism. Conjugating by S^s gives

$$S^s(RS^k(RSR)^l)S^{-s} = RS^{k-s}(RSR)^{l+s}.$$

This shows that if $k+l = 2n$, then the glide $RS^k(RSR)^l$ is conjugate with $RS^n(RSR)^n = R(SR)^{2n} = R(RS)^{2n}$. If on the other hand $k + l = 2n + 1$ the glide $RS^k(RSR)^l$ is conjugate with $RS^n(RSR)^n = RS(RSR)^n S^n = (RS)^{2n+1}$. \square

4 Groups acting on the Klein bottle

In the previous section we listed ten families of glides in KBU groups. We now consider the problem of determining the normal closure of a glide g in a KBU group. Conjugating by an orthogonal translation in view of Lemma 2.2 gives a different, parallel glide g' . Thus the normal closure always contains a subgroup isomorphic to pg .

On the other hand, any KBU group is a subgroup of finite index in pmm ([2]). This can also be seen geometrically, or by giving explicit injections:

$$\begin{aligned} i_1 : pm &\rightarrow pmm & i_1(X) &:= R_1R_3, i_1(Y) := R_2R_4, i_1(R) := R_1 \\ i_2 : pg &\rightarrow pmm & i_2(P) &:= R_1R_2R_4, i_2(Q) := R_1R_3R_1R_2R_4 \\ i_3 : cm &\rightarrow pmm & i_3(R) &:= R_1, i_3(S) := (R_1R_3)(R_2R_4) \\ i_4 : pmg &\rightarrow pmm & i_4(P) &:= R_1R_2R_4, i_4(Q) := R_1(R_1R_3)^2R_2R_4, i_4(R) = R_3 \\ i_5 : pgg &\rightarrow pmm & i_5(P) &:= R_1R_2R_4, i_5(O) := R_2R_1R_3 \\ i_6 : cm m &\rightarrow pmm & i_6(R_1) &:= R_1, i_6(R_2) := R_2, i_6(T) := R_3R_4 \end{aligned}$$

The normal closure of a glide in a KBU group will thus be a subgroup of the normal closure of a glide in pmm . Without loss of generality, consider the normal closure of the glide $g = R_1(R_2R_4)^n$ in pmm . We have

$$\begin{aligned} R_1gR_1 &= g \\ R_2gR_2 &= R_2R_1(R_2R_4)^nR_2 = R_1(R_4R_2)^n = g^{-1} \\ R_3gR_3 &= R_3R_1(R_2R_4)^nR_3 = R_3R_1R_3(R_2R_4)^n = (R_3R_1)^2g \\ R_4gR_4 &= R_4R_1(R_2R_4)^nR_4 = R_1(R_4R_2)^n = g^{-1} \end{aligned}$$

One readily checks that R_2 and R_4 fix the translation $(R_3R_1)^2$, while R_1 and R_3 invert it. It follows that the normal closure of g in pmm is the subgroup isomorphic to pg generated by g and the translation $(R_3R_1)^2$ orthogonal to g . We have arrived at the following lemma.

Lemma 4.1. The normal closure of a glide in a KBU group is a subgroup isomorphic to pg .

Proof. Indeed, the normal closure contains two parallel glides, and since it is a subgroup of pg , which contains no mirrors, nor rotations, it has to be isomorphic to pg . \square

By geometric analysis using Lemma 2.2 one can determine the normal closure of glides in KBU groups precisely.

Proposition 4.2. The normal closure of a glide g in a KBU group G for families M1–M5, G1–G5 is the group $\langle g, t \rangle$ as given in Table 2. One has $gtg^{-1} = t^{-1}$, with the indicated quotient and a set of generators of $G/\langle g, t \rangle$.

Family	Group G	glide g	translation t	$F := G/\langle g, t \rangle$	Generators of F
M1	pm	RY^n	X^2	$\mathbf{Z}_2 \times \mathbf{Z}_{2n}$	$\bar{X}; \bar{Y}$
M2	pmm	$R_1(R_2R_4)^n$	$(R_1R_3)^2$	$\mathbf{Z}_2 \times D_{4n}$	$\bar{R}_1\bar{R}_3; \bar{R}_2, \bar{R}_4$
M3	pmg	$R(T_1T_2)^n$	$(T_1R)^2$	D_{4n}	$\bar{T}_1\bar{T}_2, \bar{T}_1$
M4	cm	$R(RS)^{2n}$	$RSRS^{-1}$	\mathbf{Z}_{4n}	$\bar{R}\bar{S}$
M5	cmm	$R_1(TR_2)^{2n}$	$(TR_1)^2$	D_{8n}	$\bar{T}\bar{R}_2, \bar{R}_2$
G1	pg	P^{2n-1}	$(PQ^{-1})^2$	$\mathbf{Z}_{2(2n-1)}$	\bar{Q}
G2	pmg	$(T_1R)^{2n-1}$	$(T_1T_2)^2$	$\mathbf{Z}_2 \times D_{2(2n-1)}$	$\bar{T}_1\bar{T}_2; \bar{T}_1R, \bar{R}$
G3	pgg	O^{2n-1}	P^2	$D_{2(2n-1)}$	\bar{O}, \bar{P}
G4	cm	$(RS)^{2n-1}$	$RSRS^{-1}$	$\mathbf{Z}_{2(2n-1)}$	\bar{S}
G5	cmm	$(R_1T)^{2n-1}$	$(R_2T)^2$	$\mathbf{Z}_2 \times D_{2(2n-1)}$	$\bar{R}_2; \bar{R}_1\bar{T}, \bar{R}_1$

Table 2: Group factors of KBU groups.

Proof. Determination of normal closure in each case proceeds analogously to the case of M2 analyzed above.

Let us illustrate how one proves the rest in the case G5. First, since

$$R_2TR_1 \cdot R_1R_2T = (R_2T)^2$$

we have from (2.1) and (2.3)

$$\begin{aligned} gtg^{-1} &= (R_1T)^{2n-1}(R_2T)^2(TR_1)^{2n-1} = (R_2T)^{-2}(R_1T)^{2n-1}(TR_1)^{2n-1} \\ &= (R_2T)^{-2} = t^{-1} \end{aligned}$$

Since $R_1 \cdot R_1T = T$, the images $\bar{R}_1\bar{T}, \bar{R}_1, \bar{R}_2$ of elements R_1T, R_1, R_2 generate F . Elements $\bar{R}, \bar{T}_1, \bar{T}_2$ are all involutions. From $(\bar{R}_1\bar{R}_2)^2 = 1$ it follows that \bar{R}_2 commutes with \bar{R}_1 , and from $(\bar{R}_2\bar{T})^2 = 1$ it follows that \bar{R}_2 commutes with \bar{T} . We can now compute the quotient (omitting redundant relations):

$$\begin{aligned} cmm/\langle g, t \rangle &= \langle \bar{T}, \bar{R}_1, \bar{R}_2; \bar{T}^2 = \bar{R}_1 = \bar{R}_2^2 = 1, (\bar{R}_1\bar{T})^{2n-1} = [\bar{R}_2, \bar{T}] = [\bar{R}_2, \bar{R}_1] = 1 \rangle \\ &= \langle \bar{R}_2; \bar{R}_2^2 = 1 \rangle \times \langle \bar{R}_1, \bar{T}; \bar{R}_1^2 = \bar{T}^2 = (\bar{R}_1\bar{T})^{2n-1} = 1 \rangle \\ &= \mathbf{Z}_2 \times D_{2(2n-1)} \end{aligned}$$

□

Corollary 4.3. (Babai) A group G acts on the Klein bottle if and only if it is a subgroup of $\mathbf{Z}_2 \times D_{2n}$ for some $n \geq 1$.

Proof. It is known ([1] Lemma 7.3 and [7] Theorem 6.2.4, also [10]) that if a group H acts on a surface, then there exists a Cayley map for this group on the surface. Thus any group action can be lifted to a Cayley action of a KBU group Γ on a semiregular tiling, such that it has a normal subgroup K isomorphic to pg that preserves the canonical projection. One has $H = \Gamma/K$. The group K contains a glide γ , and since it is normal, it contains the normal closure of γ . From the previous proposition it follows that H is isomorphic to a factor of F from Table 1, all of which are subgroups of $\mathbf{Z}_2 \times D_{2n}$.

Conversely, groups $\mathbf{Z}_2 \times D_{2n}$ act on the Klein bottle for both n even (pmm with glide family M2), and n odd (pmg with glide family G2 or cmm with glide family G5). □

5 Vertex-transitive maps on the Klein bottle

In Babai’s work [1] one can find a list of vertex-transitive maps on the Klein bottle. Let us explain his terminology. He represents semiregular tilings with letters as follows:

- A the triangle tiling
- B the square tiling
- C the hexagonal tiling
- D the elongated square tiling
- E the snub square tiling
- F the truncated square tiling

Then he shows 12 families of positions for glides acting on the tilings A1, A2, B1, B2, C1–C3, D1, D2, E1, F1, F2. Each of them has the property that there is a glide g in the position, and a translation t orthogonal to g such that the quotient of the plane by the group $\langle g, t \rangle$ is a vertex-transitive map on the Klein bottle. Moreover, the vertices of the semiregular tiling fall into 4 lines modulo translations by t . He calls the last property a *4-line condition*. He notes that every position, except B1, has a natural parity condition on the glides in the

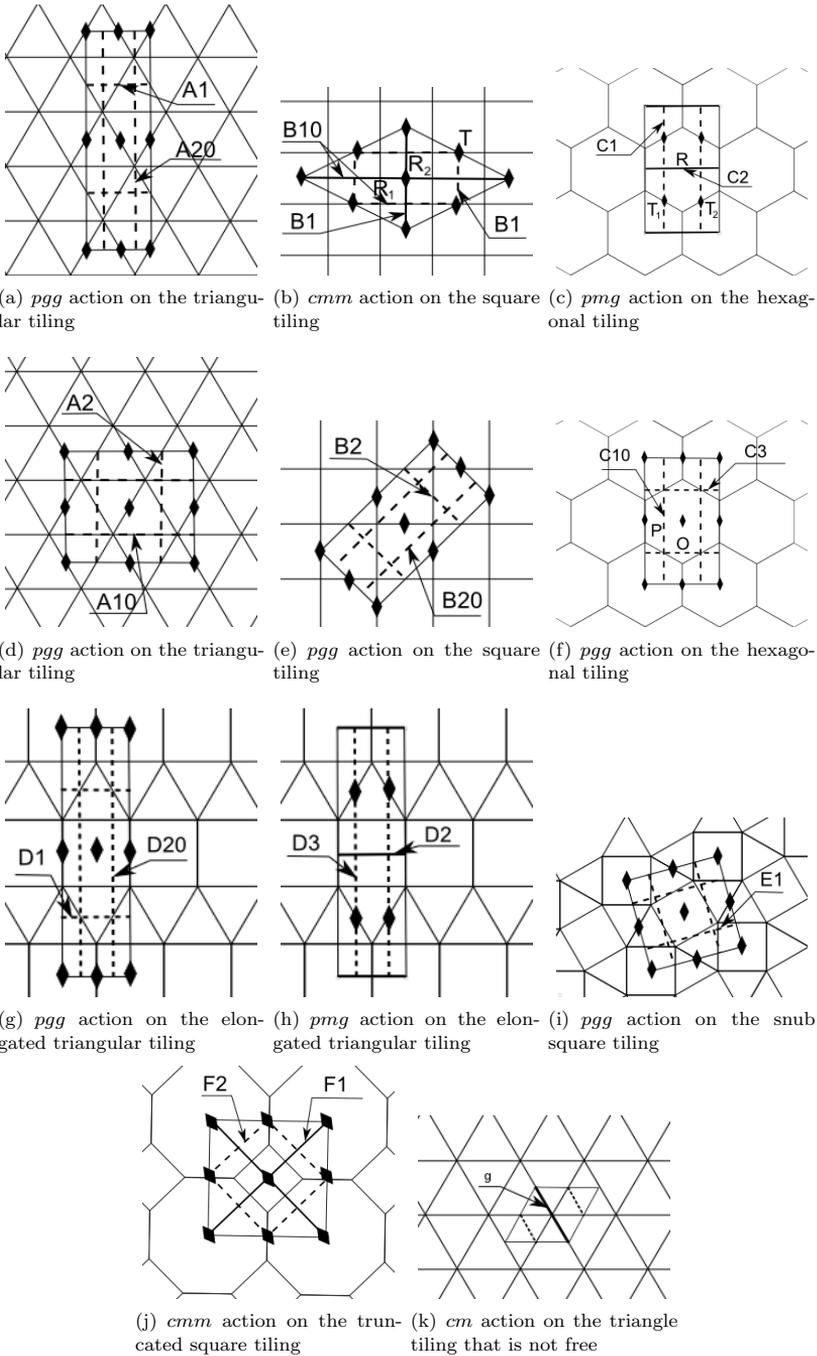


Figure 2: Vertex-transitive actions of KBU groups on semiregular tilings.

position. This results in 13 families of 4-line vertex-transitive maps, and he lists 5 2-line families (A10, A20, B10, B20, D10) and one 1-line family arising from the square tiling.

We can now state our final result.

Proposition 5.1. *All 4-line vertex transitive maps on the Klein bottle admit a free vertex transitive action. There are however vertex-transitive maps on the Klein bottle that do not admit a free vertex transitive action.*

Proof. The first statement is demonstrated by exhibiting free vertex transitive actions for each of Babai's families. This is done in Figures 2(a)-2(j). Note that for position B1, we show a free vertex transitive action for both *even* and *odd* glides [1]. Also note that for positions C1, D3, the resulting maps are indeed 4-line. This is because in the family G2, the normal closure is generated by the square of the smallest translation orthogonal to the glide.

We will prove the second statement by contradiction. Let us consider the vertex transitive action of the group $\Gamma = cm$ on the triangular tiling as shown in Figure 2(k). For any integer $n \geq 1$ one obtains a vertex transitive map on the Klein bottle by taking the quotient by the normal closure in Γ of a glide in the direction g of length $n \times (\text{length of triangle side})$. In fact, the images of translations act transitively. This shows there are vertex transitive maps arising from triangle tilings with number of vertices divisible by arbitrarily large power of 2.

Since any mirror of a triangular tiling fixes a vertex of the tiling, there are free vertex transitive actions only by groups pg and pgg on the tiling. Any maps arising from those actions are factors by *odd* multiple sized glides. Since there are only finitely many classes of actions of pg and pgg modulo automorphisms of the triangle tiling, we conclude that the power of 2 dividing the size of free vertex transitive maps has an upper bound.

This contradiction shows that there are maps that do not admit a free vertex transitive action. \square

Finally we remark there is a 2-line family of vertex transitive maps for position C10 shown in Figure 2(f), omitted in Babai's list.

References

- [1] L. Babai, Vertex-transitive graphs and vertex-transitive maps, *J. Graph Theory* **15** (1991), 587–627.
- [2] H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, 4th ed., 1980, Springer-Verlag, Berlin and New York.
- [3] B. N. Delone, Theory of planigons (in Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **23** (1959), 365–386.
- [4] B. N. Delone and N. P. Dolbilin, M. I. Štogrin, Combinatorial and metric theory of planigons (in Russian), *Algebra, number theory and their applications, Trudy Mat. Inst. Steklov.* **148** (1978), 109–140, 275.
- [5] B. Grünbaum and G. C. Shephard, The ninety-one types of isogonal tilings in the plane, *Trans. Amer. Math. Soc.* **242** (1978), 335–353.
- [6] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman and Company, 1987.
- [7] J. L. Gross and T. W. Tucker, *Topological graph theory*, Dover, Toronto, 2001.

- [8] O. Šuch, Vertex-transitive maps on a torus, *Acta Mathematica Universitatis Comenianae* **53** (2011), 1–30.
- [9] C. Thomassen, Tilings of the torus and the Klein bottle and vertex-transitive graphs on a fixed surface, *Trans. Amer. Math. Soc.* **323** (1991), 605–635.
- [10] T. W. Tucker, Finite groups acting on surfaces and the genus of a group, *J. Combin. Theory Ser. B* **34** (1983), 82–98.
- [11] http://en.wikipedia.org/wiki/Tilings_of_regular_polygons
- [12] http://en.wikipedia.org/wiki/Wallpaper_group