ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 14 (2018) 251–265

# Inherited unitals in Moulton planes*

Gábor Korchmáros , Angelo Sonnino

*Dipartimento di Matematica, Informatica ed Economia*
*Università degli Studi della Basilicata*
*Viale dell'Ateneo Lucano 10, 85100 Potenza, Italy*

Tamás Szőnyi

*ELTE Eötvös Loránd University, Institute of Mathematics and*
*MTA-ELTE Geometric and Algebraic Combinatorics Research Group*
*H-1117 Budapest, Pázmány P. s. 1/c, Hungary*

Received 11 January 2017, accepted 24 July 2017, published online 4 September 2017

**Abstract**

We prove that every Moulton plane of odd order—by duality every generalised André plane—contains a unital. We conjecture that such unitals are non-classical, that is, they are not isomorphic, as designs, to the Hermitian unital. We prove our conjecture for Moulton planes which differ from $\mathrm{PG}(2, q^2)$ by a relatively small number of point-line incidences. Up to duality, our results extend previous analogous results—due to Barwick and Grüning—concerning inherited unitals in Hall planes.

*Keywords: Unitals, Moulton planes.*

*Math. Subj. Class.: 51E20, 05B25*

## 1 Introduction

A unital is a set of $q^3+1$ points together with a family of subsets, each of size $q+1$, such that every pair of distinct points are contained in exactly one subset of the family. Such subsets are usually called blocks so that unitals are block-designs $2-(q^3+1, q+1, 1)$. The classical example of a unital arises from the unitary polarity in the Desarguesian projective plane $\mathrm{PG}(2, q^2)$ where the points are the absolute points, and the blocks are the non-absolute lines of the unitary polarity. The name of "Hermitian unital" is commonly used for the

*This research was carried out within the activities of the GNSAGA of the Italian INdAM.

*E-mail address:* gabor.korchmaros@unibas.it (Gábor Korchmáros), angelo.sonnino@unibas.it (Angelo Sonnino), szonyi@cs.elte.hu (Tamás Szőnyi)

classical example since the absolute points of the unitary polarity are the points of the Hermitian curve defined over $\mathrm{GF}(q^2)$.

A unital $\mathcal{U}$ is *embedded* in a projective plane $\Pi$ of order $q^2$, if its points are points of $\Pi$ and its blocks are intersections with lines. As usual, we adopt the term "chord" to indicate a block of $\mathcal{U}$. A line $\ell$ of $\Pi$ is either a tangent or a $(q+1)$-secant to $\mathcal{U}$ according as $|\ell\cap\mathcal{U}| = 1$ or $|\ell \cap \mathcal{U}| = q + 1$, and in the latter case $\ell \cap \mathcal{U}$ is a chord. Examples of unitals embedded in $\mathrm{PG}(2, q^2)$ other than the Hermitian ones are known to exist.

A unital is *classical* if it is isomorphic, as a block-design, to a Hermitian unital. Classical unitals contain no O'Nan configurations, and it has been conjectured that any non-classical unital embedded in $\mathrm{PG}(2, q^2)$ must contain a O'Nan configuration.

In several families of non-desarguesian planes, the problem of constructing and characterizing unitals has also been investigated; see [1, 2, 5, 6, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 27, 28]. Apart from the examples of unitals arising from a unitary polarity in a commutative semifield plane, the known examples are inherited unitals from the Hermitian unital. In a non-desarguesian plane $\Pi$ of order $q^2$ arising from $\mathrm{PG}(2, q^2)$ by altering some of the point-line incidences, the adjective "inherited" is used for those pointsets of $\mathrm{PG}(2, q^2)$ which keep their intersection properties with lines when moving from $\mathrm{PG}(2, q^2)$ to $\Pi$.

In this paper we construct inherited unitals in Moulton planes of odd order $q^2$, and, by duality, in generalised André planes of the same order; see Theorem 3.1. We also investigate the problem whether these unitals are classical; see Theorems 3.5 and 3.6. We show that if such a plane differs from $\mathrm{PG}(2, q^2)$ by a relatively small number of incidences only, then the inherited unital is non-classical. Also, we exhibit non-classical inherited unitals in case of many point-line incidence alterations. Such unitals appear to be of interest in coding theory; see [25].

What emerges from our work leads us to conjecture that the inherited unitals constructed in our paper are all non-classical. It should be noticed that our results extend previous analogous results due to Barwick and Grüning concerning inherited unitals in Hall planes which are very special André planes; see [8, 16] and Remark 3.4. The methods used in [8] are mostly geometric and involve Baer subplanes and blocking sets. In this paper, we adopt a more algebraic approach that allows us to exploit results on the number of solutions of systems of polynomial equations over a finite field.

## 2 Two new results on the Hermitian unital

We establish and prove two theorems on Hermitian unitals that will play a role in our study on unitals in Moulton planes.

Up to a change of the homogeneous coordinate system $(X_1, X_2, X_3)$ in $\mathrm{PG}(2, q^2)$, the points of the classical unital $\mathcal{U}$ are those satisfying the equation

$$X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0. \tag{2.1}$$

In the affine plane $\mathrm{AG}(2, q^2)$ arising from $\mathrm{PG}(2, q^2)$ with respect to the line $X_3 = 0$, we use the coordinates $(X, Y)$ where $X = X_1/X_3$ and $Y = X_2/X_3$). Then the points of $\mathcal{U}$ in $\mathrm{AG}(2, q^2)$ are the solutions of the equation

$$X^{q+1} + Y^{q+1} + 1 = 0. \tag{2.2}$$

Since $\mathrm{GF}(q^2)$ is the quadratic extension of $\mathrm{GF}(q)$ by adjunction of a root $i$ of the polynomial $X^2 - s$ with a non-square element $s$ of $\mathrm{GF}(q)$, every element $u$ of $\mathrm{GF}(q^2)$ can uniquely be written as $u = u_1 + iu_2$ with $u_1, u_2 \in \mathrm{GF}(q)$. Then $u^q = u_1 - iu_2$ and $\|u\| = u^{q+1} = u_1^2 - su_2^2$. Therefore, the points $P(x, y) \in \mathcal{U}$ lying in $\mathrm{AG}(2, q^2)$ are those satisfying the equation

$$x_1^2 - sx_2^2 + y_1^2 - sy_2^2 + 1 = 0. \tag{2.3}$$

For a subset $T \subseteq \mathrm{GF}(q) \setminus \{0\}$, let $\mathcal{S}_t$ denote the set of points $\{\, (x, y) \mid \|x\| = t \in T \,\}$. Hence the pointset $\mathcal{S}_t \cap \mathcal{U}$ comprises all points $P(x, y)$ such that both $x_1^2 - sx_2^2 = t$ and (2.3) hold. Therefore, a point $P(x, y) \in \mathrm{AG}(2, q^2)$ is in $\mathcal{S}_t \cap \mathcal{U}$ if and only if $P_1(x_1, x_2) \in \mathrm{AG}(2, q)$ lies on the non-degenerate conic $\mathcal{C}_1 : X^2 - sY^2 - t = 0$ while $P_2(y_1, y_2) \in \mathrm{AG}(2, q)$ does lie on the conic $\mathcal{C}_2 : X^2 - sY^2 + 1 + t = 0$. This shows that $\mathcal{S}_t \cap \mathcal{U}$ has size $(q + 1)^2$ apart from the case $t = -1$ when it consists of the $q + 1$ points of $\mathcal{U}$ lying on the $X$-axis.

**Lemma 2.1.** *Let $\ell$ be a non-vertical line in $\mathrm{AG}(2, q^2)$. Then $|\ell \cap \mathcal{U} \cap \mathcal{S}_t| \in \{0, 1, 2, q+1\}$ for every $t \in T$. If $q + 1$ occurs then $\ell$ is either a horizontal line, or it passes through the origin.*

*Proof.* The points $P(x, 0)$ with $\|x\| = t$ form a Baer subline. As $\mathcal{U}$ is classical, $\ell \cap \mathcal{U}$ is a Baer subline of $\ell$, and hence the projection of $\ell \cap \mathcal{U}$ on the $X$-axis from $Y_\infty$ is a Baer-subline, as well. Since two distinct Baer sublines have at most two common points, the first assertion follows. To prove the second one, we need some computation. If $\ell$ has equation $Y = Xm + b$, we have to count the roots $x$ of the polynomial $f(X) = X^{q+1} + (Xm + b)^{q+1} + 1$ whose norm $\|x\|$ is equal to $t$. If $\|x\| = t$, then $f(x) = bm^q x^q + b^q mx + t(1 + m^{q+1}) + b^{q+1} + 1$ and hence

$$xf(x) = b^q mx^2 + (t(1 + m^{q+1}) + b^{q+1} + 1)x + bm^q t.$$

If we have at least three such roots $x$ then either $m = 0$ and $t + 1 = -b^{q+1}$, or $b = 0$ and $t(1 + m^{q+1}) = -1$. □

Take any two distinct non-tangent lines $\ell_1$ and $\ell_2$ of $\mathcal{U}$. We are interested in the intersection of the projection of $\ell_1 \cap \mathcal{U}$ from $P$ on $\ell_2$ with $\ell_2 \cap \mathcal{U}$. For any point $P$ outside both $\ell_1$ and $\ell_2$, the projection of $\ell_1$ to $\ell_2$ from $P$ takes the chord $\ell_1 \cap \mathcal{U}$ to a Baer subline of $\ell_2$. Since two Baer sublines of $\ell_2$ intersect in $0, 1, 2$ or $q + 1$ points, one may want to determine the size of the sets $\Sigma_i$ $(i = 0, 1, 2, q+1)$ consisting of all points $P$ for which this intersection number is equal to $i$. The points in $\Sigma_i$ are called *elliptic*, *parabolic*, *hyperbolic*, or *full* with respect to the pair $(\ell_1, \ell_2)$, according as $i = 0$, $i = 1$, $i = 2$, or $i = q + 1$, respectively; see [21].

We go on to compute the size of $\Sigma_i \cap \mathcal{U}$. Since the linear collineation group $G \cong \mathrm{PGU}(3, q)$ of $\mathrm{PG}(2, q^2)$ preserving $\mathcal{U}$ acts transitively on the points outside $\mathcal{U}$, we may assume that $Y_\infty = \ell_1 \cap \ell_2$. The stabiliser of $Y_\infty$ in $G$ acts on the pencil with center in $Y_\infty$ as the general projective group $\mathrm{PGL}(2, q)$ on the projective line $\mathrm{PG}(1, q^2)$. Therefore, it has two orbits, one consisting of all tangents the other of all chords to $\mathcal{U}$ through $Y_\infty$. This allows us to assume without loss of generality that $\ell_1$ is the line at infinity. Since $\ell_2$ is not a tangent to $\mathcal{U}$, its equation is of the form $X = c$ with $c^{q+1} + 1 \neq 0$. Therefore, $c^{q+1} + 1$ is either a non-zero square or a non-square element of $\mathrm{GF}(q)$. These two cases occur depending upon whether a linear collineation $\gamma \in \mathrm{PGL}(2, q)$ taking $\ell_1$ to $\ell_2$ is in the subgroup isomorphic to the special projective group $\mathrm{PSL}(2, q)$ or not. Accordingly, $\{\ell_1, \ell_2\}$ is called a *special* pair or a *general* pair. Further, since $P$ is a point outside $\ell_1$ and $\ell_2$, it is an affine point $P = (a, b)$ with $a \neq c$.

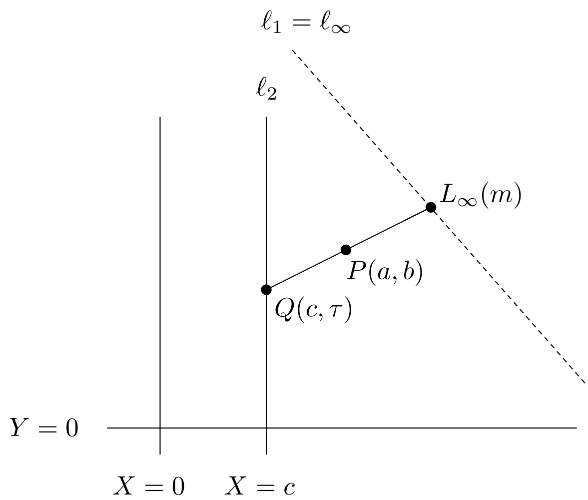


Figure 1: The initial configuration.

Let $P=(a,b)$ denote a point of $\mathcal{U}$, that is,

$$a^{q+1}+b^{q+1}+1=0. \tag{2.4}$$

Take a line $r$ of equation $Y=m(X-a)+b$ through $P=(a,b)$. A necessary and sufficient condition for $r$ to meet both $\ell_1$ and $\ell_2$ in $\mathcal{U}$ is the existence of a solution $\tau \in \mathrm{GF}(q^2)$ of the system consisting of (2.4) together with

$$c^{q+1}+\tau^{q+1}+1=0, \tag{2.5}$$

$$m^{q+1}+1=0. \tag{2.6}$$

In fact, $Q(c,\tau)$ with $\tau=m(c-a)+b$ is the point of $r$ on $\ell_2$. Then (2.5) holds if and only if $Q\in\mathcal{U}$. Furthermore, (2.6) is the necessary and sufficient condition for the infinite point of $r$ to be in $\mathcal{U}$; see Figure 1.

The above discussion also shows how to count lines through $P$ meeting both $\ell_1\cap\mathcal{U}$ and $\ell_2\cap\mathcal{U}$. Essentially, one has to find the number of solutions in the indeterminate $\tau$ of the system consisting of the equations (2.4), (2.5), and (2.6). Observe that (2.4), (2.5), (2.6) are equivalent to

$$a_1^2-sa_2^2+b_1^2-sb_2^2+1=0, \tag{2.7}$$

$$c_1^2-sc_2^2+\tau_1^2-s\tau_2^2+1=0, \tag{2.8}$$

$$b_1\tau_1-sb_2\tau_2+a_1c_1-sa_2c_2+1=0. \tag{2.9}$$

From this the following result is obtained.

**Proposition 2.2.** *The number of lines through $P$ meeting both $\ell_1\cap\mathcal{U}$ and $\ell_2\cap\mathcal{U}$ equals the number of solutions $(\tau_1,\tau_2)$, with $\tau_1,\tau_2\in\mathrm{GF}(q)$, of the system consisting of (2.7), (2.8), (2.9).*

In investigating the above system, two cases are distinguished according as $(b_1, b_2)$ is $(0, 0)$ or not.

In the former case, Equations (2.7) and (2.9) read $a_1^2 - sa_2^2 + 1 = 0$ and $a_1c_1 - sa_2c_2 + 1 = 0$. Geometrically in $\mathrm{AG}(2, q)$, the point $U = (a_1, a_2)$ is the intersection of the ellipse $\mathcal{E}$, with equation $X^2 - sY^2 + 1 = 0$, and the line $v$ with equation $c_1X - sc_2Y + 1 = 0$. Since $c^{q+1} + 1 = c_1^2 - sc_2^2 + 1$ is a non-zero element of $\mathrm{GF}(q)$, $v$ must be either a secant, or an external line to $\mathcal{E}$ and this occurs according as $c_1^2 - sc_2^2 + 1$ is a non-zero square or non-square element in $\mathrm{GF}(q)$. In fact, from (2.7) and (2.9),

$$a_1 = \frac{sc_2a_2 - 1}{c_1}, \quad a_2 = \frac{-sc_2 \pm ic_1\sqrt{c_1^2 - sc_2^2 + 1}}{s(c_1^2 - sc_2^2)}.$$

Therefore, if $P$ is on the $X$-axis, then $P$ is elliptic in general, apart from the case where $c^{q+1} + 1 = c_1^2 - sc_2^2 + 1$ is a non-square element in $\mathrm{GF}(q)$ and $P$ is one of the two common points of $\mathcal{C}$ and $v$, namely $P = P(a, 0)$ where

$$a = a_1 + ia_2 = \frac{-1 \pm \sqrt{1 + c^{q+1}}}{c^q}.$$

Further, in the exceptional case, $P$ is a full point as for any $c_1, c_2 \in \mathrm{GF}(q)$ with $c_1^2 - sc_2^2 + 1 \neq 0$, Equation (2.8) always has $q + 1$ solutions $(\tau_1, \tau_2)$ with $\tau_1, \tau_2 \in \mathrm{GF}(q)$.

In the latter case, either $b_1$ or $b_2$ is not zero. If $b_1 \neq 0$, retrieving $\tau_1$ from (2.9) and putting it in (2.8) gives a quadratic equation in the indeterminate $\tau_2$, namely

$$(s^2b_2^2 - sb_1^2)\tau_2^2 - 2sb_2(a_1c_1 - sa_2c_2 + 1)\tau_2 + \\ (a_1c_1 - sa_2c_2 + 1)^2 + b_1^2(1 + c_1^2 - sc_2^2) = 0, \quad (2.10)$$

whose discriminant is $\Delta_1 = sb_1^2\Delta$ with

$$\Delta = (b_1^2 - sb_2^2)(1 + c_1^2 - sc_2^2) + (a_1c_1 - sa_2c_2 + 1)^2$$

which can also be written by (2.7) as

$$\Delta = -(1 + c_1^2 - sc_2^2)(a_1^2 - sa_2^2 + 1) + (a_1c_1 - sa_2c_2 + 1)^2.$$

For $b_2 \neq 0$, retrieving $\tau_2$ from (2.9) and putting it in (2.8) gives the following quadratic equation in the indeterminate $\tau_1$:

$$(-b_1^2 + b_2^2)\tau_1^2 + 2a_1b_1c_1\tau_1 - \\ a_1^2 - s^2a_2^2c_2^2 - b_2^2c_1^2 + sb_2^2c_2^2 + 2sa_2c_2 - b_2^2 - 1 = 0 \quad (2.11)$$

with discriminant $\Delta_2 = s^3b_2^2\Delta$. Since $\Delta_1$ and $\Delta_2$ are simultaneously zero, or a square, or a non-square in $\mathrm{GF}(q)$, each of the equations (2.10) and (2.11) has 2, 1 or zero solutions in $\mathrm{GF}(q)$, depending upon whether $\Delta$ is a square element, zero, or a non-square element of $\mathrm{GF}(q)$, respectively. This leads to the study of the zeroes of the polynomial

$$F(X, Y, Z) = -(1 + c_1^2 - sc_2^2)(X^2 - sY^2 + 1) + (c_1X - sc_2Y + 1)^2 - Z^2. \quad (2.12)$$

Geometrically, $F(X, Y, Z) = 0$ is the equation of a quadric $\mathscr{Q}$ in $\mathrm{AG}(3, q)$. Actually, $\mathscr{Q}$

Table 1: Elliptic, parabolic, hyperbolic and full points.

| | $P(a,0)$ | | $P(a,b),\ b\neq 0$ | |
|---|---|---|---|---|
| | $1+\|c\|\in\square$ | $1+\|c\|\in\not\square$ | $1+\|c\|\in\square$ | $1+\|c\|\in\not\square$ |
| $N_{\mathcal{E}}$ | $q+1$ | $q-1$ | $\frac{-3-9q+q^2+q^3}{2}$ | $\frac{-3-5q-q^2+q^3}{2}$ |
| $N_{\mathcal{P}}$ | $0$ | $0$ | $2q-1$ | $0$ |
| $N_{\mathcal{H}}$ | $0$ | $0$ | $\frac{(q-1)^2}{2}(q+1)$ | $\frac{(q+1)^2}{2}(q-1)$ |
| $N_{\mathcal{F}}$ | $0$ | $2$ | $0$ | $0$ |

is a cone. In fact, the system $F_X = F_Y = F_Z = 0$ has a (unique) solution $(c_1, c_2, 0)$ and hence the point $V(c_1, c_2, 0)$ is the vertex of $\mathscr{Q}$. In particular, the intersection of $\mathscr{Q}$ with the plane $Z = 0$ splits into two lines over $\mathrm{GF}(q)$ or its quadratic extension $\mathrm{GF}(q^2)$, and this occurs according as the infinite points of the conic with equation

$$-(1 + c_1^2 - sc_2^2)(X^2 - sY^2) + (c_1X - sc_2Y)^2 = 0$$

lie in $\mathrm{PG}(2, q)$ or in $\mathrm{PG}(2, q^2) \setminus \mathrm{PG}(2, q)$. By a direct computation, this condition only depends on $c^{q+1}$, namely whether $1 + c^{q+1}$ is a square or a non-square element of $\mathrm{GF}(q)$. Therefore, $\mathscr{Q}$ contains either $2q-1$ or $1$ points in the plane $Z = 0$, and this occurs according as the pair $\{\ell_1, \ell_2\}$ is special or general. Also, in the former case there are exactly $2q - 1$ parabolic points $P$ but in the latter case no point $P$ is parabolic. Therefore, the following result holds.

**Theorem 2.3.** *Let $\ell_1, \ell_2$ be any two distinct non-tangent lines of the classical unital $\mathcal{U}$ in $\mathrm{PG}(2, q^2)$ whose common point is off $\mathcal{U}$. The number $N_{\mathcal{E}}, N_{\mathcal{P}}, N_{\mathcal{H}}, N_{\mathcal{S}}$, of elliptic, parabolic, hyperbolic and full points of $\mathcal{U}$ with respect to the pair $\{\ell_1, \ell_2\}$ is given in Table 1.*

We state a corollary of Theorem 2.3 that will be used in Section 3. For $i = 1, 2$ let $\Lambda_i$ be a subset of $\ell_i \cap \mathcal{U}$ such that $|\Lambda_1| = |\Lambda_2| = \lambda$.

**Theorem 2.4.** *If*

$$\lambda > \sqrt{\frac{(q+1)(q+3)}{2}} \tag{2.13}$$

*there exists a non-degenerate quadrangle $A_1B_1A_2B_2$ with vertices $A_i, B_i \in \Lambda_i$ for $i = 1, 2$ such that its diagonal point $A_1B_2 \cap B_1A_2$ lies in $\mathcal{U}$.*

*Proof.* We prove the existence of a hyperbolic point $D$ in $\mathcal{U}$ such that the projection of $\Lambda_1$ from $D$ on $\ell_2$ share two points with $\Lambda_2$. From Theorem 2.3, we have at least $\frac{1}{2}(q-1)^2(q+1)$ hyperbolic points in $\mathcal{U}$. We omit those hyperbolic points projecting $\overline{\Lambda}_1 = (\ell_1 \cap \mathcal{U}) \setminus \Lambda_1$ to a pointset of $\ell_2$ meeting $\ell_2 \cap \mathcal{U}$ nontrivially. The number of such hyperbolic points is

$\bar{\lambda}(q-1)(q+1)$ with $\bar{\lambda} = q+1-\lambda$. Similarly we omit all hyperbolic points projecting $\overline{\Lambda}_2 = (\ell_2 \cap \mathcal{U}) \setminus \Lambda_2$ to a pointset of $\ell_1$ meeting $\ell_1 \cap \mathcal{U}$ nontrivially. Therefore, the total number of omitted hyperbolic points is $2\bar{\lambda}(q^2-1)-\bar{\lambda}^2(q-1) = (q-1)\bar{\lambda}(2q+2-\bar{\lambda}(q-1))$. From Theorem 2.3, this number is smaller than the total number of hyperbolic points as far as (2.13) holds. □

To state the other new result on the classical unital a couple of *ad hoc* notation in $\mathrm{AG}(2,q^2)$ will be useful: For a non-vertical line $\ell$ with equation $Y = Xm + b$, $\bar{\ell}$ denotes the non-vertical line with equation $Y = Xm^q + b$. Given a point $P(a,b)$ outside $\mathcal{U}$, two lines $\ell_1$ and $\ell_2$ are said to be a *good line-pair* whenever the lines $\bar{\ell_1}$ and $\bar{\ell_2}$ meet in a point of $\mathcal{U}$. Our goal is to show that if $a \neq 0$ then there exist many good pairs.

For $i = 1,2$, write the equations of $\ell_i$ in the form $Y = (X-a)m_i + b$. Then $\bar{\ell_i}$ has equation $Y = Xm_i^q - am_i + b$. Hence $\bar{P}(x,y) = \bar{\ell_1} \cap \bar{\ell_2}$ where

$$x = \frac{a(m_1-m_2)}{m_1^q - m_2^q},$$

and hence

$$y = \frac{a(m_1-m_2)}{m_1^q - m_2^q} m_1^q - am_1 + b.$$

Note that

$$\|x\| = x^{q+1} = a^{q+1}\left(\frac{1}{(m_1-m_2)^{q-1}}\right)^{q+1} = \frac{\|a\|}{(m_1-m_2)^{q^2-1}} = \|a\| \neq 0.$$

The condition for $\overline{P}(x,y)$ to lie in $\mathcal{U}$ is

$$x^{q+1} + y^{q+1} + 1 = a^{q+1} + a^{q+1}\left(\frac{(m_1-m_2)}{(m_1-m_2)^q} m_1^q - m_1 + \frac{b}{a}\right)^{q+1} + 1 = 0.$$

Let

$$\xi = -\frac{a^{q+1}+1}{a^{q+1}} \in \mathrm{GF}(q).$$

Then the last equation reads

$$\left(\frac{(m_1-m_2)}{(m_1-m_2)^q} m_1^q - m_1 + \frac{b}{a}\right)^{q+1} = \xi. \tag{2.14}$$

Henceforth we assume that

$$\|a\| \neq -1.$$

With

$$m_1 = \alpha + i\beta, \quad m_2 = \gamma + i\delta, \quad \frac{b}{a} = u + iv,$$

(2.14) reads

$$\left(\frac{(\alpha-\gamma)+i(\beta-\delta)}{(\alpha-\gamma)-i(\beta-\delta)}(\alpha - i\beta) - (\alpha + i\beta) + u + iv\right)^{q+1} = \xi,$$

whence

$$(u\alpha - u\gamma - sv\beta + sv\delta)^2 - s(2\beta\gamma - 2\alpha\delta - u(\beta-\delta) + v(\alpha-\gamma))^2 \\ - \xi((\alpha-\gamma)^2 - s(\beta-\delta)^2) = 0,$$

that is,

$$(u(\alpha-\gamma) - sv(\beta-\delta))^2 - s(2\beta\gamma - 2\alpha\delta - u(\beta-\delta) + v(\alpha-\gamma))^2 \\ - \xi((\alpha-\gamma)^2 - s(\beta-\delta)^2) = 0. \quad (2.15)$$

With

$$\gamma = \alpha - \overline{\gamma}, \qquad \delta = \beta - \overline{\delta},$$

Equation (2.15) becomes

$$(u\overline{\gamma} - sv\overline{\delta})^2 - s(-2\beta\overline{\gamma} - 2\alpha\overline{\delta} - u\overline{\delta} + v\overline{\gamma})^2 - \xi(\overline{\gamma}^2 - s\overline{\delta}^2) = 0, \quad (2.16)$$

which can be viewed as a quadratic form in $\overline{\gamma}$ and $\overline{\delta}$:

$$F(\overline{\gamma},\overline{\delta}) = (u^2 - v^2 s + 4v\beta s - 4\beta^2 s - \xi)\,\overline{\gamma}^2 + 2(-2u\beta s + 2v\alpha s - 4\alpha\beta s)\,\overline{\gamma}\overline{\delta} \\ + (-u^2 s - 4u\alpha s + v^2 s^2 - 4\alpha^2 s + s\xi)\,\overline{\delta}^2 \quad (2.17)$$

with discriminant

$$\Delta = -u^4 s + 2u^2 v^2 s^2 + 2u^2 s\xi - v^4 s^3 - 2v^2 s^2 \xi - s\xi^2 \\ + (-4u^3 s + 4uv^2 s^2 + 4us\xi)\,\alpha + (-4u^2 vs^2 + 4v^3 s^3 + 4vs^2\xi)\,\beta \\ - 8uvs^2\,\alpha\beta + (-4u^2 s + 4s\xi)\,\alpha^2 + (-4v^2 s^3 - 4s^2\xi)\,\beta^2.$$

Note that $\overline{P}(x,y) \in \mathcal{U}$ if and only if $\Delta = \lambda^2$ for some $\lambda \in \mathrm{GF}(q)$. This leads us to consider the quadric $\mathcal{Q}$ in $\mathrm{AG}(3,q)$ of equation

$$a_{00} + a_{01}X + a_{02}Y + a_{12}XY + a_{11}X^2 + a_{22}Y^2 - Z^2 = 0,$$

where

$$\begin{aligned} a_{00} &= -u^4 s + 2u^2 v^2 s^2 + 2u^2 s\xi - v^4 s^3 - 2v^2 s^2\xi - s\xi^2, \\ a_{01} &= -4u^3 s + 4uv^2 s^2 + 4us\xi, \\ a_{02} &= -4u^2 vs^2 + 4v^3 s^3 + 4vs^2\xi, \\ a_{12} &= -8uvs^2, \\ a_{11} &= -4u^2 s + 4s\xi, \\ a_{22} &= -4v^2 s^3 - 4s^2\xi. \end{aligned}$$

The above coefficients are related by the following equations:

(i) $a_{00} - \frac{1}{2}(\frac{1}{2}a_{01}u - \frac{1}{2}a_{02}v) = s\xi(u^2 - sv^2 - \xi)$;

(ii) $\frac{1}{2}a_{01} - \frac{1}{2}(a_{11}u - \frac{1}{2}a_{12}v) = 0$;

(iii) $\frac{1}{2}a_{02} - \frac{1}{2}(\frac{1}{2}a_{12}u - a_{22}v) = 0$.

Therefore, the determinant $D$ of the $4 \times 4$ matrix associated with $\mathcal{Q}$ is equal to $-s\xi(u^2 - sv^2 - \xi)$ multiplied by the determinant of the cofactor of $a_{00}$. The latter determinant $a_{11}a_{22} - \frac{1}{4}a_{12}^2$ is equal to

$$D_0 = s^3\xi(u^2 - sv^2 - \xi) = s^3(a^{q+1} + b^{q+1} + 1)(a^{q+1} + 1). \tag{2.18}$$

It turns out that

$$D = -(s^2\xi(u^2 - sv^2 - \xi))^2.$$

Observe that $\xi = 0$ if and only if $a^{q+1} = -1$, while

$$u^2 - sv^2 - \xi = \frac{b^{q+1}}{a^{q+1}} + \frac{a^{q+1}+1}{a^{q+1}} = \frac{a^{q+1} + b^{q+1} + 1}{a^{q+1}}$$

vanishes only for $P(a, b) \in \mathcal{U}$. Therefore, $\mathcal{Q}$ is non-degenerate. More precisely, the quadric $\mathcal{Q}$ is either elliptic or hyperbolic according as $q \equiv -1 \pmod 4$ or $q \equiv 1 \pmod 4$. The plane at infinity cuts out from $\mathcal{Q}$ a conic $\mathcal{C}$ with homogeneous equation $a_{11}X^2 + a_{12}XY + a_{22}Y^2 - Z^2 = 0$. Observe that $\mathcal{C}$ is non-degenerate by $D_0 \neq 0$. Thus, the number of points of $\mathcal{Q}$ in $AG(3, q)$ is $q^2 \pm q$ with $q \equiv \pm 1 \pmod 4$. Furthermore, the point at infinity $Z_\infty$ on the $Z$-axis does not lie on $\mathcal{Q}$, and it is an external point or an internal point to $\mathcal{C}$ according as $-D_0$ is a non-zero square or a non-square in $\mathrm{GF}(q)$. Therefore, the number of tangents to $\mathcal{Q}$ through $Z_\infty$ in $AG(3, q)$ is equal to $q - 1$ or $q + 1$ according as $-D_0$ is a (non-zero) square or a non-square in $\mathrm{GF}(q)$. From the above discussion, the numbers $N_s$ and $N_t$ of secants and tangents to $\mathcal{Q}$ through $Z_\infty$ are those given in the following lemma:

**Lemma 2.5.** *For $q \equiv -1 \pmod 4$, either $N_t = q + 1$, $N_s = \frac{1}{2}(q-1)^2$, or $N_t = q - 1$, $N_s = \frac{1}{2}(q^2 - 2q - 1)$, according as $D_0$ is a (non-zero) square or a non-square in* $\mathrm{GF}(q)$. *For $q \equiv 1 \pmod 4$, either $N_t = q - 1$, $N_s = \frac{1}{2}(q^2 + 1)$, or $N_t = q + 1$, $N_s = \frac{1}{2}(q^2 - 1)$, according as $D_0$ is a (non-zero) square or a non-square in* $\mathrm{GF}(q)$.

Going back to the discriminant $\Delta$, we see that $\Delta$ vanishes for $N_s + N_t$ ordered pairs $(\alpha, \beta)$, that is, $N_s + N_t$ is the number of lines $\ell_1$ through $P(a, b)$ for which there exists a line $\ell_2$ such that $(\ell_1, \ell_2)$ is a good line-pair. For each $\ell_1$ counted in $N_t$ (resp. $N_s$), we have $q - 1$ (resp. $2(q - 1)$) such lines $\ell_2$, since if (2.17) has a non-trivial solution $(\bar{\gamma}, \bar{\delta})$ in $\mathrm{GF}(q) \times \mathrm{GF}(q)$ then it has exactly $q - 1$ solutions, the multiples of $(\bar{\gamma}, \bar{\delta})$ by the non-zero elements of $\mathrm{GF}(q)$.

If we do not count the $q + 1$ tangents to $\mathcal{U}$ through $P(a, b)$, each of the lines through $P(a, b)$ counted in $N_s$ is in at least $2(q - 1) - (q + 1) = q - 3$ good line-pairs. Therefore Lemma 2.5 has the following corollary:

**Theorem 2.6.** *Let $P(a, b)$ be a point of* $\mathrm{AG}(2, q^2)$ *outside $\mathcal{U}$. If $a \neq 0$, $\|a\| \neq -1$ and $q > 3$, then there exist at least two non-tangent lines $\ell_1, \ell_2$ of $\mathcal{U}$ through $P$, such that the non-tangent lines $\bar{\ell}_1$ and $\bar{\ell}_2$ meet in a point of $\mathcal{U}$. Further, if $q > 5$ then $\ell_1$ and $\ell_2$ may be chosen among the lines through $P(a, b)$ other than the horizontal lines and those passing through the origin.*

## 3 Unitals in Moulton planes

Let $T$ be a non-empty subset of the multiplicative group of $\mathrm{GF}(q)$. The (affine) *Moulton plane* $\mathfrak{M}_T(q^2)$ which is considered in our paper is the affine plane coordinatized by the left quasifield $\mathrm{GF}(q^2)(+,\circ)$ where

$$x \circ y = \begin{cases} xy & \text{if } \|x\| \notin T, \\ xy^q & \text{if } \|x\| \in T, \end{cases}$$

with $\|x\| = x^{q+1}$ being the norm of $x \in \mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$. Geometrically, $\mathfrak{M}_T(q^2)$ is constructed on $\mathrm{AG}(2, q^2)$ by replacing the non-vertical lines with the graphs of the functions

$$Y = X \circ m + b. \tag{3.1}$$

This also shows that to the non-vertical line $\ell$ of equation $Y = Xm + b$ there corresponds the line of equation $\tilde{\ell}$ of equation $Y = X \circ m + b$ in $\mathfrak{M}_T(q^2)$, and viceversa. It is useful to look at the partition of the points outside the $Y$-axis into $q-1$ subsets $\mathcal{S}_i$, called stripes, where $P(x,y) \in \mathcal{S}_i$ if and only if $\|x\| = \omega^i$ with $\omega$ a fixed primitive element of $\mathrm{GF}(q)$. Such stripes were already defined in Section 2; here we just abbreviate the subscript $\omega^i$ by $i$. In fact, moving to $\mathfrak{M}_T(q^2)$ the point-line incidences $P \in \ell$ in $\mathrm{AG}(2, q^2)$ do not alter as long as $P \in \mathcal{S}_i$ with $\omega^i \notin T$. The projective Moulton plane is the projective closure of $\mathfrak{M}_T(q^2)$ and it has the same points at infinity as $\mathrm{AG}(2, q^2)$. For a similar description of Moulton planes see also [3, 4, 26].

The dual of the Moulton plane is the André plane $\mathfrak{A}_T(q^2)$ coordinatized by the right quasifield $\mathrm{GF}(q^2)(+,*)$ where

$$x * y = \begin{cases} xy & \text{if } \|x\| \notin T, \\ x^q y & \text{if } \|x\| \in T. \end{cases}$$

In this duality, the correspondence occurs between the point $(u, v)$ of $\mathfrak{M}_T(q^2)$ and the line of equation $Y = u * X - v$, as well as between the line of equation $Y = X \circ m + b$ and the point $(m, -b)$ of $\mathfrak{A}_T(q^2)$. The correspondence between points at infinity and lines through $Y_\infty$, and viceversa, is the same as the canonical duality between $\mathrm{PG}(2, q^2)$ and its dual plane $\mathrm{PG}^*(2, q^2)$. If $T$ consists of just one element, then the arising André planes are pairwise isomorphic and they are also known as Hall planes.

Let $\mathcal{U}$ be the classical unital in $\mathrm{PG}(2, q^2)$ given in its canonical form (2.1). We prove that $\mathcal{U}$ is an inherited unital in the Moulton plane, that is, the point-set of $\mathcal{U}$ is a unital in $\mathfrak{M}_T(q^2)$ as well.

**Theorem 3.1.** *Let $\mathcal{U}$ be the classical unital in $\mathrm{PG}(2, q^2)$ given in its canonical form (2.1). Then, for any $T$, $\mathcal{U}$ is a unital in the projective Moulton plane $\mathfrak{M}_T(q^2)$ as well.*

*Proof.* In the very special case $T = \{-1\}$, the proof is straightforward. It is enough to show that if a non-vertical line $\ell$ of equation $Y = Xm + b$ meets $\mathcal{U}$ in a point $P(x, y)$ with $\|x\| = -1$ then $y = 0$ and $x = -b/m$ with $(-b/m)^{q+1} = 1$. In fact, the corresponding line $\tilde{\ell}$ in $\mathfrak{M}_T(q^2)$ has the same property: if $P(x, y) \in \tilde{\ell} \cap \mathcal{U}$ then $y = 0$ and $x = (-b/m^q)^{q+1}$. Since $(-b/m)^{q+1} = (-b/m^q)^{q+1}$, the assertion follows for $T = \{-1\}$.

In the general case, it suffices to exhibit a bijective map from $\ell \cap \mathcal{U}$ to $\tilde{\ell} \cap \mathcal{U}$ for every line $\ell$ of $\mathrm{AG}(2, q^2)$. We may limit ourselves to non-vertical lines with non zero slopes. Let

$Y = Xm + b$ be the equation of such a line $\ell$ and take any point $P(x, y)$ lying in $\ell \cap \mathcal{U}$. Then $m \neq 0$ and $x = (y - b)m^{-1}$. Define the map $\varphi \colon \ell \mapsto \tilde{\ell}$ by

$$\varphi(P) = \begin{cases} \overline{P}((y-b)m^{-1}, y) & \text{for } \|x\| \notin T, \\ \overline{P}((y-b)m^{-q}, y) & \text{for } \|x\| \in T. \end{cases}$$

Obviously, $\varphi(P) = P$ whenever $\|x\| \notin T$.

Since $\varphi$ is bijective, it suffices to show that $P \in \mathcal{U}$ yields $\varphi(P) \in \mathcal{U}$, and the converse also holds. $P(x, y) = ((y - b)m^{-1}, y) \in \mathcal{U}$ if and only if

$$((y-b)m^{-1})^{q+1} + y^{q+1} - 1 = (y-b)^{q+1}(m^{-1})^{q+1} + y^{q+1} - 1 = 0.$$

By $(m^q)^{q+1} = m^{q+1}$, the latter equation is equivalent to

$$((y-b)^{q+1}(m^{-q})^{q+1} + y^{q+1} - 1 = ((y-b)m^{-q})^{q+1} + y^{q+1} - 1 = 0,$$

whence the claim follows. □

Theorem 3.1 and its proof also show that if $\ell$ is a tangent to $\mathcal{U}$ in $\mathrm{AG}(2, q^2)$ then the corresponding line $\tilde{\ell}$ is a tangent to $\mathcal{U}$ in the projective Moulton plane, and the converse also holds. In particular, the tangent to $\mathcal{U}$ at a point outside the $X$-axis is the line $\ell$ of equation $Y = X(-cd^{-1})^q - d^{-q}$ with tangency point $P(c, d)$. Therefore, the corresponding line $\tilde{\ell}$ of equation $Y = X \circ (-cd^{-1})^q - d^{-q}$ is a tangent to $\mathcal{U}$ at the point $\varphi(P) = \bar{P}(\bar{c}, d)$ with $\bar{c} = c$ or $\bar{c} = c(cd^{-1})^{q-1}$ according as $\|c\| \notin T$ or $\|c\| \in T$. Since $\|\bar{c}\| = \|c\|$, the tangency points of $\ell$ and $\tilde{\ell}$ lie in the same stripe. The tangents of $\mathcal{U}$ with tangency point at infinity contain the origin and each of them has equation $Y = Xm$ with $m^{q+1} + 1 = 0$. By the proof of Theorem 3.1, the corresponding lines $Y = X \circ m$ are the tangents of $\mathcal{U}$ in the projective Moulton plane.

Now look at dual plane of the projective Moulton plane $\mathfrak{M}_T(q^2)$ which is the projective André plane $\mathfrak{A}_T(q^2)$. In this duality, the tangent line $\tilde{\ell}$ of $\mathcal{U}$ with equation $Y = X \circ (-cd^{-1})^q - d^{-q}$ corresponds to the point $P^*(u^*, v^*) \in \mathfrak{A}_T(q^2)$ where $u^* = -(-cd^{-1})^q$ and $v^* = d^{-q}$. Since $((-cd^{-1})^q)^{q+1} + (d^{-q})^{q+1} + 1 = 0$, we have $u^{*q+1} + v^{*q+1} + 1 = 0$. Similarly, the tangent line $\tilde{\ell}$ of $\mathcal{U}$ with equation $Y = X \circ m$, $m^{q+1} + 1 = 0$, corresponds to the point $P^*(u^*, v^*) \in \mathfrak{A}_T(q^2)$ where $u^* = u$ and $v^* = 0$. Therefore $u^{*q+1} + v^{*q+1} + 1 = 0$. In terms of $\mathrm{PG}^*(2, q^2)$, the Desarguesian plane which gives rise to the projective André plane $\mathfrak{A}_T(q^2)$, the points $P^*(u^*, v^*)$ lie on the classical unital $\mathcal{U}^*$ given in its canonical form. This shows that $\mathcal{U}^*$ can be viewed as an inherited unital in the projective André plane $\mathfrak{A}_T(q^2)$.

**Remark 3.2.** If $T = \{-1\}$ then the unique stripe where incidence are altered meets $\mathcal{U}$ in $q + 1$ points lying on the $X$-axis. The unital $\mathcal{U}^*$ in the Hall plane is the Grüning unital [16] while for $T = \{i\}$ with $\omega^i \neq -1$, $\mathcal{U}^*$ in the Hall plane is the Barwick unital [7].

A O'Nan configuration of a unital consists of four blocks $b_1$, $b_2$, $b_3$ and $b_4$ intersecting in six points $P_1$, $P_2$, $P_3$, $P_4$, $P_5$ and $P_6$ as in Figure 2. As mentioned in the introduction, the Hermitian unital contains no O'Nan configuration. This fundamental result due to O'Nan dates back to 1972, see [22] and [9, Section 4.2].

**Lemma 3.3.** *If $T = \{-1\}$ then the unital $\mathcal{U}$ of $\mathfrak{M}_T(q^2)$ is non-classical.*

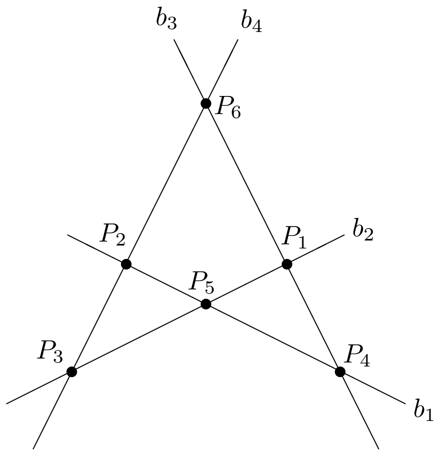


Figure 2: O'Nan configuration of four blocks and six points.

*Proof.* We show that the unital $\mathcal{U}$ in $\mathfrak{M}_T(q^2)$ with $T = \{-1\}$ contains a O'Nan configuration. Take $\alpha \in \mathrm{GF}(q^2)$ such that $\|\alpha\| = -1$. The line $\ell_1$ of equation $Y = X - \alpha$ meets $\mathcal{U}$ in $Q(\alpha, 0)$ and $q$ more points. Take $m \in \mathrm{GF}(q^2)$ such that $m^{q-1} = -1$. The line $\ell_2$ of equation $Y = Xm + \alpha m$ meets $\mathcal{U}$ in $R\,(-\alpha, 0)$ and $q$ more points. Further, the common point of $\ell_1$ and $\ell_2$ is

$$S = \left(\frac{-\alpha(m+1)}{(m-1)}, \frac{-2\alpha m}{(m-1)}\right).$$

Since

$$\left\|\frac{-\alpha(m+1)}{(m-1)}\right\| = -\alpha^{q+1}\frac{(m+1)^{q+1}}{(m-1)^{q+1}} = \\ -\frac{m^{q+1}+m^q+m+1}{m^{q+1}-m^q-m+1} = -\frac{-m^2-m+m+1}{-m^2+m-m+1} = -1,$$

the point $S$ is outside $\mathcal{U}$. Further, in the Moulton plane $\mathfrak{M}_T(q^2)$ with $T = \{-1\}$, the corresponding lines $\tilde{\ell}_1$ and $\tilde{\ell}_2$ meet in $Q(\alpha, 0)$ which is a point of $\mathcal{U}$.

To show that $\mathcal{U}$ is not a classical unital in our Moulton plane $\mathfrak{M}_T(q^2)$, it suffices to exhibit a O'Nan configuration $\{P_0, P_1, P_2, P_3, P_4, P_5\}$ lying in $\mathcal{U}$. The idea is to start off with $P_0 = Q(\alpha, 0)$, and to find four more affine points $P_1, P_2 \in \tilde{\ell}_1$ and $P_3, P_4 \in \tilde{\ell}_2$ each lying in $\mathcal{U}$, so that $\mathcal{U}$ also contains one of the two diagonal points $P_5$ of the quadrangle $P_1P_2P_3P_4$ that are different from $P_0$. First we show that $P_1 \in \ell_1$. Let $P_1 = P_1(x_1, y_1)$. Then, $\|x_1\| \neq -1$. In fact, otherwise, we would have $y_1^{q+1} = 0$ and hence $y_1 = 0$, contradicting $P_0 \neq P_1$. Similarly, $P_2 \in \ell_1$ and $P_3, P_4 \in \ell_2$. Now we use a counting argument in $\mathrm{PG}(2, q^2)$ to show that the quadrangle $P_1P_2P_3P_4$ can be chosen in such a way that $P_5 \in \mathcal{U}$. Since $S = \ell_1 \cap \ell_2$ is outside $\mathcal{U}$, the lines of $\mathcal{U}$ joining a point of $\bar{\ell}_1$ with a point of $\bar{\ell}_2$ cover $(q+1)^2(q-1)$ points of $\mathcal{U}$ other than those lying in $\bar{\ell}_1 \cup \bar{\ell}_2$. From $(q+1)^2(q-1) > q^3 + 1 - 2q$, there exists a quadrangle $P_1P_2P_3P_4$ in $\mathrm{PG}(2, q^2)$ such that

$$P_1, P_2 \in \ell_1 \cap \mathcal{U},\ P_3, P_4 \in \ell_2 \cap \mathcal{U},\ P_5 = P_1P_3 \cap P_2P_4 \in \mathcal{U}.$$

Since $(q+1)^2(q-1) > q^3+1-2q+(q+1)$ we may also assume that either $P_5 \in \ell_\infty \cap \mathcal{U}$, or $P_5 = (x_5, y_5)$ with $\|x_5\| \neq -1$. In particular, $P_5$ is not on the $X$-axis.

If $P_1, P_2 \neq Q$ and $P_3, P_4 \neq R$ then $P_5$ remains a diagonal point of the quadrangle $P_1P_2P_3P_4$ in $\mathfrak{M}_T(q^2)$, and we are done.

Otherwise, take the cyclic subgroup $G$ of $PGU(3,q)$ of order $q+1$ fixing the point $S$ and preserving each line through $S$. Since $|G| \geq 4$, $G$ contains an element $g$ such that $Q \notin \{g(P_1), g(P_2)\}$ and $R \notin \{g(P_3), g(P_4)\}$. Then $g$ takes the quadrangle $P_1P_2P_3P_4$ to another one, whose vertices are different from both $Q$ and $R$. The image $g(P_5)$ is on the line $r$ through $S$ and $P_5$. Since $r \cap \mathcal{U}$ has at most one point on the $X$-axis, there exists at most one $g \in G$ such that $g(P_5)$ lies on the $X$-axis. Therefore, if $|G| \geq 5$, some $g \in G$ also takes $P_5$ either to a point of infinity or a point $(x'_5, y'_5)$ with $\|x'_5\| \neq -1$. In the Moulton plane $\mathfrak{M}_T(q^2)$, the O'Nan configuration $P_0, g(P_1), g(P_2), g(P_3), g(P_4), g(P_5)$ arising from the quadrangle $g(P_1)g(P_2)g(P_3)g(P_4)$ lying in $\mathcal{U}$ has also two diagonal points, namely $P_0$ and $g(P_5)$, belonging to $\mathcal{U}$. $\square$

**Remark 3.4.** Lemma 3.3 can also be obtained from Grüning's work. In fact, if $T = \{-1\}$ then $\mathcal{U}$ is isomorphic to its dual, see [16, Theorem 4.2], and the dual of $\mathcal{U}$ contains some O'Nan configuration, see [16, Lemma 5.4c].

We conjecture that Lemma 3.3 holds true for any $T$. Theorem 3.5 proves this as long as $T$ is small enough. On the other end, Theorem 3.6 provides Moulton planes with large $T$ for which the conjecture holds.

**Theorem 3.5.** *If $q > 5$ and*

$$|T| < \tfrac{1}{2}\left((q+1) - \sqrt{\tfrac{1}{2}(q+1)(q+3)}\right), \tag{3.2}$$

*then $\mathcal{U}$ in the Moulton plane $\mathfrak{M}_T(q^2)$ is a non-classical unital.*

*Proof.* As in the proof of Lemma 3.3, we show the existence of a O'Nan-configuration $\{P_0, P_1, P_2, P_3, P_4, P_5\}$ lying in $\mathcal{U}$. For a point $P(a,b) \in \mathrm{AG}(2, q^2)$ with $a \neq 0$ and $\|a\| \in T \setminus \{-1\}$, Theorem 2.6 ensures the existence of two non-vertical lines $\ell_1$ and $\ell_2$ through $P$ such that

(i) neither $\ell_1$ nor $\ell_2$ is horizontal or passes through the origin,

(ii) $P_0 = \bar{\ell}_1 \cap \bar{\ell}_2 \in \mathcal{U}$.

From Lemma 2.1, there exist at least $q+1-2|T|$ points $P(x,y)$ lying on $\ell_1 \cap \mathcal{U}$ such that $\|x\| \notin T$, and the same holds for $\ell_2 \cap \mathcal{U}$. Therefore, Theorem 2.4 applies with $\lambda = q+1-2|T|$ showing that if (3.2) is assumed, then the unital $\mathcal{U}$ in $\mathfrak{M}_T(q^2)$ contains a O'Nan configuration. $\square$

**Theorem 3.6.** *If $q > 5$, then there exists a $T$ with $|T| > q-4$ such that $\mathcal{U}$ is a non-classical unital in $\mathfrak{M}_T(q^2)$.*

*Proof.* From the proof of Theorem 3.5, some Moulton plane $\mathfrak{M}_T(q^2)$ contains O'Nan configurations lying in $\mathcal{U}$. If $\{P_0, P_1, P_2, P_3, P_4, P_5\}$ one of them, add each non-zero element $s \in \mathrm{GF}(q)$ to $T$ which satisfies the condition $s \neq \|x_i\|$ for $P_i = P_i(x_i, y_i)$ with $1 \leq i \leq 5$. Then $T$ expands and its size becomes at least $q-4$. In the resulting Moulton plane $\mathfrak{M}_T(q^2)$, the above hexagon $\{P_0, P_1, P_2, P_3, P_4, P_5\}$ is still a O'Nan configuration lying in the unital $\mathcal{U}$. $\square$

## References


[1] V. Abatangelo, M. R. Enea, G. Korchmáros and B. Larato, Ovals and unitals in commutative twisted field planes, *Discrete Math.* **208/209** (1999), 3–8, doi:10.1016/s0012-365x(99)00055-2.

[2] V. Abatangelo, G. Korchmáros and B. Larato, Transitive parabolic unitals in translation planes of odd order, *Discrete Math.* **231** (2001), 3–10, doi:10.1016/s0012-365x(00)00301-0.

[3] V. Abatangelo and B. Larato, Canonically inherited arcs in Moulton planes of odd order, *Innov. Incidence Geom.* **6/7** (2007/08), 3–21, http://www.iig.ugent.be/online/6/volume-6-article-1-online.pdf.

[4] V. Abatangelo and B. Larato, Complete arcs in Moulton planes of odd order, *Ars Combin.* **98** (2011), 521–527.

[5] A. Barlotti and G. Lunardon, Una classe di unitals nei $\Delta$-piani, *Riv. Mat. Univ. Parma (4)* **5** (1979), 781–785, http://www.rivmat.unipr.it/fulltext/1979-5s-5ss/1979-5ss-781.pdf.

[6] S. G. Barwick, A characterization of the classical unital, *Geom. Dedicata* **52** (1994), 175–180, doi:10.1007/bf01263605.

[7] S. G. Barwick, A class of Buekenhout unitals in the Hall plane, *Bull. Belg. Math. Soc. Simon Stevin* **3** (1996), 113–124, http://projecteuclid.org/euclid.bbms/1105540762.

[8] S. G. Barwick, Unitals in the Hall plane, *J. Geom.* **58** (1997), 26–42, doi:10.1007/bf01222924.

[9] S. G. Barwick and G. L. Ebert, *Unitals in Projective Planes*, Springer Monographs in Mathematics, Springer, New York, 2008, doi:10.1007/978-0-387-76366-8.

[10] S. G. Barwick and D. J. Marshall, Unitals and replaceable $t$-nests, *Australas. J. Combin.* **43** (2009), 115–126, https://ajc.maths.uq.edu.au/pdf/43/ajc_v43_p115.pdf.

[11] S. G. Barwick and C. T. Quinn, Generalising a characterisation of Hermitian curves, *J. Geom.* **70** (2001), 1–7, doi:10.1007/pl00000978.

[12] A. Beutelspacher, Embedding the complement of a Baer subplane or a unital in a finite projective plane, *Mitt. Math. Sem. Giessen* **163** (1984), 189–202.

[13] F. Buekenhout, Existence of unitals in finite translation planes of order $q^2$ with a kernel of order $q$, *Geom. Dedicata* **5** (1976), 189–194, doi:10.1007/bf00145956.

[14] M. J. de Resmini and N. Hamilton, Hyperovals and unitals in Figueroa planes, *European J. Combin.* **19** (1998), 215–220, doi:10.1006/eujc.1997.0166.

[15] T. Grundhöfer, B. Krinn and M. Stroppel, Non-existence of isomorphisms between certain unitals, *Des. Codes Cryptogr.* **60** (2011), 197–201, doi:10.1007/s10623-010-9428-2.

[16] K. Grüning, A class of unitals of order $q$ which can be embedded in two different planes of order $q^2$, *J. Geom.* **29** (1987), 61–77, doi:10.1007/bf01234988.

[17] A. M. W. Hui, H. F. Law, Y. K. Tai and P. P. W. Wong, Non-classical polar unitals in finite Dickson semifield planes, *J. Geom.* **104** (2013), 469–493, doi:10.1007/s00022-013-0174-2.

[18] A. M. W. Hui, H. F. Law, Y. K. Tai and P. P. W. Wong, A note on unitary polarities in finite Dickson semifield planes, *J. Geom.* **106** (2015), 175–183, doi:10.1007/s00022-014-0254-y.

[19] M. W. Hui and P. P. W. Wong, Non-classical polar unitals in finite Figueroa planes, *J. Geom.* **103** (2012), 263–273, doi:10.1007/s00022-012-0121-7.

[20] N. L. Johnson and G. Lunardon, On the Bose-Barlotti $\Delta$-planes, *Geom. Dedicata* **49** (1994), 173–182, doi:10.1007/bf01610619.

[21] G. Korchmáros, A. Siciliano and T. Szőnyi, Embedding of classical polar unitals in $\mathrm{PG}(2, q^2)$, *J. Combin. Theory Ser. A* (2017), doi:10.1016/j.jcta.2017.08.002.

[22] M. E. O'Nan, Automorphisms of unitary block designs, *J. Algebra* **20** (1972), 495–511, doi:10.1016/0021-8693(72)90070-1.

[23] G. Rinaldi, Construction of unitals in the Hall planes, *Geom. Dedicata* **56** (1995), 249–255, doi:10.1007/bf01263565.

[24] G. Rinaldi, Complete unital-derived arcs in the Hall planes, *Abh. Math. Sem. Univ. Hamburg* **71** (2001), 197–203, doi:10.1007/bf02941471.

[25] A. Sonnino, Non-classical unitals may be code words, submitted.

[26] A. Sonnino, Existence of canonically inherited arcs in Moulton planes of odd order, *Finite Fields Appl.* **33** (2015), 187–197, doi:10.1016/j.ffa.2014.11.011.

[27] S. D. Stoichev and V. D. Tonchev, Unital designs in planes of order 16, *Discrete Appl. Math.* **102** (2000), 151–158, doi:10.1016/s0166-218x(99)00236-x.

[28] K. L. Wantz, Unitals in the regular nearfield planes, *J. Geom.* **88** (2008), 169–177, doi:10.1007/s00022-007-2021-9.