Every finite group has a normal bi-Cayley graph*

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Abstract

A graph $\Gamma$ with a group $H$ of automorphisms acting semiregularly on the vertices with two orbits is called a bi-Cayley graph over $H$. When $H$ is a normal subgroup of $\text{Aut}(\Gamma)$, we say that $\Gamma$ is normal with respect to $H$. In this paper, we show that every finite group has a connected normal bi-Cayley graph. This improves a theorem by Arezoomand and Taeri and provides a positive answer to a question reported in the literature.

Keywords: Normal, bi-Cayley, Cartesian product.

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1 Introduction

Throughout this paper, groups are assumed to be finite, and graphs are assumed to be finite, connected, simple and undirected. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [5, 23].

Let $G$ be a permutation group on a set $\Omega$ and $\alpha \in \Omega$. Denote by $G_\alpha$ the stabilizer of $\alpha$ in $G$, that is, the subgroup of $G$ fixing the point $\alpha$. We say that $G$ is semiregular on $\Omega$ if $G_\alpha = 1$ for every $\alpha \in \Omega$ and regular if $G$ is transitive and semiregular. It is well-known that a graph $\Gamma$ is a Cayley graph if it has an automorphism group acting regularly on its vertex set (see [4, Lemma 16.3]). If we, instead, require that the graph $\Gamma$ admits a group of automorphisms acting semiregularly on its vertex set with two orbits, then we obtain the so-called bi-Cayley graph.

Cayley graph is usually defined in the following way. Given a finite group $G$ and an inverse closed subset $S \subseteq G \setminus \{1\}$, the Cayley graph $\text{Cay}(G, S)$ on $G$ with respect to $S$ is a

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graph with vertex set $G$ and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. For any $g \in G$, $R(g)$ is the permutation of $G$ defined by $R(g) : x \mapsto xg$ for $x \in G$. Set $R(G) := \{R(g) \mid g \in G\}$. It is well-known that $R(G)$ is a subgroup of Aut(Cay($G, S$)). In 1981, Godsil [10] proved that the normalizer of $R(G)$ in Aut(Cay($G, S$)) is $\bar{R}(G) \rtimes \text{Aut}(G, S)$, where Aut($G, S$) is the group of automorphisms of $G$ fixing the set $S$ set-wise. This result has been successfully used in characterizing various families of GRRs, namely, Cayley graphs Cay($G, S$) such that $R(G) = \text{Aut}(\text{Cay}(G, S))$ (see, for example, [10, 11]). A Cayley graph Cay($G, S$) is said to be normal if $R(G)$ is normal in Aut(Cay($G, S$)). This concept was introduced by Xu in [24], and for more results about normal Cayley graphs, we refer the reader to [8].

Similarly, every bi-Cayley graph admits the following concrete realization. Given a group $H$, let $R, L$ and $S$ be subsets of $H$ such that $R^{-1} = R, L^{-1} = L$ and $R \cup L$ does not contain the identity element of $H$. The bi-Cayley graph over $H$ relative to the triple $(R, L, S)$, denoted by BiCay($H, R, L, S$), is the graph having vertex set the union of the right part $H_0 = \{h_0 \mid h \in H\}$ and the left part $H_1 = \{h_1 \mid h \in H\}$, and edge set the union of the right edges $\{(h_0, g_0) \mid gh^{-1} \in R\}$, the left edges $\{(h_1, g_1) \mid gh^{-1} \in L\}$ and the spokes $\{(h_0, g_1) \mid gh^{-1} \in S\}$. Let $\Gamma = \text{BiCay}(H, R, L, S)$. For $g \in H$, define a permutation $BR(g)$ on the vertices of $\Gamma$ by the rule

$$h_i^{BR(g)} = (hg)_i, \forall i \in \mathbb{Z}_2, h \in H.$$  

Then $BR(H) = \{BR(g) \mid g \in H\}$ is a semiregular subgroup of Aut($\Gamma$) which is isomorphic to $H$ and has $H_0$ and $H_1$ as its two orbits. When $BR(H)$ is normal in Aut($\Gamma$), the bi-Cayley graph $\Gamma = \text{BiCay}(H, R, L, S)$ will be called a normal bi-Cayley graph over $H$ (see [3] or [27])

Wang et al. in [22] determined the groups having a connected normal Cayley graph.

**Proposition 1.1.** Every finite group $G$ has a normal Cayley graph unless $G \cong C_4 \times C_2$ or $G \cong Q_8 \times C_2^r (r \geq 0)$.

Following up this result, Arezoomand and Taeri in [3] asked: Which finite groups have normal bi-Cayley graphs? They also gave a partial answer to this question by proving that every finite group $G \cong Q_8 \times C_2^r (r \geq 0)$ has at least one normal bi-Cayley graph. At the end of [3], the authors asked the following question:

**Question 1.2 ([3, Question]).** Is there any normal bi-Cayley graph over $G \cong Q_8 \times C_2^r$ for each $r \geq 0$?

We remark that for every finite group $G \cong Q_8 \times C_2^r (r \geq 0)$, the normal bi-Cayley graph over $G$ constructed in the proof of [3, Theorem 5] is not of regular valency, and so is not vertex-transitive. So it is natural to ask the following question.

**Question 1.3.** Is there any vertex-transitive normal bi-Cayley graph over a finite group $G$?

In this paper, Questions 1.2 and 1.3 are answered in positive. The following is the main result of this paper.

**Theorem 1.4.** Every finite group has a vertex-transitive normal bi-Cayley graph.

To end this section we give some notation which is used in this paper. For a positive integer $n$, denote by $C_n$ the cyclic group of order $n$, by $\mathbb{Z}_n$ the ring of integers modulo $n$, by $D_{2n}$ the dihedral group of order $2n$, and by Alt($n$) and Sym($n$) the alternating group
and symmetric group of degree \( n \), respectively. Denote by \( Q_8 \) the quaternion group. For two groups \( M \) and \( N \), \( N \rtimes M \) denotes a semidirect product of \( N \) by \( M \). The identity element of a finite group \( G \) is denoted by \( 1 \).

For a finite, simple and undirected graph \( \Gamma \), we use \( V(\Gamma) \), \( E(\Gamma) \) and \( \text{Aut}(\Gamma) \) to denote its vertex set, edge set and full automorphism group, respectively, and for any \( u, v \in V(\Gamma) \), \( u \sim v \) means that \( u \) and \( v \) are adjacent. A graph \( \Gamma \) is said to be vertex-transitive if its full automorphism group \( \text{Aut}(\Gamma) \) acts transitively on its vertex set. For any subset \( B \) of \( V(\Gamma) \), the subgraph of \( \Gamma \) induced by \( B \) will be denoted by \( \Gamma[B] \).

2 Cartesian products

The Cartesian product \( X \square Y \) of graphs \( X \) and \( Y \) is a graph with vertex set \( V(X) \times V(Y) \), and with vertices \( (u, x) \) and \( (v, y) \) being adjacent if and only if \( u = v \) and \( x \sim y \) in \( Y \), or \( x = y \) and \( u \sim v \) in \( X \).

A non-trivial graph \( X \) is prime if it is not isomorphic to a Cartesian product of two smaller graphs. The following proposition shows the uniqueness of the prime factor decomposition of connected graphs with respect to the Cartesian product.

**Proposition 2.1** ([12, Theorem 6.6]). Every connected finite graph can be decomposed as a Cartesian product of prime graphs, uniquely up to isomorphism and the order of the factors.

Two non-trivial graphs are relatively prime (w.r.t. Cartesian product) if they have no non-trivial common factor. Now we consider the automorphisms of Cartesian product of graphs.

**Proposition 2.2** ([12, Theorem 6.10]). Suppose \( \phi \) is an automorphism of a connected graph \( \Gamma \) with prime factor decomposition \( \Gamma = \Gamma_1 \square \Gamma_2 \square \cdots \square \Gamma_k \). Then there is a permutation \( \pi \) of \( \{1, 2, \ldots, k\} \) and isomorphisms \( \phi_i : \Gamma_{\pi(i)} \rightarrow \Gamma_i \) for which

\[
\phi(x_1, x_2, \ldots, x_k) = (\phi_1(x_{\pi(1)}), \phi_2(x_{\pi(2)}), \ldots, \phi_k(x_{\pi(k)})).
\]

**Corollary 2.3** ([12, Corollary 6.12]). Let \( \Gamma \) be a connected graph with prime factor decomposition \( \Gamma = \Gamma_1 \square \Gamma_2 \square \cdots \square \Gamma_k \). If \( \Gamma_1, \Gamma_2, \ldots, \Gamma_k \) are relatively prime, then \( \text{Aut}(\Gamma) = \text{Aut}(\Gamma_1) \times \text{Aut}(\Gamma_2) \times \cdots \times \text{Aut}(\Gamma_k) \).

The following theorem provides a method of constructing normal bi-Cayley graphs.

**Theorem 2.4.** Let \( X \) be a connected normal bi-Cayley graph over a group \( H \), and let \( Y \) be a connected normal Cayley graph over a group \( K \). If \( X \) and \( Y \) are relatively prime, then \( X \square Y \) is also a normal bi-Cayley graph over the group \( H \times K \).

**Proof.** Assume that \( X \) and \( Y \) are relatively prime. By Corollary 2.3, \( \text{Aut}(X \square Y) = \text{Aut}(X) \times \text{Aut}(Y) \). Since \( X \) is a connected normal bi-Cayley graph over \( H \), one has \( BR(H) \leq \text{Aut}(X) \), and since \( Y \) is a connected normal Cayley graph over a group \( K \), one has \( R(K) \leq \text{Aut}(Y) \). Then \( BR(H) \times R(K) \) is a normal subgroup of \( \text{Aut}(X \square Y) = \text{Aut}(X) \times \text{Aut}(Y) \). Note that \( BR(H) \) acts semiregularly on \( V(X) \) with two orbits, and \( R(K) \) acts regularly on \( V(Y) \). It follows that \( BR(H) \times R(K) \) acts semiregularly on \( V(X) \times V(Y) \) with two orbits, and thereby \( X \square Y \) is also a normal bi-Cayley graph over the group \( H \times K \). \( \square \)
3 Normal bi-Cayley graphs over $Q_8 \times C_2^r (r \geq 0)$

In this section, we shall answer Question 1.2 in positive.

3.1 The $n$-dimensional hypercube

For $n \geq 1$, the $n$-dimensional hypercube, denoted by $Q_n$, is the graph whose vertices are all the $n$-tuples of 0’s and 1’s with two $n$-tuples being adjacent if and only if they differ in exactly one place.

Let $N = C_2^n$ be an elementary abelian 2-group of order $2^n$ with a minimum generating set $S = \{s_1, s_2, s_3, \ldots, s_n\}$. By the definition of $Q_n$, we have $\text{Cay}(N, S) \cong Q_n$. For convenience of the statement, we assume that $Q_n = \text{Cay}(N, S)$. If $n = 1$, then $Q_1 = K_2$ and so $\text{Aut}(Q_1) = N$. In what follows, assume that $n \geq 2$. It is easy to observe that for any distinct $s_i, s_j$ there is a unique 4-cycle in $Q_n$ passing through $1, s_i, s_j, 1$, where 1 is the identity element of $N$. So if a subgroup of $\text{Aut}(Q_n)$ fixes $S$ pointwise, then it also fixes every vertex at distance 2 from 1. By the connectedness and vertex-transitivity of $Q_n$, we have $\text{Aut}(Q_n)_1$ acts faithfully on $S$. It follows that $\text{Aut}(Q_n)_1 \leq \text{Sym}(n)$. On the other hand, it is easy to see that each permutation on $S$ induces an automorphism of $N$, and so $\text{Aut}(N, S) \cong \text{Sym}(n)$. Since $\text{Aut}(N, S) \leq \text{Aut}(Q_n)_1$, one has $\text{Aut}(Q_n)_1 = \text{Aut}(N, S) \cong \text{Sym}(n)$. Consequently, we have $\text{Aut}(Q_n) = R(N) \rtimes \text{Aut}(N, S) \cong N \rtimes \text{Sym}(n)$ (see also [25, Lemma 1.1]).

Note that $Q_n$ is bipartite. Let $\text{Aut}(Q_n)^*$ be the kernel of $\text{Aut}(Q_n)$ acting on the two partition sets of $Q_n$. Let $E = R(N) \cap \text{Aut}(Q_n)^*$. Then $E \trianglelefteq \text{Aut}(Q_n)^*$ and $E \leq R(N)$. It follows that $E \trianglelefteq \text{Aut}(Q_n)^* R(N) = \text{Aut}(Q_n)$. Clearly, $E$ acts semiregularly on $V(Q_n)$ with two orbits. Thus, we have the following lemma.

**Lemma 3.1.** Use the same notation as in the above three paragraphs. For any $n \geq 1$, $Q_n$ is a normal Cayley graph over $N$, and $Q_n$ is also a normal bi-Cayley graph over $E$.

3.2 The Möbius-Kantor graph

The Möbius-Kantor graph $GP(8, 3)$ is a graph with vertex set $V = \{i, i' \mid i \in \mathbb{Z}_8\}$ and edge set the union of the outer edges $\{(i, i+1) \mid i \in \mathbb{Z}_8\}$, the inner edges $\{(i', (i+3)') \mid i \in \mathbb{Z}_8\}$, and the spokes $\{(i, i') \mid i \in \mathbb{Z}_8\}$ (see Figure 1). Note that $GP(8, 3)$ is a bipartite graph with bipartition sets $B_1 = \{1, 3, 5, 7, 0', 2', 4', 6'\}$ and $B_2 = \{0, 2, 4, 6, 1', 3', 5', 7'\}$.

In [26], the edge-transitive groups of automorphisms of $\text{Aut}(GP(8, 3))$ were determined. We first introduce the following automorphisms of $GP(8, 3)$, represented as permutations on the vertex set $V$:

$$
\begin{align*}
\alpha &= (1 3 5 7)(0 2 4 6)(1' 3' 5' 7')(0' 2' 4' 6'), \\
\beta &= (0 1')(0' 2')(3 4' 5')(7 6')(4 5')(7 6')(2 4'), \\
\gamma &= (1 1')(2 6')(3 3')(4 0')(5 5')(6 2')(7 7')(0 4'), \\
\delta &= (1 1')(2 6')(3 3')(4 2')(5 5')(6 0')(7 3')(0 6').
\end{align*}
$$

By [26, Lemma 3.1], we have $\langle \alpha, \beta \rangle = \langle \alpha, \alpha^\beta \rangle \times \langle \beta \rangle \cong Q_8 \times \mathbb{Z}_3$, where $Q_8$ is the quaternion group, and moreover, $\langle \alpha, \beta \rangle \leq \text{Aut}(GP(8, 3))$. Clearly, $\langle \alpha, \alpha^\beta \rangle \cong Q_8$ is the Sylow 2-subgroup of $\langle \alpha, \beta \rangle$, so $\langle \alpha, \alpha^\beta \rangle$ is characteristic in $\langle \alpha, \beta \rangle$, and then it is normal in $\text{Aut}(GP(8, 3))$ because $\langle \alpha, \beta \rangle \leq \text{Aut}(GP(8, 3))$. For convenience of the statement, we let $Q_8 = \langle \alpha, \alpha^\beta \rangle$. It is easy to see that $Q_8$ acts semiregularly on $V$ with two orbits $B_1$ and $B_2$. Thus we have the following lemma.
Lemma 3.2. **GP(8, 3) is a normal bi-Cayley graph over Q₈.**

### 3.3 An answer to Question 1.2

Noting that GP(8, 3) is of girth 6, GP(8, 3) is prime. For each \( r \geq 1 \), it is easy to see that \( Q_r = K_2 \Box K_2 \Box \cdots \Box K_2 \). So, \( Q_n \) and GP(8, 3) are relatively prime. Now combining together Lemmas 3.1 and 3.2 and Theorem 2.4, we can obtain the following theorem.

**Theorem 3.3.** For each \( r \geq 1 \), GP(8, 3) \( \times Q_r \) is a vertex-transitive normal bi-Cayley graph over \( Q_8 \times N \), where \( N \cong C_2^r \).

### 4 Proof of Theorem 1.4

The proof of Theorem 1.4 will be completed by the following lemmas. Let \( G \) be a group. A Cayley graph \( \Gamma = \text{Cay}(G, S) \) on \( G \) is said to be a *graphical regular representation* (or *GRR* for short) of \( G \) if \( \text{Aut}(\Gamma) = R(G) \).

**Lemma 4.1.** Let \( G \) be a group admitting a GRR \( \Gamma \). Then \( \Gamma \Box K_2 \) is a normal bi-Cayley graph over the group \( G \).

**Proof.** If \( K_2 \) and \( \Gamma \) are relatively prime, then by Corollary 2.3, we have \( \text{Aut}(\Gamma \Box K_2) = \text{Aut}(\Gamma) \times \text{Aut}(K_2) \). Clearly, \( R(G) \times \{1\} \) acts semiregularly on \( V(\Gamma \Box K_2) \) with two orbits, and \( R(G) \times \{1\} \trianglelefteq \text{Aut}(\Gamma \Box K_2) \), where \( 1 \) is the identity of \( \text{Aut}(K_2) \). It follows that \( \Gamma \Box K_2 \) is a normal bi-Cayley graph over the group \( G \).

Suppose that \( K_2 \) is also a prime factor of \( \Gamma \). Let \( \Gamma = \Gamma_1 \Box \underbrace{K_2 \Box \cdots \Box K_2}_{m \text{ times}} \) be such that \( \Gamma_1 \) is coprime to \( K_2 \). From Corollary 2.3 it follows that \( G = \text{Aut}(\Gamma) = \text{Aut}(\Gamma_1) \times \text{Aut}(K_2 \Box \cdots \Box K_2) \). Since \( \Gamma \) is a GRR of \( G \), one has \( m = 1 \), and therefore \( \Gamma \Box K_2 = \Gamma_1 \Box K_2 \Box K_2 \). Then \( G = \text{Aut}(\Gamma_1) \times \text{Aut}(K_2 \Box K_2) \), and \( \Gamma_1 \) is a GRR of \( \text{Aut}(\Gamma_1) \). By Lemma 3.1, \( K_2 \Box K_2 \) is a normal bi-Cayley graph over \( C_2 \), and by Theorem 2.4, \( \Gamma \Box K_2 \) is a normal bi-Cayley graph over \( \text{Aut}(\Gamma_1) \times C_2 \cong G \).

A group \( G \) is called *generalized dicyclic group* if it is non-abelian and has an abelian
Corollary 2.3, \( \Gamma \) is a normal bi-Cayley graph. In what follows, we assume that \( G \) is a subgroup of \( \text{Aut}(\Gamma) \).

Proof. By Proposition 1.1, \( G \) is a normal bi-Cayley graph. Let \( \Gamma \) be the subgroups of \( \text{Aut}(\Gamma) \) and \( x \) be the element of order \( 2 \) in \( \text{Aut}(\Gamma) \).

Theorem 4.2. A finite group \( G \) admits a GRR unless \( G \) belongs to one of the following classes of groups:

(I) Class C: abelian groups of exponent greater than two;

(II) Class D: the generalized dicyclic groups;

(III) Class E: the following thirteen “exceptional groups”:

1. \( \mathbb{Z}_2^2 \), \( \mathbb{Z}_2^3 \), \( \mathbb{Z}_2^4 \);
2. \( D_6, D_8, D_{10} \);
3. \( A_4 \);
4. \( \langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle \);
5. \( \langle a, b \mid a^8 = b^2 = 1, bab = a^5 \rangle \);
6. \( \langle a, b, c \mid a^3 = b^3 = c^2 = 1, ab = ba, (ac)^2 = (cb)^2 = 1 \rangle \);
7. \( \langle a, b, c \mid a^3 = b^3 = c^3 = 1, ac = ca, bc = cb, c = a^{-1}b^{-1}c \rangle \);
8. \( Q_8 \times \mathbb{Z}_3, Q_8 \times \mathbb{Z}_4 \).

Lemma 4.3. Let \( G \) be a group in Class D of Theorem 4.2. Then \( G \) has a normal bi-Cayley graph.

Proof. If \( G \cong Q_8 \times C_2^r \) for some \( r \geq 0 \), then by Theorem 3.3 and Lemma 3.2, \( G \) has a normal bi-Cayley graph. In what follows, we assume that \( G \neq Q_8 \times C_2^r \) for any \( r \geq 0 \).

By Proposition 1.1, \( G \) has a normal Cayley graph, say \( \Gamma \). If \( \Gamma \) is a normal Cayley graph over a group which is isomorphic to \( K_2 \), then by Corollary 2.3, \( \Gamma \square K_2 \) is a normal bi-Cayley graph over \( G \).

Now suppose that \( K_2 \) is a prime factor of \( \Gamma \). Let \( \Gamma = \Gamma_1 \square Q_m \), where \( Q_m = K_2 \square \cdots \square K_2 \) and \( \Gamma_1 \) is a normal Cayley graph over a group which is isomorphic to \( G_{V_x} \) and \( G_{U_y} \) be the subgroups of \( G \) fixing \( V_x \) and \( U_y \) setwise, respectively. We shall prove the following claim.

Claim. \( G = G_{V_x} \times G_{U_y} \), \( \Gamma_1 \) is a normal Cayley graph over a group which is isomorphic to \( G_{V_x} \), and \( G_{U_y} \cong C_2^m \).

Since \( \Gamma \) is vertex-transitive, by Proposition 2.2, \( V_x \) is an orbit of \( \text{Aut}(\Gamma_1) \times \{1\} \) and \( \text{Aut}(\Gamma_1) \times \{1\} \leq \text{Aut}(\Gamma) \). As \( \text{Aut}(\Gamma_1) \times \{1\} \leq \text{Aut}(\Gamma) \), each \( V_x \) is a block of imprimitivity of \( \text{Aut}(\Gamma) \) (namely, either \( V_x^g = V_x \) or \( V_x^g \cap V_x = \emptyset \) for any \( g \in \text{Aut}(\Gamma) \)). Consider the quotient graph \( \Gamma' \) with vertex set \( \{V_x \mid x \in V(Q_m)\} \), and \( V_x \) is adjacent to \( V_{x'} \) if and only if \( x \) is adjacent to \( x' \) in \( Q_m \). Then \( \Gamma' \cong Q_m \), and \( \text{Aut}(\Gamma_1) \times \{1\} \) is just the kernel of \( \text{Aut}(\Gamma) \) acting on \( V(\Gamma') \). This implies that the subgroup \( \text{Aut}(\Gamma_1)_x \) of \( \text{Aut}(\Gamma)_x \) fixing \( V_x \) set-wise is just \( \text{Aut}(\Gamma_1) \times \text{Aut}(Q_m)_x \). Since \( G \) is regular on \( V(\Gamma) \), \( G_{V_x} \) is also regular on \( V_x \), and so \( \Gamma_1 \cong \Gamma[V_x] \) may be viewed as a Cayley graph on \( G_{V_x} \). Since \( G \leq \text{Aut}(\Gamma) \), one has \( G_{V_x} = G \cap \text{Aut}(\Gamma)_x \leq \text{Aut}(\Gamma)_x \). Note that \( \{1\} \times \text{Aut}(Q_m)_x \) fixes every vertex in \( V_x \). It follows that \( G_{V_x} \cap (\{1\} \times \text{Aut}(Q_m)_x) \) is trivial, and so \( G_{V_x} \) can be viewed as a
normal regular subgroup of $\text{Aut}(\Gamma_1) \times \{1\}$. Therefore, $\Gamma_1$ is a normal Cayley graph over some group, say $H \cong G_{V_x}$.

With a similar argument as above, we can show that $Q_m$ is also a normal Cayley graph over some group, say $K \cong G_{U_y}$. From the argument in Section 3.1, we have $\text{Aut}(Q_m) = N \times \text{Sym}(m)$ with $N \cong C_{2^m}$. We claim that $K = N$. If this is not true, then we would have $1 \neq KN/N \leq \text{Aut}(Q_m)/N \cong \text{Sym}(m)$, and since $K$ is a 2-group, the only possibility is $m = 4$. However, by Magma [6], $\text{Aut}(Q_4)$ has only one normal regular subgroup which is isomorphic to $C_2^4$, a contradiction. Thus, $K = N \cong C_2^m$, and hence $G_{U_y} \cong C_{2^m}$.

For any $g \in G_{V_x} \cap G_{U_y}$, we have $g$ fixes $(y, x)$ and so $g = 1$ because $G$ is regular on $V(\Gamma)$. Thus, $G_{V_x} \cap G_{U_y} = \{1\}$. Then $|G_{V_x}G_{U_y}| = |G_{V_x}| |G_{U_y}| = |V_x||U_y| = |V(\Gamma)| = |G|$. It follows that $G = G_{V_x}G_{U_y}$. To show that $G = G_{V_x} \times G_{U_y}$, it suffices to show that both $G_{V_x}$ and $G_{U_y}$ are normal in $G$. As $G$ is a generalized dicyclic group, it is non-abelian and has an abelian subgroup $L$ of index 2 and an element $b \in G \setminus L$ of order 4 such that $b^{-1}ab = a^{-1}$ for every $a \in L$.

Suppose that $G_{U_y} \nsubseteq L$. Then there exists $g \in G_{U_y}$ such that $g = ab^i$ for some $a \in L$ and $i = 1$ or $-1$. Since $G_{U_y} \cong C_{2^m}$, $g$ is also an involution, and so $G = L \times \langle g \rangle$. Clearly, for any $a \in L$, we have $g^{-1}ag = a^{-1}$, and so $(ga)^2 = 1$. This would force that every element of $G$ outside $L$ is an involution, a contradiction. Thus, $G_{U_y} \leq L$, and hence $G_{U_y} \leq G$.

Since $G = G_{V_x}G_{U_y}$, $G_{U_y} \leq L$ implies that $G_{V_x} \nsubseteq L$. Then $|G_{V_x} : G_{V_x} \cap L| = 2$ since $|G : L| = 2$. It then follows that $G_{V_x} \cap L \leq G$, and hence $G_{V_x} \cap L = (G_{U_y}(G_{V_x} \cap L)/G_{V_x} \cap L) \times (G_{V_x}/G_{V_x} \cap L)$.

Again as $G$ is a generalized dicyclic group and since $G_{U_y} \leq L$, the non-trivial element of $G_{V_x}/G_{V_x} \cap L$ maps every element of $G_{U_y}(G_{V_x} \cap L)/G_{V_x} \cap L$ to its inverse. Since $G_{U_y} \cong C_{2^m}$, one has $G/G_{V_x} \cap L$ is abelian, and so $G_{V_x} \leq G$, completing the proof of the Claim.

By Lemma 3.1, we may let $Q_{m+1} = K_2 \boxtimes \cdots \boxtimes K_2$ be a connected normal bi-Cayley graph over $G_{V_x} \cong C_{2^m}$. By Claim, we may view $\Gamma_1$ as a normal Cayley graph over $G_{V_x}$. Since $\Gamma_1$ is coprime to $K_2$, by Theorem 2.4, $\Gamma_1 \boxtimes Q_{m+1}$ is a connected normal bi-Cayley graph over $G_{V_x} \times G_{U_y} = G$.

**Lemma 4.4.** Let $G$ be a group in Class E of Theorem 4.2. Then $G$ has a normal bi-Cayley graph.

**Proof.** By Lemma 3.1, each of the groups in Class E (1) has a connected normal bi-Cayley graph.

Let $G = D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ with $n \geq 3$. Let $\Gamma = \text{Cay}(G, \{ab, b\})$. Then $\Gamma$ is a cycle of length $2n$, and so $\Gamma$ is coprime to $K_2$. By Theorem 2.4, $\Gamma \boxtimes K_2$ is a connected normal bi-Cayley graph over $G$. Thus, each of the groups in Class E (2) has a connected normal bi-Cayley graph.

Let $G = \text{Alt}(4)$ and let $\Gamma = \text{Cay}(G, \{(1 2 3), (1 3 2), (1 2 4), (1 4 2)\})$. By Magma [6], we have $\Gamma \boxtimes K_2$ is a connected normal bi-Cayley graph over $\text{Alt}(4)$.

Let $G = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$ be the group in Class E (4). Let $\Gamma = \text{Cay}(G, \{a, b, c\})$. By Magma [6], $\Gamma$ is a connected trivalent normal Cayley graph over $G$ and $\Gamma$ has girth 6. Hence, $\Gamma$ is coprime to $K_2$. By Theorem 2.4, $\Gamma \boxtimes K_2$ is a connected normal bi-Cayley graph over $G$.  

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Let \( G = \langle a, b \mid a^8 = b^2 = 1, bab = a^5 \rangle \) be the group in Class E (5). Let \( \Gamma = \text{Cay}(G, \{a, a^{-1}, b, a^4, a^4b\}) \). By [22, Lemma 6], \( \Gamma \) is a connected normal Cayley graph over \( G \), and by Magma, \( \text{Aut}(\Gamma \square K_2) = \text{Aut}(\Gamma) \times \mathbb{Z}_2 \). Thus, \( \Gamma \square K_2 \) is a normal bi-Cayley graph over \( G \).

Let \( G = \langle a, b, c \mid a^3 = b^3 = c^2 = 1, ac = ca, (ab)^2 = (cb)^2 = 1 \rangle \) be the group in Class E (6). Let \( \Gamma = \text{Cay}(G, \{c, ca, cb\}) \). By Magma [6], \( \Gamma \) is a connected trivalent normal Cayley graph over \( G \) and \( \Gamma \) has girth 6. Hence, \( \Gamma \) is coprime to \( K_2 \). By Lemma 2.4, \( \Gamma \square K_2 \) is a connected normal bi-Cayley graph over \( G \).

Let \( G = \langle a, b, c \mid a^3 = b^3 = c^2 = 1, ac = ca, bc = cb, c = a^{-1}b^{-1}cb \rangle \) be the group in Class E (7). Let \( \Gamma = \text{Cay}(G, \{a, b, a^{-1}, b^{-1}\}) \). By Magma [6], \( \Gamma \) is a connected trivalent normal Cayley graph over \( G \). Since \( G \) has order 27, \( \Gamma \) is coprime to \( K_2 \). By Theorem 2.4, \( \Gamma \square K_2 \) is a connected normal bi-Cayley graph over \( G \).

Finally, we consider the groups in Class E (8). By Lemma 3.2, \( \text{GP}(8, 3) \) is a normal bi-Cayley graph over \( Q_8 \). For \( n \geq 3 \), let \( C_n = \langle a \rangle \) and let \( \Gamma = \text{Cay}(C_n, \{a, a^{-1}\}) \). Clearly, \( \Gamma \) is a normal Cayley graph over \( C_n \). Since \( \text{GP}(8, 3) \) is of girth 6, \( \text{GP}(8, 3) \) is coprime to \( \Gamma \). By Theorem 2.4, \( \text{GP}(8, 3) \square \Gamma \) is a connected normal bi-Cayley graph over \( Q_8 \times C_n \). Thus each of the groups in Class E (8) has a connected normal bi-Cayley graph.

**Lemma 4.5.** Let \( G \) be a group in Class C of Theorem 4.2. Then \( G \) has a normal bi-Cayley graph.

**Proof.** Since \( G \) is abelian, \( G \) has an automorphism \( \alpha \) such that \( \alpha \) maps every element of \( G \) to its inverse. Set \( H = G \rtimes \langle \alpha \rangle \). If \( H \) has a GRR \( \Gamma \), then \( \Gamma \) is also a normal bi-Cayley graph over \( G \). Suppose that \( H \) has no GRR. Then by Theorem 4.2 we have \( H \) is one of the groups in Class E (2) and (6). By Lemma 4.4, \( G \) has a normal bi-Cayley graph.

**Proof of Theorem 1.4.** Let \( G \) be a finite group. If \( G \) has a GRR, then by Lemma 4.1, \( G \) has a connected normal bi-Cayley graph. If \( G \) does not have a GRR, then the theorem follows from Lemmas 4.3, 4.4, 4.5 and 3.2.

5 **Final remarks**

This paper would not be complete without mentioning some related work, namely on some special families of bi-Cayley graphs such as bi-circulants, bi-abelians etc. Numerous papers on the topic have been published (see, for instance, [1, 2, 7, 13, 14, 15, 16, 17, 18, 19, 20, 21]). In view of these, the following problem arises naturally.

**Problem 5.1.** For a given finite group \( H \), classify or characterize bi-Cayley graphs with specific symmetry properties over \( H \).

Let \( H \) be a finite group. We say that a bi-Cayley graph \( \Gamma \) of regular valency over \( H \) is a bi-graphical regular representation (or bi-GRR for short) if \( \text{Aut}(\Gamma) = BR(H) \). Motivated by the classification of finite groups having no GRR (see Theorem 4.2), we would like to pose the following problem.

**Problem 5.2.** Determine finite groups which have no bi-GRR.

**References**


