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# Facial parity edge colouring

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**Abstract**

A *facial parity edge colouring* of a connected bridgeless plane graph is an edge colouring in which no two face-adjacent edges (consecutive edges of a facial walk of some face) receive the same colour, in addition, for each face $\alpha$ and each colour $c$, either no edge or an odd number of edges incident with $\alpha$ is coloured with $c$. From Vizing's theorem it follows that every 3-connected plane graph has a such colouring with at most $\Delta^* + 1$ colours, where $\Delta^*$ is the size of the largest face. In this paper we prove that any connected bridgeless plane graph has a facial parity edge colouring with at most 92 colours.

*Keywords: plane graph, facial walk, edge colouring*

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## 1 Introduction

One of the motivations for this paper has come from recent papers of Bunde et al. [1, 2] who introduced parity edge colourings of simple graphs. Studying the parity of the usage of colours along walks suggested two edge-colouring parameters that have interesting properties and applications. A *parity walk* in an edge colouring of a simple graph is a walk along which each colour is used an even number of times. Bunde et al. [2] introduced two parameters. Let $p(G)$ be the minimum number of colours in an edge colouring of $G$ having no parity path (a *parity edge colouring*). Let $\widehat{p}(G)$ be the minimum number of colours in an edge colouring of $G$ in which every parity walk is closed (a *strong parity edge colouring*). Since incident edges of the same colour would form a parity path of length 2, every parity edge colouring is a proper edge colouring, and hence $p(G) \geq \chi'(G)$, where $\chi'(G)$ is the chromatic index of $G$. Since a path is an open walk, no strong parity edge colouring has a parity path. Hence, every strong parity edge colouring is a parity edge colouring and $\widehat{p}(G) \geq p(G)$ for every graph $G$. Although there are graphs $G$ with $\widehat{p}(G) > p(G)$ [2], it remains unknown how large $\widehat{p}(G)$ can be when $p(G) = k$. Elementary results on these parameters appear in [2]. In [1] it is proved that $\widehat{p}(K_n) = 2^{\lceil log(n) \rceil} - 1$ for all $n$. Moreover, the optimal strong parity edge colouring of the complete $n$-vertex graph $K_n$ is unique when $n$ is a power of 2. The authors of [2] mentioned that computing $p(G)$ or $\widehat{p}(G)$ is NP-hard even when $G$ is a tree. Clearly, the parity edge colouring is such a colouring that each path uses at least one colour an odd number of times.

The vertex version of this problem (*strong parity vertex colouring*) with some restrictions was introduced in [3]. The authors of [3] conjectured that there is a constant $K$ such that the vertices of any 2-connected plane graph can be coloured with at most $K$ colours in such a way that for each face $\alpha$ and each colour $c$, either no vertex or an odd number of vertices incident with $\alpha$ is coloured with $c$.

Another motivation for this work has come from the papers of Pyber [5] and Mátrai [4]. A graph is called *odd* if the degree of its vertices is odd or zero. Pyber raises the problem of edge covering with odd subgraphs in [5] as the counterpart of even subgraph covering problems. He proved that the edges of every finite simple graph can be covered by at most 4 edge-disjoint odd subgraphs; moreover, if the number of vertices is even then 3 odd subgraphs are sufficient. For not necessarily edge-disjoint coverings we have the following question: Is it true that every graph can be covered by at most 3 odd subgraphs? Mátrai in [4] showed that every finite simple graph can be covered by 3 odd subgraphs and he found an infinite sequence of finite simple connected graphs not coverable by 3 edge-disjoint odd subgraphs.

Pyber's result implies the following: The edges of any 3-connected plane graph $G$ can be coloured by at most 4 colours in such a way that for each face $\alpha$ and each colour $c$, either no edge or an odd number of edges incident with $\alpha$ is coloured with $c$. It is sufficient to consider the dual $G^*$ of $G$ and its edge cover with at most 4 edge-disjoint odd subgraphs. This cover induces the required colouring of $G$.

If we add a requirement that such a colouring must be proper, then it is not clear whether there exists a colouring with $K$ colours, where $K$ is an absolute constant. From Vizing's theorem [6] it follows that every 3-connected plane graph $G$ has such a colouring which uses at most $\Delta^* + 1$ colours, where $\Delta^*$ is the size of the largest face. Consider a proper edge colouring of the dual graph $G^*$. This colouring induces a colouring of $G$ in a natural way. It is such a colouring in which, for each face $\alpha$ of $G$, all the edges in the boundary of $\alpha$ have distinct colours.

In this paper we show that each connected bridgeless plane graph has a facial parity edge colouring using at most 92 colours.

## 2 Notation

Let us introduce the notation used in this paper. A graph which can be embedded in the plane is called *planar graph*; a fixed embedding of a planar graph is called *plane graph*.

A *bridge* is an edge whose removal increases the number of components. A graph which contains no bridge is said to be *bridgeless*. In this paper we consider connected bridgeless plane graphs, multiple edges and loops are allowed.

Let $G = (V, E, F)$ be a connected plane graph with the vertex set $V$, the edge set $E$, and the face set $F$. The *degree* of a vertex $v$, denoted by $\deg(v)$, is the number of edges incident with $v$, each loop counting as two edges. A $k$-vertex is a vertex of degree $k$. For a face $\alpha$, the *size* of $\alpha$, $\deg(\alpha)$, is defined to be the length of its *facial walk*, i.e. the shortest closed walk containing all edges from the boundary of $\alpha$. We often say $k$-face for a face of size $k$.

Given a graph $G$ and one of its edges $e = uv$ (the vertices $u$ and $v$ do not have to be different), the *contraction* of $e$ denoted by $G/e$ consists of replacing $u$ and $v$ by a new vertex adjacent to all the former neighbours of $u$ and $v$, and removing the loop corresponding to the edge $e$. (We keep multiple edges if they arise.) Analogously we define the contraction of the set of edges $H = \{e_1, \ldots, e_k\}$ and we denote it by $G/\{e_1, \ldots, e_k\}$ or $G/H$.

Two faces are *adjacent* if they share an edge. Two (distinct) edges are *face-adjacent* if they are consecutive edges of a facial walk of some face $\alpha$.

A *$k$-edge colouring* of a graph $G$ is a mapping $\varphi : E(G) \to \{1, \ldots, k\}$. The *facial parity edge* $(FPE)$ *colouring* of a connected bridgeless plane graph is such an edge colouring that no two face-adjacent edges receive the same colour, in addition, for each face $\alpha$ and each colour $c$, either no edge or an odd number of edges incident with $\alpha$ is coloured with $c$.

**Question 2.1.** What is the minimum number of colours $\chi'_{fp}(G)$ that a connected bridgeless plane graph $G$ has a facial parity edge colouring with at most $\chi'_{fp}(G)$ colours?

The number $\chi'_{fp}(G)$ is called the *facial parity chromatic index* of $G$.

## 3 Results

**Theorem 3.1.** *Let $G$ be a connected bridgeless plane graph. Then*

$$\chi'_{fp}(G) \leq 92.$$

The proof uses the method of discharging. Let $G$ be a counterexample with minimal number of edges, then minimal number of 1-faces, and then minimal number of 2-faces. If $G$ is a single cycle of length $d$, $d \leq 5$, we use exactly $d$ colours. We consider this to be the first step of induction and call this case trivial.

First, we prove several structural properties of $G$.

We say that a face $\alpha$ is *small* if $1 \leq \deg(\alpha) \leq 44$ and a face $\beta$ is *big* if $\deg(\beta) \geq 45$.

### 3.1 Reducible configurations

We find such (forbidden) subgraphs $H$ of $G$ that the facial parity edge colouring of $G/H$ using at most 92 colours can be extended to a required colouring of $G$ using at most 92 colours, which is a contradiction to $G$ being a counterexample. In the sequel, whenever we speak about an FPE colouring, we always mean an FPE colouring using at most 92 colours.

#### 3.1.1 1-faces

**Claim 3.2.** *Each vertex of $G$ is incident with at most one 1-face.*

*Proof.* Let $v$ be a vertex incident with at least two 1-faces $\alpha_1$ and $\alpha_2$. If we split $v$ into two vertices $v_1$ and $v_2$ in such a way that $\alpha_1$ and $\alpha_2$ become a 2-face $\alpha$ and face-adjacency of all edges incident with $v$ is preserved, we obtain a graph $G'$, see Figure 1. It has the same number of edges as $G$, but fewer 1-faces. Thus, it is not a counterexample and we can find an FPE colouring $\varphi'$ of $G'$. It induces an edge colouring $\varphi$ of $G$ in a natural way. It is easy to see that $\varphi$ is an FPE colouring. □


Figure 1: A vertex incident with at least two 1-faces can be split into two vertices, reducing the number of 1-faces. Similarly, one can reduce a 1-face and a $d$-face ($2 \leq d \leq 4$) incident with the same vertex.

**Claim 3.3.** *Each vertex of $G$ incident with a 1-face is not incident with any other $d$-face for $2 \leq d \leq 4$.*

*Proof.* We use the same reduction as in the proof of Claim 3.2. We split the vertex $v$ incident with a 1-face $\gamma$ and a $d$-face $\alpha$ ($2 \leq d \leq 4$) in such a way that the faces $\alpha$ and $\gamma$ become a $(d+1)$-face $\alpha'$ and face-adjacency of all edges incident with $v$ is preserved, see Figure 1 for illustration. Let the reduced graph be $G'$. It has fewer 1-faces than $G$, therefore, it has an FPE colouring $\varphi'$. The face $\alpha'$ has size at most five, therefore, its edges are coloured using $d+1$ different colours. Thus, the colouring $\varphi$ of $G$ induced by the colouring $\varphi'$ of $G'$ is an FPE colouring, too. □

**Claim 3.4.** *Each vertex of $G$ incident with a 2-face is not incident with any $d$-face for $2 \leq d \leq 3$.*

*Proof.* We use the same reduction as in the proofs of Claims 3.2 and 3.3. We omit the details. □

#### 3.1.2 Small faces

**Claim 3.5.** *There are no two adjacent small faces in $G$.*

*Proof.* Let $\alpha_1$ and $\alpha_2$ be two adjacent small faces in $G$.

If both $\alpha_1$ and $\alpha_2$ are 1-faces, the graph consists of a single vertex and a loop; it has an FPE colouring using 1 colour.

Let $\alpha_1$ be a 1-face and $\alpha_2$ be a $d$-face, $d \geq 2$; let $e$ be the loop they share, see Figure 2. Then the graph $G' = G/e$ has fewer edges than $G$, therefore, it has an FPE colouring $\varphi'$. Let $\alpha'$ be a face in $G'$ corresponding to $\alpha_1$ and $\alpha_2$ in $G$. Since $\alpha_2$ is a small face, at most 43 colours occur on the edges incident with $\alpha'$. To extend the colouring $\varphi'$ of $G'$ to an FPE colouring of $G$, it suffices to colour the edge $e$ with any colour that does not occur on $\alpha'$.

Let $\alpha_1$ and $\alpha_2$ be two small faces of size at least 2 and let $e$ be the edge they share, see Figure 2. The graph $G' = G/e$ has fewer edges than $G$, therefore, it has an FPE colouring $\varphi'$. Let $\alpha_1'$ and $\alpha_2'$ be the faces of $G'$ corresponding to the faces $\alpha_1$ and $\alpha_2$ in $G$. (Since $\alpha_1$ and $\alpha_2$ are faces of size at least 2, the size of $\alpha_1'$ and $\alpha_2'$ is at least 1.)

Consider the set of colours different from the colours occurring on the edges of $\alpha_1'$ and $\alpha_2'$ (the colours admissible for the edge $e$). Since $\alpha_1$ and $\alpha_2$ are small, at most $2 \cdot (44 - 1) = 86$ colours occur on the edges incident with $\alpha_1'$ and $\alpha_2'$. Hence, there is an admissible colour $c$. We can extend $\varphi'$ to an FPE colouring $\varphi$ of $G$ by setting $\varphi(e) = c$. □

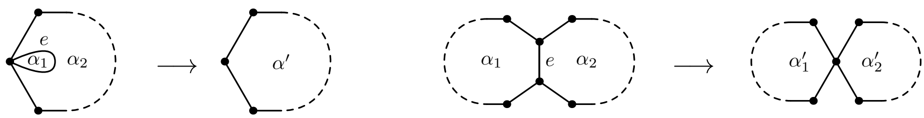


Figure 2: Two adjacent small faces form a reducible configuration: one can contract the edge they share.

**Claim 3.6.** *Let $\beta$ be a big face adjacent to two small faces $\alpha_1$ and $\alpha_2$. Let $e_i$ be an edge incident with $\beta$ and $\alpha_i$, $i = 1, 2$. Then $e_1$ and $e_2$ are face-adjacent.*

*Proof.* Let $e_1$ and $e_2$ not be face-adjacent. See Figure 3 for illustration. The graph $G' = G/\{e_1, e_2\}$ has fewer edges than $G$, therefore, it has an FPE colouring $\varphi'$. Let $\alpha_1'$, $\alpha_2'$, $\beta'$ be the faces of $G'$ corresponding to the faces $\alpha_1$, $\alpha_2$, $\beta$ in $G$, respectively.

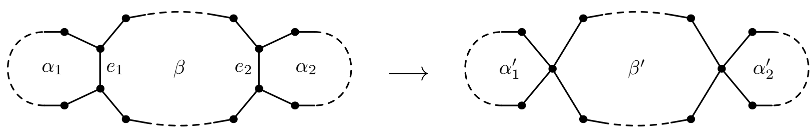


Figure 3: A big face $\beta$ adjacent to two different small faces $\alpha_1$ and $\alpha_2$ forms a reducible configuration unless the edges $e_1$ and $e_2$ are face-adjacent.

We extend the colouring $\varphi'$ of $G'$ to an FPE colouring of $G$ in the following way: Consider the set of colours different from the colours occurring on the edges of $\alpha_1'$ and $\alpha_2'$; also different from the colours occurring on the edges of $G'$ corresponding to the edges of $G$ face-adjacent to $e_1$ and $e_2$. There are at least $92 - 2 \cdot (44 - 1) - 4 = 2$ such colours, say $c_1$ and $c_2$. If at least one of them, say $c_i$, already occurs on $\beta'$, we set $\varphi(e_1) = \varphi(e_2) = c_i$. If none of them occurs on $\beta'$, we set $\varphi(e_i) = c_i$, $i = 1, 2$. □

**Claim 3.7.** *Each big face is adjacent to at most one 1-face.*

*Proof.* It follows from Claims 3.2 and 3.6. □

**Claim 3.8.** *Each big face is adjacent to at most two small faces.*

*Proof.* Let a big face $\beta$ be adjacent to small faces $\alpha_1$, $\alpha_2$, and $\alpha_3$. Consider the edges that $\beta$ shares with $\alpha_i$, $i = 1, 2, 3$. It is easy to see that there must be a pair of edges $e_i$ and $e_j$, incident to $\alpha_i$ and $\alpha_j$, respectively ($i \neq j$), which are not face-adjacent. It is a contradiction with Claim 3.6. □

### 3.1.3 Chains of 2-vertices

**Claim 3.9.** *There is no chain consisting of at least* $5$ *consecutive* $2$*-vertices in* $G$*.*

*Proof.* Let $v_0 e_0 v_1 e_1 \ldots v_p e_p v_{p+1}$ be a chain consisting of $p$ vertices $v_1, \ldots, v_p$ of degree 2, where $p \geq 5$. The graph $G' = G/\{e_1, e_2, e_3, e_4\}$ has an FPE colouring $\varphi'$. Let $e_0'$ and $e_5'$ be the edges in $G'$ corresponding to the edges $e_0$ and $e_5$ in $G$. Let $\varphi'(e_0') = c_1$ and $\varphi'(e_5') = c_2$. The FPE colouring $\varphi'$ of $G'$ can be extended to an FPE colouring $\varphi$ of $G$ by setting $\varphi(e_2) = \varphi(e_4) = c_1$ and $\varphi(e_1) = \varphi(e_3) = c_2$, see Figure 4. □

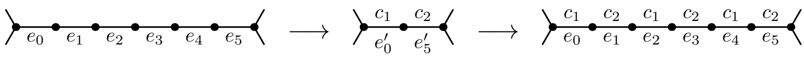


Figure 4: A chain of (at least) five 2-vertices is a reducible configuration.

A $d$-face $\alpha$ is *hanging* on a vertex $v$, if all vertices incident with $\alpha$ are 2-vertices except for the vertex $v$. By Claim 3.9 we have $d \leq 5$. (If $\deg(v) = 2$, the graph $G$ consists of a single cycle of length at most 5, which is the trivial case).

We colour the vertices of $G$ with black, blue and white colour in the following way:

Let all 2-vertices be black, all 3-vertices be blue and all $k$-vertices for $k \geq 6$ be white.

A $4$-vertex $v$ is black if there is a face hanging on it, else it is white. See Figure 5 for illustration of all types of black vertices.

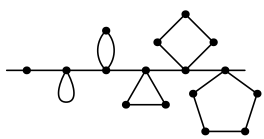

Figure 5: A vertex is black, if it is a 2-vertex, or a 4-vertex with a hanging face.

A $5$-vertex $v$ is blue, if there is a face hanging on it, else it is white.

Observe that any 2-vertex $v$ is incident with two faces. We say $v$ is *bad* for both faces it is incident with. Any black 4-vertex $v$ is incident with a small face $\alpha$ of size at most 5 and two other faces. The face $\beta$ adjacent to $\alpha$ must be big (see Claim 3.5); the vertex $v$ occurs twice on the facial walk of $\beta$. The other face $\gamma$ can be big or small. We say $v$ is *bad* for the face $\gamma$.

**Claim 3.10.** *For any face* $\alpha$*, there is no chain of at least* $5$ *consecutive vertices bad for* $\alpha$*. Each chain of bad vertices contains at most one bad* $4$*-vertex.*

*Proof.* Let $v_0e_0v_1e_1v_2e_2v_3e_3v_4e_4v_5e_5v_6$ be a subpath of the facial walk of $\alpha$ containing 5 vertices $v_1, \dots, v_5$ bad for $\alpha$. It is easy to see that all the edges $e_0, \dots, e_5$ are incident with the same face $\beta$ ($\beta \neq \alpha$). If at least two of $v_1, \dots, v_5$ are bad 4-vertices, we come to a contradiction with Claim 3.5 or with Claim 3.6. If none of them is a 4-vertex, we are in the case of Claim 3.9. If precisely one of them is a 4-vertex, say $v_i$, it is incident with a small face $\gamma$ of size $d \leq 5$.

The graph $G' = G/\{e_1, e_2, e_3, e_4\}$ has an FPE colouring $\varphi'$. Let $e'_0$ and $e'_5$ be the edges in $G'$ corresponding to the edges $e_0$ and $e_5$ in $G$; let $\alpha'$, $\beta'$, $\gamma'$ be the faces in $G'$ corresponding to the faces $\alpha, \beta, \gamma$ in $G$; let $v'$ be the vertex in $G'$ corresponding to $v_1, \dots, v_5$ in $G$. Let $\varphi'(e'_0) = c_1$ and $\varphi'(e'_5) = c_2$. Since $e'_0$ and $e'_5$ are face-adjacent in $G'$, $c_1 \neq c_2$. To extend $\varphi'$ to an FPE colouring of $G$, we proceed in the following way:

If $i = 1$, 3, or 5, we simply set $\varphi(e_2) = \varphi(e_4) = c_1$ and $\varphi(e_1) = \varphi(e_3) = c_2$.

If $i = 2$ or $i = 4$, we set $\varphi(e_2) = \varphi(e_4) = c_1$ and $\varphi(e_1) = \varphi(e_3) = c_2$ and switch the order of colours of edges incident with $\gamma'$, see Figure 6 for illustration (here the small face $\gamma$ is a bigon).

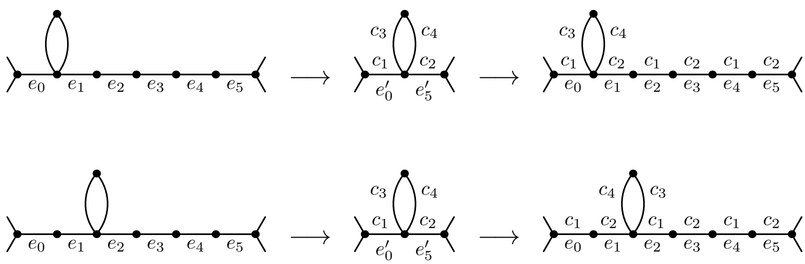


Figure 6: A chain of (at least) five bad black vertices is a reducible configuration as well.

□

**Claim 3.11.** *Let $v$ be a black vertex bad for a small face $\alpha$. If $v$ is a 4-vertex, then the face hanging on it is a 1-face.*

*Proof.* Let $v$ be a black 4-vertex, let $\gamma$ be the face hanging on $v$ of size at least 2 and let $e_1$ and $e_2$ be the edges of $\alpha$ incident with $v$. It is easy to see that the edges $e_1$ and $e_2$ are incident with the same big face, say $\beta$. See Figure 7 for illustration.

There is an edge $e_\gamma$ incident with $\gamma$ and $\beta$, which is not face-adjacent to $e_\alpha \in \{e_1, e_2\}$, which is a contradiction with Claim 3.6. □


Figure 7: Reducible pairs of edges ($e_\gamma$ and $e_\alpha$). For details see Claims 3.11 and 3.12.

**Claim 3.12.** *Let $\alpha$ be a small face sharing at least two bad black vertices with a big face $\beta$. Then all the bad black vertices incident with $\alpha$ and $\beta$ are 2-vertices.*

*Proof.* Let $v_1$ and $v_2$ be black vertices incident with $\alpha$ and $\beta$. If $v_1$ is a 4-vertex, it is bad for $\alpha$, therefore, there is a small face $\gamma$ hanging on $v_1$, adjacent to $\beta$. By Claim 3.11, the face $\gamma$ is a 1-face. Thus, we can find an edge (a loop) $e_\gamma$ incident with $\gamma$ and $\beta$ and an edge $e_\alpha$ incident with $\alpha$ and $\beta$, which are not face-adjacent. This is a contradiction with Claim 3.6. See Figure 7 for illustration. □

**Claim 3.13.** *Let $v$ be a black vertex bad for a small $d$-face $\alpha$, $d \in \{2, 3, 4\}$. Then $v$ is a 2-vertex.*

*Proof.* Let $v$ be a black 4-vertex, let $\gamma$ be the face hanging on $v$. By Claim 3.11, the face $\gamma$ is a 1-face. On the other hand, by Claim 3.3, the face $\gamma$ cannot be a 1-face, which is a contradiction. □

**Claim 3.14.** *If a big face $\beta$ shares a 2-vertex $v$ with a small face $\alpha$, then $\beta$ is not adjacent to any other small face.*

*Proof.* It follows immediately from Claim 3.6. □

**Claim 3.15.** *Let $\gamma$ be a $d$-face, $d \in \{2, 3, 4, 5\}$, hanging on a vertex $v$, adjacent to a big face $\beta$. Then $\beta$ is not adjacent to any other small face.*

*Proof.* It follows immediately from Claim 3.14. □

### 3.2 Discharging rules

If $G$ is a minimal counterexample then it contains no reducible configuration. Let the initial charge of each vertex be $\psi(v) = \deg(v) - 6$ and the initial charge of each face be $\psi(\alpha) = 2\deg(\alpha) - 6$. From Euler's formula we can easily derive that

$$\sum_{\alpha \in F} (2\deg(\alpha) - 6) + \sum_{v \in V} (\deg(v) - 6) = -12.$$

It is obvious that all the negative charge is in the vertices of degree 2, 3, 4, and 5 and in the faces of size 1 and 2.

**Rule 1:** Let $\beta$ be a big face.

- If $\beta$ is adjacent to a single small face $\alpha$, it sends 3 units of charge to $\alpha$.
- If $\beta$ is adjacent to two small faces $\alpha_1$ and $\alpha_2$, such that $\deg(\alpha_1) \le \deg(\alpha_2)$, it sends 2 units of charge to $\alpha_1$ and 1 unit of charge to $\alpha_2$. (If $\deg(\alpha_1) = \deg(\alpha_2)$, it is decided arbitrarily.)

**Rule 2:** Let $\beta$ be a big face.

- It sends 2 units of charge to any black vertex bad for $\beta$.
- It sends 1 unit of charge to any other black, blue, or white vertex incident with $\beta$. (Multiply incident vertices are considered as different.)

**Rule 3:** Let $\alpha$ be a small face.

- It sends 2 units of charge to any black vertex bad for $\alpha$.
- It sends 1 unit of charge to any other black or blue vertex incident with $\alpha$.

**Rule 4:** Let $v$ be a black 4-vertex.

- It sends 2 units of charge to the incident small hanging face $\gamma$.

**Rule 5:** Let $v$ be a blue 5-vertex.

- It sends 2 units of charge to the incident small hanging face $\gamma$.

**Rule 6:** Let $v$ be a $k$-vertex, $k \geq 6$.

- It sends 2 units of charge to any incident small hanging face $\gamma$.

## 3.3 Analysis of the discharging process

### 3.3.1 Vertices

Every 2-vertex is black and bad for both faces incident with it, hence it receives 2 units of charge from both incident faces (Rules 2 and 3). Its new charge is $-4 + 2 + 2 = 0$.

Every 3-vertex is blue, hence it receives 1 unit of charge from all the three incident faces (Rules 2 and 3). Its new charge is $-3 + 3 \cdot 1 = 0$.

Every black 4-vertex $v$ receives 2 units of charge from the face it is bad for (Rules 2 and 3) and $2 \cdot 1$ units of charge from the doubly-incident big face $\beta$ (Rule 2). It sends 2 units of charge to the hanging face $\gamma$ (Rule 4). The new charge of $v$ is $-2 + 2 + 2 \cdot 1 - 2 = 0$.

Every white 4-vertex is incident with at least 2 big faces (see Claim 3.5), therefore, its new charge is at least $-2 + 2 \cdot 1 = 0$.

Every blue 5-vertex $v$ is incident with a hanging face $\gamma$, doubly-incident with a big face $\beta$ and incident with two more faces $\alpha_1$ and $\alpha_2$. Therefore, $v$ receives $4 \cdot 1$ units of charge from the incident faces (Rules 2 and 3). It sends 2 units of charge to $\gamma$ (Rule 5). Therefore, the new charge of $v$ is $-1 + 4 \cdot 1 - 2 = 1$.

Every white 5-vertex is incident with at least 3 big faces (see Claim 3.5), therefore, its new charge is at least $-1 + 3 \cdot 1 = 2$.

Every (white) $k$-vertex $v$, $k \geq 6$, has non-negative initial charge. It receives charge from big faces (Rule 2) and sends charge to the hanging faces $\gamma_1, \ldots, \gamma_r$ (Rule 6). For each hanging face $\gamma_i$, the adjacent big face $\beta_i$ is doubly-incident to $v$. The faces $\beta_i$ and $\beta_j$ are different for different $\gamma_i$ and $\gamma_j$ (see Claims 3.6 and 3.7). Therefore, the new charge of $v$ is at least $r \cdot (2 \cdot 1 - 2) = 0$.

### 3.3.2 1-faces

Let $\gamma$ be a 1-face. It is adjacent to a big face $\beta$ and it is hanging on a vertex $v$. The face $\beta$ is adjacent to at most one 1-face (Claim 3.7). Therefore, it sends at least 2 units of charge to the face $\gamma$ (Rule 1).

If the vertex $v$ is a 2-vertex, we get the trivial graph, which is not a counterexample. If the vertex $v$ is a 3-vertex, the third edge incident with $v$ is a bridge in $G$, which is not allowed. Therefore, $\deg(v) \geq 4$ and the vertex $v$ sends 2 units of charge to the face $\gamma$ (Rules 4 – 6). The new charge of $\gamma$ is at least $-4 + 2 + 2 = 0$.

### 3.3.3 2-faces

Let $\alpha$ be a 2-face. Its initial charge is $-2$. Let $v_1$ and $v_2$ be the vertices incident with $\alpha$. The face $\alpha$ is adjacent to at most two faces, which must be big (see Claim 3.5). Consider the number of black vertices bad for $\alpha$. Note that by Claim 3.13 each such vertex is a 2-vertex. See Figure 8 for illustration.

1. Let both $v_1$ and $v_2$ be black and bad. Then the graph $G$ consists of a single cycle on two vertices, which is not a counterexample.
2. Let $v_1$ be a black 2-vertex. Then $\alpha$ is adjacent to a single big face $\beta$. The big face $\beta$ is not adjacent to any other small face (see Claim 3.14). Therefore, $\beta$ sends 3 units of charge to $\alpha$ (Rule 1). The face $\alpha$ sends 2 units of charge to $v_1$ and at most 1 unit of charge to $v_2$ (Rule 3). On the other hand, $\alpha$ is hanging on $v_2$, therefore, it receives 2 units of charge from $v_2$ (Rules 4 – 6). Note that $v_2$ cannot be a 3-vertex, otherwise there would be a bridge in $G$. The new charge of $\alpha$ is at least $-2-2-1+2+3=0$.
3. Let none of $v_1$ and $v_2$ be black and bad. Consider the number of faces adjacent to $\alpha$. If $\alpha$ is adjacent to a single big face $\beta$, then $\beta$ is not adjacent to any other small face. Therefore, $\beta$ sends 3 units of charge to $\alpha$. Moreover, in this case none of $v_1$ and $v_2$ can be neither black nor blue. The new charge of $\alpha$ is $-2+3=1$.

   If $\alpha$ is adjacent to two big faces $\beta_1$ and $\beta_2$, the face $\alpha$ sends at most $2 \cdot 1$ units of charge to $v_1$ and $v_2$. The big face $\beta_i$, $i=1,2$, is not adjacent to any other small face of size at most 2 (see Claims 3.3, 3.4, and 3.6). Therefore, by Rule 1, the face $\beta_i$, $i=1,2$, sends at least 2 units of charge to $\alpha$. The new charge of $\alpha$ is at least $-2-2\cdot 1+2\cdot 2=0$.

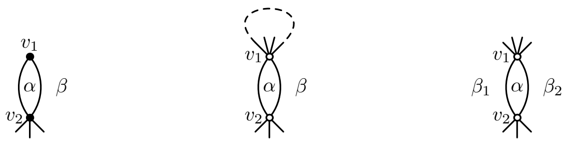


Figure 8: Different possible neighbourhoods of a 2-face $\alpha$.

### 3.3.4 3-faces

Let $\alpha$ be a 3-face. Its initial charge is 0. Let $v_1, v_2, v_3$ be the vertices incident with $\alpha$. (They do not have to be pairwise different.) Consider the number of black vertices bad for $\alpha$. Note that by Claim 3.13 each such vertex is a 2-vertex. See Figure 9 for illustration.

1. Let all the three vertices $v_1, v_2, v_3$ be black and bad. Then the graph $G$ consists of a single cycle on three vertices, which is not a counterexample.
2. Let $v_2$ and $v_3$ be black and bad. Then all the three edges of $\alpha$ are incident with the same big face $\beta$ and $\alpha$ is hanging on $v_1$. Then by Claim 3.15 the face $\beta$ sends 3 units of charge to $\alpha$ (Rule 1). The face $\alpha$ then sends $2 \cdot 2$ units of charge to the 2-vertices $v_2, v_3$ and at most 1 unit of charge to the vertex $v_1$ (Rule 3). If the vertex $v_1$ is a 3-vertex, the third edge incident with it is a bridge in $G$. Therefore

$\deg(v_1) \geq 4$ and $v_1$ sends 2 units of charge to $\alpha$ (Rules 4 – 6). The new charge of $\alpha$ is $3 - 2 \cdot 2 - 1 + 2 = 0$.

3. Let $v_3$ be black and bad. Let $\beta_1$ be the big face incident with the edge $v_1v_2$ and $\beta_2$ be the big face incident with the edges $v_2v_3$ and $v_3v_1$.

   The face $\beta_2$ sends 3 units of charge to $\alpha$, $\beta_1$ sends at least 1 unit of charge to $\alpha$. The face $\alpha$ then sends 2 units of charge to $v_3$ and at most 1 unit of charge to $v_1$ and $v_2$. The new charge of $\alpha$ is at least $3 + 1 - 2 - 2 \cdot 1 = 0$.

4. Let none of the vertices $v_1, v_2, v_3$ be black and bad. Consider the number of faces adjacent to $\alpha$.

   If there are three different faces adjacent to $\alpha$ (they must be big, see Claim 3.5) then the face $\alpha$ receives at least 1 unit of charge from each of them and sends at most 1 unit of charge to each incident vertex. Hence, the new charge of $\alpha$ is at least 0.

   If there is a big face $\beta$ sharing at least two edges with $\alpha$, these edges are not face-adjacent in $\beta$. Therefore, $\beta$ is not adjacent to any other small face, thus, it sends 3 units of charge to $\alpha$. The new charge of $\alpha$ is non-negative again.

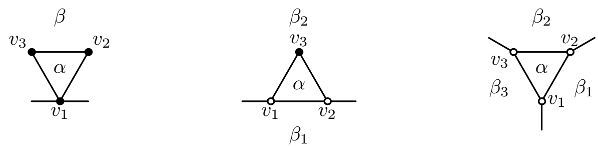


Figure 9: Different possible neighbourhoods of a 3-face $\alpha$.

### 3.3.5 4-faces

Let $\alpha$ be a 4-face. Its initial charge is 2. Let $v_1, v_2, v_3, v_4$ be the vertices incident with $\alpha$. (They do not have to be pairwise different.) Consider the number of black vertices bad for $\alpha$. Note that by Claim 3.13 each such vertex is a 2-vertex. See Figure 10 for illustration.

1. Let all the four vertices $v_1, v_2, v_3, v_4$ be black and bad. Then the graph $G$ consists of a single cycle on four vertices, which is not a counterexample.
2. Let $v_1, v_2$, and $v_3$ be black and bad. Then all the four edges of $\alpha$ are incident with the same big face $\beta$ and $\alpha$ is hanging on $v_4$. Hence, $\beta$ sends 3 units of charge to $\alpha$ (Rule 1). The face $\alpha$ then sends $3 \cdot 2$ units of charge to the 2-vertices $v_1, v_2, v_3$ and at most 1 unit of charge to $v_4$ (Rule 3). If the vertex $v_4$ is a 3-vertex, the third edge incident with it is a bridge in $G$. Therefore, $\deg(v_4) \geq 4$ and $v_4$ sends 2 units of charge to $\alpha$ (Rules 4 – 6). The new charge of $\alpha$ is $2 + 3 - 3 \cdot 2 - 1 + 2 = 0$.
3. Let $v_1$ and $v_3$ be black and bad. Let $\beta_1$ be the big face incident with $v_1$, let $\beta_2$ be the big face incident with $v_3$.

   If $\beta_1 \neq \beta_2$, both $\beta_1$ and $\beta_2$ send 3 units of charge to $\alpha$. The face $\alpha$ then sends 2 units of charge to the vertices $v_1$ and $v_3$ and at most 1 unit of charge to the vertices $v_2$ and $v_4$. The new charge of $\alpha$ is at least $2 + 2 \cdot 3 - 2 \cdot 2 - 2 \cdot 1 = 2$. If $\beta_1 = \beta_2$, then $v_2$ and $v_4$ are not blue, and the new charge of $\alpha$ is at least $2 + 3 - 2 \cdot 2 = 1$.

4. Let $v_1$ and $v_2$ be black and bad. Let $\beta_1$ be the big face incident with the edge $v_3v_4$ and $\beta_2$ be the big face incident with the vertices $v_1$ and $v_2$. If $\beta_1 \neq \beta_2$, $\beta_2$ sends 3 units of charge to $\alpha$ and $\beta_1$ sends at least 1 unit of charge to $\alpha$. The face $\alpha$ then sends 2 units of charge both to $v_1$ and $v_2$ and at most 1 unit of charge to $v_3$ and $v_4$. The new charge of $\alpha$ is at least $2 + 3 + 1 - 2 \cdot 2 - 2 \cdot 1 = 0$. If $\beta_1 = \beta_2$, then $v_3$ and $v_4$ are not blue, and the new charge of $\alpha$ is at least $2 + 3 - 2 \cdot 2 = 1$.
5. Let $v_1$ be black and bad. The big face $\beta_1$ incident with $v_1$ sends 3 units of charge to $\alpha$. The face $\alpha$ then sends 2 units of charge to $v_1$ and at most 1 unit of charge to $v_2, v_3$, and $v_4$. The new charge of $\alpha$ is at least $2 + 3 - 2 - 3 \cdot 1 = 0$.
6. Let no black and bad vertex be incident with $\alpha$. Then the big faces adjacent to $\alpha$ send together at least 2 units of charge (if there was only one big face, it would send 3 units of charge), and $\alpha$ sends at most 1 unit of charge to each incident vertex. The new charge of $\alpha$ is at least $2 + 2 - 4 = 0$.

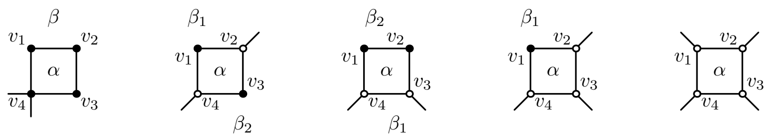


Figure 10: Different possible neighbourhoods of a 4-face $\alpha$.

### 3.3.6 5-faces

Let $\alpha$ be a 5-face. Its initial charge is 4. Let $v_1, v_2, v_3, v_4, v_5$ be the vertices incident with $\alpha$. (They do not have to be pairwise different.) Consider the number of black vertices bad for $\alpha$. Note that by Claim 3.11 each such vertex is either a 2-vertex or a 4-vertex with a hanging 1-face. See Figure 11 for illustration.

1. Let all the five vertices $v_1, v_2, v_3, v_4, v_5$ be black and bad. Then the graph $G$ contains only 5 vertices, hence, it is not a counterexample.
2. Let $v_1, v_2, v_3$, and $v_4$ be black and bad. From Claim 3.6 it follows that none of them is incident with a 1-face. Then all the five edges of $\alpha$ are incident with the same big face $\beta$ and $\alpha$ is hanging on $v_5$. Hence, $\beta$ sends 3 units of charge to $\alpha$ (Rule 1). The face $\alpha$ then sends $4 \cdot 2$ units of charge to the 2-vertices $v_1, v_2, v_3$, and $v_4$ (Rule 3) and at most 1 unit of charge to the vertex $v_5$ (Rule 3). If the vertex $v_5$ is a 3-vertex, the third edge incident with it is a bridge in $G$. Therefore $\deg(v_5) \geq 4$ and $v_5$ sends 2 units of charge to $\alpha$ (Rules 4 – 6). The new charge of $\alpha$ is $4 + 3 - 4 \cdot 2 - 1 + 2 = 0$.
3. Let $v_1, v_2$, and $v_3$ be black and bad. Let $\beta_1$ be the big face incident with the vertices $v_1, v_2$, and $v_3$ and $\beta_2$ be the big face incident with the edge $v_4v_5$. By Claim 3.12 $v_1, v_2$, and $v_3$ are 2-vertices. If $\beta_1 \neq \beta_2$, $\beta_1$ sends 3 units of charge to $\alpha$ and $\beta_2$ sends at least 1 unit of charge to $\alpha$. The face $\alpha$ then sends 2 units of charge to $v_1, v_2$, and $v_3$ and at most 1 unit of charge to $v_4$ and $v_5$. The new charge of $\alpha$ is at least $4 + 3 + 1 - 3 \cdot 2 - 2 \cdot 1 = 0$. If $\beta_1 = \beta_2$, then $v_4$ and $v_5$ are not blue, and the new charge of $\alpha$ is at least $4 + 3 - 3 \cdot 2 = 1$.

4. Let $v_1, v_2$, and $v_4$ be black and bad. Let $\beta_1$ be the big face incident with $v_1$ and $v_2$, let $\beta_2$ be the big face incident with $v_4$. By Claim 3.12 $v_1$ and $v_2$ are 2-vertices, thus $\beta_1$ sends 3 units of charge to $\alpha$.

   If $\beta_1 \neq \beta_2$, then $\beta_2$ sends at least 1 unit of charge to $\alpha$. The face $\alpha$ then sends 2 units of charge to the vertices $v_1, v_2$, and $v_4$ and at most 1 unit of charge to the vertices $v_3$ and $v_5$. The new charge of $\alpha$ is at least $4+3+1-3\cdot 2-2\cdot 1=0$. If $\beta_1=\beta_2$, then $v_3$ and $v_5$ are not blue, and the new charge of $\alpha$ is $4+3-3\cdot 2=1$.

5. Let $v_1$ and $v_2$ be black and bad. Let $\beta_1$ be the big face incident with the vertices $v_1$ and $v_2$. The face $\beta_1$ sends 3 units of charge to $\alpha$. The face $\alpha$ then sends 2 units of charge both to $v_1$ and $v_2$ and at most 1 unit of charge to $v_3, v_4$, and $v_5$. The new charge of $\alpha$ is at least $4+3-2\cdot 2-3\cdot 1=0$.

6. Let $v_1$ and $v_3$ be black and bad. Let $\beta_1$ be the big face incident with $v_1$, $\beta_2$ be the big face incident with $v_3$, and $\beta_3$ be the big face incident with the edge $v_4v_5$. If $\beta_1, \beta_2$, and $\beta_3$ are three different faces, the new charge of $\alpha$ is at least $4+3-2\cdot 2-3\cdot 1=0$. If two of them coincide, then at least one of the vertices $v_2, v_4$, and $v_5$ is not blue and the new charge of $\alpha$ is at least $4+2-2\cdot 2-2\cdot 1=0$. If $\beta_1=\beta_2=\beta_3$, then $v_2, v_4$, and $v_5$ are not blue and the new charge of $\alpha$ is at least $4+1-2\cdot 2=1$.

7. Let $v_1$ be black and bad. Then the big faces adjacent to $\alpha$ send together at least 2 units of charge (if there was only one big face, it would send 3 units of charge). The new charge of $\alpha$ is at least $4+2-2-4\cdot 1=0$.

8. Let no black and bad vertex be incident with $\alpha$. Then the big faces adjacent to $\alpha$ send together at least 1 unit of charge. The new charge of $\alpha$ is at least $4+1-5=0$.

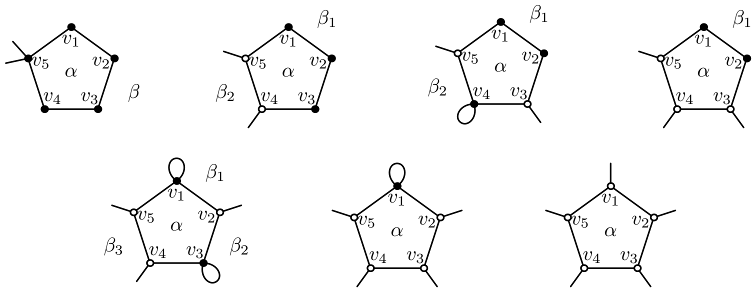


Figure 11: Different possible neighbourhoods of a 5-face $\alpha$.

### 3.3.7 Small faces of size at least 6

Let $\alpha$ be a $d$-face, $6 \leq d \leq 44$. Its initial charge is $2d-6$. Let $v_1, \ldots, v_d$ be the vertices incident with $\alpha$. (They do not have to be pairwise different.) Consider the black vertices incident with $\alpha$. Let $v_i$ be a black 4-vertex. It cannot be good for $\alpha$, since no two small faces are adjacent (see Claim 3.5). Therefore, each black 4-vertex is bad for $\alpha$. By Claim 3.10 at most $d-2$ vertices incident with $\alpha$ are bad. Let $k \leq d-2$ be the number of black vertices incident with $\alpha$. We can divide the facial walk of $\alpha$ into $d-k \geq 2$ parts, each beginning and ending in a blue or white vertex, each incident with $\alpha$ and a big face $\beta_i$,

$i \in \{1, \ldots, d-k\}$. Each of these big faces sends at least 1 unit of charge to $\alpha$. (If $\beta_i = \beta_j$ for some $1 \le i < j \le d-k$, the face $\beta_i$ cannot be adjacent to another small face but $\alpha$, therefore, it sends 3 units to $\alpha$, which is even more than what two different big faces would send.)

The face $\alpha$ then sends 2 units of charge to each of the $k$ incident black vertices, and at most 1 unit of charge to each of the other incident vertices. Together, the new charge of $\alpha$ is at least

$$2d - 6 + (d-k) \cdot 1 - k \cdot 2 - (d-k) \cdot 1 = 2(d-k) - 6.$$

If $d - k \ge 3$, the new charge of $\alpha$ is non-negative.

Let $d - k = 2$. It means there are only two vertices which are not black. Since $d \ge 6$, at least one big face $\beta$ shares at least 2 black vertices with $\alpha$, say $v_1$ and $v_2$. By Claim 3.10 at least one from $v_1$ and $v_2$ is a 2-vertex, hence, by Claim 3.14 the face $\beta$ sends 3 units of charge to $\alpha$. The new charge of $\alpha$ is therefore at least

$$2d - 6 + 3 + 1 - (d-2) \cdot 2 - 2 \cdot 1 = 0.$$

### 3.3.8 Big faces

Let $\beta$ be a $d$-face, $d \ge 45$. Its initial charge is $2d - 6$. It sends 3 units of charge to the small faces it is adjacent to (Rule 1). It sends 2 units of charge to all bad black vertices; 1 unit of charge to all other vertices. Let $k$ be the number of black vertices bad for $\beta$. By Claim 3.10, $k \le \frac{4}{5} \cdot d$. The new charge of $\beta$ is therefore at least

$$2d - 6 - 3 - k \cdot 2 - (d-k) \cdot 1 = d - k - 9 \ge d - \frac{4d}{5} - 9 = \frac{d}{5} - 9 = \frac{d-45}{5} \ge 0.$$

The new charge of all elements of the graph is non-negative, but the sum of all the charges is $-12$, which is a contradiction. This contradiction implies that the minimum counterexample does not exist.

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