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Every finite group has a normal bi-Cayley graph*

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Abstract

A graph Γ with a group H of automorphisms acting semiregularly on the vertices with two orbits is called a *bi-Cayley graph* over H. When H is a normal subgroup of $\mathrm{Aut}(\Gamma)$, we say that Γ is *normal* with respect to H. In this paper, we show that every finite group has a connected normal bi-Cayley graph. This improves a theorem by Arezoomand and Taeri and provides a positive answer to a question reported in the literature.

Keywords: Normal, bi-Cayley, Cartesian product.

Math. Subj. Class.: 05C25, 20B25

1 Introduction

Throughout this paper, groups are assumed to be finite, and graphs are assumed to be finite, connected, simple and undirected. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [5, 23].

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. It is well-known that a graph Γ is a *Cayley graph* if it has an automorphism group acting regularly on its vertex set (see [4, Lemma 16.3]). If we, instead, require that the graph Γ admits a group of automorphisms acting semiregularly on its vertex set with two orbits, then we obtain the so-called *bi-Cayley graph*.

Cayley graph is usually defined in the following way. Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is a

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graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. For any $g \in G$, $R(g)$ is the permutation of G defined by $R(g)\colon x \mapsto xg$ for $x \in G$. Set $R(G) := \{R(g) \mid g \in G\}$. It is well-known that $R(G)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. In 1981, Godsil [10] proved that the normalizer of $R(G)$ in $\mathrm{Aut}(\mathrm{Cay}(G, S))$ is $R(G) \rtimes \mathrm{Aut}(G, S)$, where $\mathrm{Aut}(G, S)$ is the group of automorphisms of G fixing the set S set-wise. This result has been successfully used in characterizing various families of GRRs, namely, Cayley graphs $\mathrm{Cay}(G, S)$ such that $R(G) = \mathrm{Aut}(\mathrm{Cay}(G, S))$ (see, for example, [10, 11]). A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. This concept was introduced by Xu in [24], and for more results about normal Cayley graphs, we refer the reader to [8].

Similarly, every bi-Cayley graph admits the following concrete realization. Given a group H, let R, L and S be subsets of H such that $R^{-1} = R$, $L^{-1} = L$ and $R \cup L$ does not contain the identity element of H. The *bi-Cayley graph* over H relative to the triple (R, L, S), denoted by $\mathrm{BiCay}(H,\ R,\ L,\ S)$, is the graph having vertex set the union of the right part $H_0 = \{h_0 \mid h \in H\}$ and the left part $H_1 = \{h_1 \mid h \in H\}$, and edge set the union of the right edges $\{\{h_0,\ g_0\} \mid gh^{-1} \in R\}$, the left edges $\{\{h_1,\ g_1\} \mid gh^{-1} \in L\}$ and the spokes $\{\{h_0,\ g_1\} \mid gh^{-1} \in S\}$. Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$. For $g \in H$, define a permutation $BR(g)$ on the vertices of Γ by the rule

$$h_i^{BR(g)} = (hg)_i, \forall i \in \mathbb{Z}_2, h \in H.$$

Then $BR(H) = \{BR(g) \mid g \in H\}$ is a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ which is isomorphic to H and has H_0 and H_1 as its two orbits. When $BR(H)$ is normal in $\mathrm{Aut}(\Gamma)$, the bi-Cayley graph $\Gamma = \mathrm{BiCay}(H, R, L, S)$ will be called a *normal bi-Cayley graph* over H (see [3] or [27]).

Wang et al. in [22] determined the groups having a connected normal Cayley graph.

Proposition 1.1. *Every finite group G has a normal Cayley graph unless $G \cong C_4 \times C_2$ or $G \cong \mathrm{Q}_8 \times C_2^r (r \geq 0)$.*

Following up this result, Arezoomand and Taeri in [3] asked: Which finite groups have normal bi-Cayley graphs? They also gave a partial answer to this question by proving that every finite group $G \ncong \mathrm{Q}_8 \times C_2^r (r \geq 0)$ has at least one normal bi-Cayley graph. At the end of [3], the authors asked the following question:

Question 1.2 ([3, Question])**.** Is there any normal bi-Cayley graph over $G \cong \mathrm{Q}_8 \times C_2^r$ for each $r \geq 0$?

We remark that for every finite group $G \ncong \mathrm{Q}_8 \times C_2^r (r \geq 0)$, the normal bi-Cayley graph over G constructed in the proof of [3, Theorem 5] is not of regular valency, and so is not vertex-transitive. So it is natural to ask the following question.

Question 1.3. Is there any vertex-transitive normal bi-Cayley graph over a finite group G?

In this paper, Questions 1.2 and 1.3 are answered in positive. The following is the main result of this paper.

Theorem 1.4. *Every finite group has a vertex-transitive normal bi-Cayley graph.*

To end this section we give some notation which is used in this paper. For a positive integer n, denote by C_n the cyclic group of order n, by $\mathbb{Z}_n$ the ring of integers modulo n, by D_{2n} the dihedral group of order $2n$, and by $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$ the alternating group and symmetric group of degree n, respectively. Denote by Q_8 the quaternion group. For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. The identity element of a finite group G is denoted by $\mathbf{1}$.

For a finite, simple and undirected graph Γ, we use $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its vertex set, edge set and full automorphism group, respectively, and for any $u, v \in V(\Gamma)$, $u \sim v$ means that u and v are adjacent. A graph Γ is said to be *vertex-transitive* if its full automorphism group $\mathrm{Aut}(\Gamma)$ acts transitively on its vertex set. For any subset B of $V(\Gamma)$, the subgraph of Γ induced by B will be denoted by $\Gamma[B]$.

2 Cartesian products

The *Cartesian product* $X \square Y$ of graphs X and Y is a graph with vertex set $V(X) \times V(Y)$, and with vertices (u, x) and (v, y) being adjacent if and only if $u = v$ and $x \sim y$ in Y, or $x = y$ and $u \sim v$ in X.

A non-trivial graph X is *prime* if it is not isomorphic to a Cartesian product of two smaller graphs. The following proposition shows the uniqueness of the prime factor decomposition of connected graphs with respect to the Cartesian product.

Proposition 2.1 ([12, Theorem 6.6]). *Every connected finite graph can be decomposed as a Cartesian product of prime graphs, uniquely up to isomorphism and the order of the factors.*

Two non-trivial graphs are *relatively prime* (w.r.t. Cartesian product) if they have no non-trivial common factor. Now we consider the automorphisms of Cartesian product of graphs.

Proposition 2.2 ([12, Theorem 6.10]). *Suppose ϕ is an automorphism of a connected graph Γ with prime factor decomposition $\Gamma = \Gamma_1 \square \Gamma_2 \square \cdots \square \Gamma_k$. Then there is a permutation π of $\{1, 2, \ldots, k\}$ and isomorphisms $\phi_i \colon \Gamma_{\pi(i)} \to \Gamma_i$ for which*

$$\phi(x_1, x_2, \ldots, x_k) = (\phi_1(x_{\pi(1)}), \phi_2(x_{\pi(2)}), \ldots, \phi_k(x_{\pi(k)})).$$

Corollary 2.3 ([12, Corollary 6.12]). *Let Γ be a connected graph with prime factor decomposition $\Gamma = \Gamma_1 \square \Gamma_2 \square \cdots \square \Gamma_k$. If $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ are relatively prime, then* $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2) \times \cdots \times \mathrm{Aut}(\Gamma_k)$.

The following theorem provides a method of constructing normal bi-Cayley graphs.

Theorem 2.4. *Let X be a connected normal bi-Cayley graph over a group H, and let Y be a connected normal Cayley graph over a group K. If X and Y are relatively prime, then $X \square Y$ is also a normal bi-Cayley graph over the group $H \times K$.*

Proof. Assume that X and Y are relatively prime. By Corollary 2.3, $\mathrm{Aut}(X \square Y) = \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$. Since X is a connected normal bi-Cayley graph over H, one has $BR(H) \trianglelefteq \mathrm{Aut}(X)$, and since Y is a connected normal Cayley graph over a group K, one has $R(K) \trianglelefteq \mathrm{Aut}(Y)$. Then $BR(H) \times R(K)$ is a normal subgroup of $\mathrm{Aut}(X \square Y) = \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$. Note that $BR(H)$ acts semiregularly on $V(X)$ with two orbits, and $R(K)$ acts regularly on $V(Y)$. It follows that $BR(H) \times R(K)$ acts semiregularly on $V(X) \times V(Y)$ with two orbits, and thereby $X \square Y$ is also a normal bi-Cayley graph over the group $H \times K$. $\square$

3 Normal bi-Cayley graphs over $Q_8 \times C_2^r (r \geq 0)$

In this section, we shall answer Question 1.2 in positive.

3.1 The n-dimensional hypercube

For $n \geq 1$, the *n-dimensional hypercube*, denoted by Q_n, is the graph whose vertices are all the n-tuples of 0's and 1's with two n-tuples being adjacent if and only if they differ in exactly one place.

Let $N = C_2^n$ be an elementary abelian 2-group of order 2^n with a minimum generating set $S = \{s_1, s_2, s_3, \ldots, s_n\}$. By the definition of Q_n, we have $\mathrm{Cay}(N, S) \cong Q_n$. For convenience of the statement, we assume that $Q_n = \mathrm{Cay}(N, S)$. If $n = 1$, then $Q_1 = \mathbf{K}_2$ and so $\mathrm{Aut}(Q_1) = N$. In what follows, assume that $n \geq 2$. It is easy to observe that for any distinct s_i, s_j there is a unique 4-cycle in Q_n passing through $\mathbf{1}, s_i, s_j$, where $\mathbf{1}$ is the identity element of N. So if a subgroup of $\mathrm{Aut}(Q_n)$ fixes S pointwise, then it also fixes every vertex at distance 2 from $\mathbf{1}$. By the connectedness and vertex-transitivity of Q_n, we have $\mathrm{Aut}(Q_n)_{\mathbf{1}}$ acts faithfully on S. It follows that $\mathrm{Aut}(Q_n)_{\mathbf{1}} \lesssim \mathrm{Sym}(n)$. On the other hand, it is easy to see that each permutation on S induces an automorphism of N, and so $\mathrm{Aut}(N, S) \cong \mathrm{Sym}(n)$. Since $\mathrm{Aut}(N, S) \leq \mathrm{Aut}(Q_n)_{\mathbf{1}}$, one has $\mathrm{Aut}(Q_n)_{\mathbf{1}} = \mathrm{Aut}(N, S) \cong \mathrm{Sym}(n)$. Consequently, we have $\mathrm{Aut}(Q_n) = R(N) \rtimes \mathrm{Aut}(N, S) \cong N \rtimes \mathrm{Sym}(n)$ (see also [25, Lemma 1.1]).

Note that Q_n is bipartite. Let $\mathrm{Aut}(Q_n)^*$ be the kernel of $\mathrm{Aut}(Q_n)$ acting on the two partition sets of Q_n. Let $E = R(N) \cap \mathrm{Aut}(Q_n)^*$. Then $E \trianglelefteq \mathrm{Aut}(Q_n)^*$ and $E \trianglelefteq R(N)$. It follows that $E \trianglelefteq \mathrm{Aut}(Q_n)^* R(N) = \mathrm{Aut}(Q_n)$. Clearly, E acts semiregularly on $V(Q_n)$ with two orbits. Thus, we have the following lemma.

Lemma 3.1. *Use the same notation as in the above three paragraphs. For any $n \geq 1$, Q_n is a normal Cayley graph over N, and Q_n is also a normal bi-Cayley graph over E.*

3.2 The Möbius-Kantor graph

The Möbius-Kantor graph $\mathrm{GP}(8, 3)$ is a graph with vertex set $V = \{i, i' \mid i \in \mathbb{Z}_8\}$ and edge set the union of the *outer edges* $\{\{i, i{+}1\} \mid i \in \mathbb{Z}_8\}$, the *inner edges* $\{\{i', (i{+}3)'\} \mid i \in \mathbb{Z}_8\}$, and the *spokes* $\{\{i, i'\} \mid i \in \mathbb{Z}_8\}$ (see Figure 1). Note that $\mathrm{GP}(8, 3)$ is a bipartite graph with bipartition sets $B_1 = \{1, 3, 5, 7, 0', 2', 4', 6'\}$ and $B_2 = \{0, 2, 4, 6, 1', 3', 5', 7'\}$.

In [26], the edge-transitive groups of automorphisms of $\mathrm{Aut}(\mathrm{GP}(8, 3))$ were determined. We first introduce the following automorphisms of $\mathrm{GP}(8, 3)$, represented as permutations on the vertex set V:

$$\begin{aligned}
\alpha &= (1\ 3\ 5\ 7)(0\ 2\ 4\ 6)(1'\ 3'\ 5'\ 7')(0'\ 2'\ 4'\ 6'),\\
\beta &= (0\ 1'\ 2)(0'\ 6'\ 3)(4\ 5'\ 6)(7\ 4'\ 2'),\\
\gamma &= (1\ 1')(2\ 6')(3\ 3')(4\ 0')(5\ 5')(6\ 2')(7\ 7')(0\ 4'),\\
\delta &= (1\ 1')(2\ 4')(3\ 7')(4\ 2')(5\ 5')(6\ 0')(7\ 3')(0\ 6').
\end{aligned}$$

By [26, Lemma 3.1], we have $\langle \alpha, \beta \rangle = \langle \alpha, \alpha^\beta \rangle \rtimes \langle \beta \rangle \cong Q_8 \rtimes \mathbb{Z}_3$, where Q_8 is the quaternion group, and moreover, $\langle \alpha, \beta \rangle \trianglelefteq \mathrm{Aut}(\mathrm{GP}(8, 3))$. Clearly, $\langle \alpha, \alpha^\beta \rangle \cong Q_8$ is the Sylow 2-subgroup of $\langle \alpha, \beta \rangle$, so $\langle \alpha, \alpha^\beta \rangle$ is characteristic in $\langle \alpha, \beta \rangle$, and then it is normal in $\mathrm{Aut}(\mathrm{GP}(8, 3))$ because $\langle \alpha, \beta \rangle \trianglelefteq \mathrm{Aut}(\mathrm{GP}(8, 3))$. For convenience of the statement, we let $Q_8 = \langle \alpha, \alpha^\beta \rangle$. It is easy to see that Q_8 acts semiregularly on V with two orbits B_1 and B_2. Thus we have the following lemma.

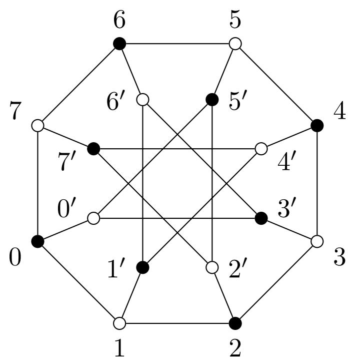

Figure 1: The Möbius-Kantor graph $\mathrm{GP}(8,3)$.

Lemma 3.2. $\mathrm{GP}(8,3)$ *is a normal bi-Cayley graph over* Q_8.

3.3 An answer to Question 1.2

Noting that $\mathrm{GP}(8,3)$ is of girth 6, $\mathrm{GP}(8,3)$ is prime. For each $r \geq 1$, it is easy to see that $Q_r = \underbrace{\mathbf{K}_2 \,\Box\, \mathbf{K}_2 \,\Box \cdots \Box\, \mathbf{K}_2}_{n \text{ times}}$. So, Q_n and $\mathrm{GP}(8,3)$ are relatively prime. Now combining together Lemmas 3.1 and 3.2 and Theorem 2.4, we can obtain the following theorem.

Theorem 3.3. *For each* $r \geq 1$, $\mathrm{GP}(8,3) \times Q_r$ *is a vertex-transitive normal bi-Cayley graph over* $\mathrm{Q}_8 \times N$, *where* $N \cong C_2^r$.

4 Proof of Theorem 1.4

The proof of Theorem 1.4 will be completed by the following lemmas. Let G be a group. A Cayley graph $\Gamma = \mathrm{Cay}(G,S)$ on G is said to be a *graphical regular representation* (or *GRR* for short) of G if $\mathrm{Aut}(\Gamma) = R(G)$.

Lemma 4.1. *Let G be a group admitting a GRR Γ. Then $\Gamma \,\Box\, \mathbf{K}_2$ is a normal bi-Cayley graph over the group G.*

Proof. If $\mathbf{K}_2$ and Γ are relatively prime, then by Corollary 2.3, we have $\mathrm{Aut}(\Gamma \,\Box\, \mathbf{K}_2) = \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\mathbf{K}_2)$. Clearly, $R(G) \times \{\mathbf{1}\}$ acts semiregularly on $V(\Gamma \,\Box\, \mathbf{K}_2)$ with two orbits, and $R(G) \times \{\mathbf{1}\} \trianglelefteq \mathrm{Aut}(\Gamma \,\Box\, \mathbf{K}_2)$, where $\mathbf{1}$ is the identity of $\mathrm{Aut}(\mathbf{K}_2)$. It follows that $\Gamma \,\Box\, \mathbf{K}_2$ is a normal bi-Cayley graph over the group G.

Suppose that $\mathbf{K}_2$ is also a prime factor of Γ. Let $\Gamma = \Gamma_1 \,\Box\, \underbrace{\mathbf{K}_2 \,\Box \cdots \Box\, \mathbf{K}_2}_{m \text{ times}}$ be such that Γ_1 is coprime to $\mathbf{K}_2$. From Corollary 2.3 it follows that $G = \mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\mathbf{K}_2 \,\Box \cdots \Box\, \mathbf{K}_2)$. Since Γ is a GRR of G, one has $m = 1$, and therefore $\Gamma \,\Box\, \mathbf{K}_2 = \Gamma_1 \,\Box\, \mathbf{K}_2 \,\Box\, \mathbf{K}_2$. Then $G = \mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\mathbf{K}_2 \,\Box\, \mathbf{K}_2)$, and Γ_1 is a GRR of $\mathrm{Aut}(\Gamma_1)$. By Lemma 3.1, $\mathbf{K}_2 \,\Box\, \mathbf{K}_2$ is a normal bi-Cayley graph over C_2, and by Theorem 2.4, $\Gamma \,\Box\, \mathbf{K}_2$ is a normal bi-Cayley graph over $\mathrm{Aut}(\Gamma_1) \times C_2 \cong G$. □

A group G is called *generalized dicyclic group* if it is non-abelian and has an abelian subgroup L of index 2 and an element $b \in G \setminus L$ of order 4 such that $b^{-1}xb = x^{-1}$ for every $x \in L$.

The following theorem gives a list of groups having no GRR (see [9]).

Theorem 4.2. *A finite group G admits a GRR unless G belongs to one of the following classes of groups:*

(I) *Class C: abelian groups of exponent greater than two;*

(II) *Class D: the generalized dicyclic groups;*

(III) *Class E: the following thirteen "exceptional groups":*

(1) $\mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_2^4$;
(2) D_6, D_8, D_{10};
(3) A_4;
(4) $\langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$;
(5) $\langle a, b \mid a^8 = b^2 = 1, bab = a^5 \rangle$;
(6) $\langle a, b, c \mid a^3 = b^3 = c^2 = 1, ab = ba, (ac)^2 = (cb)^2 = 1 \rangle$;
(7) $\langle a, b, c \mid a^3 = b^3 = c^3 = 1, ac = ca, bc = cb, c = a^{-1}b^{-1}cb \rangle$;
(8) $\mathrm{Q}_8 \times \mathbb{Z}_3, \mathrm{Q}_8 \times \mathbb{Z}_4$.

Lemma 4.3. *Let G be a group in Class D of Theorem 4.2. Then G has a normal bi-Cayley graph.*

Proof. If $G \cong \mathrm{Q}_8 \times C_2^r$ for some $r \geq 0$, then by Theorem 3.3 and Lemma 3.2, G has a normal bi-Cayley graph. In what follows, we assume that $G \not\cong \mathrm{Q}_8 \times C_2^r$ for any $r \geq 0$. By Proposition 1.1, G has a normal Cayley graph, say Γ. If Γ is coprime to $\mathbf{K}_2$, then by Corollary 2.3, $\Gamma \,\square\, \mathbf{K}_2$ is a normal bi-Cayley graph over G.

Now suppose that $\mathbf{K}_2$ is a prime factor of Γ. Let $\Gamma = \Gamma_1 \,\square\, Q_m$, where $Q_m = \underbrace{\mathbf{K}_2 \,\square \cdots \square\, \mathbf{K}_2}_{m \text{ times}}$ and Γ_1 is coprime to $\mathbf{K}_2$. Again by Corollary 2.3, we have $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(Q_m)$. For any $x \in V(Q_m)$, set $V_x = \{(u, x) \mid u \in V(\Gamma_1)\}$, and for any $y \in V(\Gamma_1)$, set $U_y = \{(y, v) \mid v \in V(Q_m)\}$. Then $\Gamma[V_x] \cong \Gamma_1$ and $\Gamma[V_y] \cong Q_m$. Let G_{V_x} and G_{U_y} be the subgroups of G fixing V_x and U_y setwise, respectively. We shall prove the following claim.

Claim. $G = G_{V_x} \times G_{U_y}$, Γ_1 is a normal Cayley graph over a group which is isomorphic to G_{V_x}, and $G_{U_y} \cong C_2^m$.

Since Γ is vertex-transitive, by Proposition 2.2, V_x is an orbit of $\mathrm{Aut}(\Gamma_1) \times \{\mathbf{1}\}$ and $\mathrm{Aut}(\Gamma_1) \times \{\mathbf{1}\} = \mathrm{Aut}(\Gamma[V_x])$. As $\mathrm{Aut}(\Gamma_1) \times \{\mathbf{1}\} \trianglelefteq \mathrm{Aut}(\Gamma)$, each V_x is a block of imprimitivity of $\mathrm{Aut}(\Gamma)$ (namely, either $V_x^g = V_x$ or $V_x^g \cap V_x = \emptyset$ for any $g \in \mathrm{Aut}(\Gamma)$). Consider the quotient graph Γ' with vertex set $\{V_x \mid x \in V(Q_m)\}$, and V_x is adjacent to $V_{x'}$ if and only if x is adjacent to x' in Q_m. Then $\Gamma' \cong Q_m$, and $\mathrm{Aut}(\Gamma_1) \times \{\mathbf{1}\}$ is just the kernel of $\mathrm{Aut}(\Gamma)$ acting on $V(\Gamma')$. This implies that the subgroup $\mathrm{Aut}(\Gamma)_{V_x}$ of $\mathrm{Aut}(\Gamma)$ fixing V_x set-wise is just $\mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(Q_m)_x$. Since G is regular on $V(\Gamma)$, G_{V_x} is also regular on V_x, and so $\Gamma_1 \cong \Gamma[V_x]$ may be viewed as a Cayley graph on G_{V_x}. Since $G \trianglelefteq \mathrm{Aut}(\Gamma)$, one has $G_{V_x} = G \cap \mathrm{Aut}(\Gamma)_{V_x} \trianglelefteq \mathrm{Aut}(\Gamma)_{V_x}$. Note that $\{\mathbf{1}\} \times \mathrm{Aut}(Q_m)_x$ fixes every vertex in V_x. It follows that $G_{V_x} \cap (\{\mathbf{1}\} \times \mathrm{Aut}(Q_m)_x)$ is trivial, and so G_{V_x} can be viewed as a normal regular subgroup of $\mathrm{Aut}(\Gamma_1) \times \{\mathbf{1}\}$. Therefore, Γ_1 is a normal Cayley graph over some group, say $H \cong G_{V_x}$.

With a similar argument as above, we can show that Q_m is also a normal Cayley graph over some group, say $K \cong G_{U_y}$. From the argument in Section 3.1, we have $\mathrm{Aut}(Q_m) = N \rtimes \mathrm{Sym}(m)$ with $N \cong C_2^m$. We claim that $K = N$. If this is not true, then we would have $1 \neq KN/N \trianglelefteq \mathrm{Aut}(Q_m)/N \cong \mathrm{Sym}(m)$, and since K is a 2-group, the only possibility is $m = 4$. However, by Magma [6], $\mathrm{Aut}(Q_4)$ has only one normal regular subgroup which is isomorphic to C_2^4, a contradiction. Thus, $K = N \cong C_2^m$, and hence $G_{U_y} \cong C_{2^m}$.

For any $g \in G_{V_x} \cap G_{U_y}$, we have g fixes (y, x) and so $g = \mathbf{1}$ because G is regular on $V(\Gamma)$. Thus, $G_{V_x} \cap G_{U_y} = \{\mathbf{1}\}$. Then $|G_{V_x} G_{U_y}| = |G_{V_x}||G_{U_y}| = |V_x||U_y| = |V(\Gamma)| = |G|$. It follows that $G = G_{V_x} G_{U_y}$. To show that $G = G_{V_x} \times G_{U_y}$, it suffices to show that both G_{V_x} and G_{U_y} are normal in G. As G is a generalized dicyclic group, it is non-abelian and has an abelian subgroup L of index 2 and an element $b \in G \setminus L$ of order 4 such that $b^{-1}ab = a^{-1}$ for every $a \in L$.

Suppose that $G_{U_y} \nleq L$. Then there exists $g \in G_{U_y}$ such that $g = ab^i$ for some $a \in L$ and $i = 1$ or -1. Since $G_{U_y} \cong C_2^m$, g is also an involution, and so $G = L \rtimes \langle g \rangle$. Clearly, for any $a \in L$, we have $g^{-1}ag = a^{-1}$, and so $(ga)^2 = 1$. This would force that every element of G outside L is an involution, a contradiction. Thus, $G_{U_y} \leq L$, and hence $G_{U_y} \trianglelefteq G$.

Since $G = G_{V_x} G_{U_y}$, $G_{U_y} \leq L$ implies that $G_{V_x} \nleq L$. Then $|G_{V_x} : G_{V_x} \cap L| = 2$ since $|G : L| = 2$. It then follows that $G_{V_x} \cap L \trianglelefteq G$, and hence

$$G/G_{V_x} \cap L = (G_{U_y}(G_{V_x} \cap L)/G_{V_x} \cap L) \rtimes (G_{V_x}/G_{V_x} \cap L).$$

Again as G is a generalized dicyclic group and since $G_{U_y} \leq L$, the non-trivial element of $G_{V_x}/G_{V_x} \cap L$ maps every element of $G_{U_y}(G_{V_x} \cap L)/G_{V_x} \cap L$ to its inverse. Since $G_{U_y} \cong C_{2^m}$, one has $G/G_{V_x} \cap L$ is abelian, and so $G_{V_x} \trianglelefteq G$, completing the proof of the Claim.

By Lemma 3.1, we may let $Q_{m+1} = \underbrace{\mathbf{K}_2 \,\Box \cdots \Box\, \mathbf{K}_2}_{m+1 \text{ times}}$ be a connected normal bi-Cayley graph over $G_{U_y} \cong C_2^m$. By Claim, we may view Γ_1 as a normal Cayley graph over G_{V_x}. Since Γ_1 is coprime to $\mathbf{K}_2$, by Theorem 2.4, $\Gamma_1 \,\Box\, Q_{m+1}$ is a connected normal bi-Cayley graph over $G_{V_x} \times G_{U_y} = G$. □

Lemma 4.4. *Let G be a group in Class E of Theorem 4.2. Then G has a normal bi-Cayley graph.*

Proof. By Lemma 3.1, each of the groups in Class E (1) has a connected normal bi-Cayley graph.

Let $G = D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ with $n \geq 3$. Let $\Gamma = \mathrm{Cay}(G, \{ab, b\})$. Then Γ is a cycle of length $2n$, and so Γ is coprime to $\mathbf{K}_2$. By Theorem 2.4, $\Gamma \,\Box\, \mathbf{K}_2$ is a connected normal bi-Cayley graph over G. Thus, each of the groups in Class E (2) has a connected normal bi-Cayley graph.

Let $G = \mathrm{Alt}(4)$ and let $\Gamma = \mathrm{Cay}(G, \{(1\,2\,3), (1\,3\,2), (1\,2\,4), (1\,4\,2)\})$. By Magma [6], we have $\Gamma \,\Box\, \mathbf{K}_2$ is a connected normal bi-Cayley graph over $\mathrm{Alt}(4)$.

Let $G = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$ be the group in Class E (4). Let $\Gamma = \mathrm{Cay}(G, \{a, b, c\})$. By Magma [6], Γ is a connected trivalent normal Cayley graph over G and Γ has girth 6. Hence, Γ is coprime to $\mathbf{K}_2$. By Theorem 2.4, $\Gamma \,\Box\, \mathbf{K}_2$ is a connected normal bi-Cayley graph over G.

Let $G = \langle a, b \mid a^8 = b^2 = 1, bab = a^5\rangle$ be the group in Class E (5). Let $\Gamma = \mathrm{Cay}(G, \{a, a^{-1}, b, a^4, a^4 b\})$. By [22, Lemma 6], Γ is a connected normal Cayley graph over G, and by Magma, $\mathrm{Aut}(\Gamma \,\square\, \mathbf{K}_2) = \mathrm{Aut}(\Gamma) \times \mathbb{Z}_2$. Thus, $\Gamma \,\square\, \mathbf{K}_2$ is a normal bi-Cayley graph over G.

Let $G = \langle a, b, c \mid a^3 = b^3 = c^2 = 1, ac = ca, (ab)^2 = (cb)^2 = 1\rangle$ be the group in Class E (6). Let $\Gamma = \mathrm{Cay}(G, \{c, ca, cb\})$. By Magma [6], Γ is a connected trivalent normal Cayley graph over G and Γ has girth 6. Hence, Γ is coprime to $\mathbf{K}_2$. By Lemma 2.4, $\Gamma \,\square\, \mathbf{K}_2$ is a connected normal bi-Cayley graph over G.

Let $G = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, ac = ca, bc = cb, c = a^{-1}b^{-1}cb\rangle$ be the group in Class E (7). Let $\Gamma = \mathrm{Cay}(G, \{a, b, a^{-1}, b^{-1}\})$. By Magma [6], Γ is a connected trivalent normal Cayley graph over G. Since G has order 27, Γ is coprime to $\mathbf{K}_2$. By Theorem 2.4, $\Gamma \,\square\, \mathbf{K}_2$ is a connected normal bi-Cayley graph over G.

Finally, we consider the groups in Class E (8). By Lemma 3.2, $\mathrm{GP}(8, 3)$ is a normal bi-Cayley graph over Q_8. For $n \geq 3$, let $C_n = \langle a\rangle$ and let $\Gamma = \mathrm{Cay}(C_n, \{a, a^{-1}\})$. Clearly, Γ is a normal Cayley graph over C_n. Since $\mathrm{GP}(8, 3)$ is of girth 6, $\mathrm{GP}(8, 3)$ is coprime to Γ. By Theorem 2.4, $\mathrm{GP}(8, 3) \,\square\, \Gamma$ is a connected normal bi-Cayley graph over $\mathrm{Q}_8 \times C_n$. Thus each of the groups in Class E (8) has a connected normal bi-Cayley graph. □

Lemma 4.5. *Let G be a group in Class C of Theorem 4.2. Then G has a normal bi-Cayley graph.*

Proof. Since G is abelian, G has an automorphism α such that α maps every element of G to its inverse. Set $H = G \rtimes \langle\alpha\rangle$. If H has a GRR Γ, then Γ is also a normal bi-Cayley graph over G. Suppose that H has no GRR. Then by Theorem 4.2 we have H is one of the groups in Class E (2) and (6). By Lemma 4.4, G has a normal bi-Cayley graph □

Proof of Theorem 1.4. Let G be a finite group. If G has a GRR, then by Lemma 4.1, G has a connected normal bi-Cayley graph. If G does not have a GRR, then the theorem follows from Lemmas 4.3, 4.4, 4.5 and 3.2. □

5 Final remarks

This paper would not be complete without mentioning some related work, namely on some special families of bi-Cayley graphs such as bi-circulants, bi-abelians etc. Numerous papers on the topic have been published (see, for instance, [1, 2, 7, 13, 14, 15, 16, 17, 18, 19, 20, 21]). In view of these, the following problem arises naturally.

Problem 5.1. For a given finite group H, classify or characterize bi-Cayley graphs with specific symmetry properties over H.

Let H be a finite group. We say that a bi-Cayley graph Γ of regular valency over H is a *bi-graphical regular representation* (or *bi-GRR* for short) if $\mathrm{Aut}(\Gamma) = BR(H)$. Motivated by the classification of finite groups having no GRR (see Theorem 4.2), we would like to pose the following problem.

Problem 5.2. Determine finite groups which have no bi-GRR.

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