Tridiagonal pairs of q -Racah type, the Bockting operator ψ , and L -operators for $U_q(L(\mathfrak{sl}_2))$

Paul Terwilliger

*Department of Mathematics, University of Wisconsin, 480 Lincoln Drive,
Madison, WI 53706-1388 USA*

Received 11 January 2017, accepted 18 March 2017, published online 5 May 2017

Abstract

We describe the Bockting operator ψ for a tridiagonal pair of q -Racah type, in terms of a certain L -operator for the quantum loop algebra $U_q(L(\mathfrak{sl}_2))$.

Keywords: Bockting operator; tridiagonal pair; Leonard pair.

Math. Subj. Class.: 17B37, 15A21

1 Introduction

In the theory of quantum groups there exists the concept of an L -operator; this was introduced in [20] to obtain solutions for the Yang-Baxter equation. In linear algebra there exists the concept of a tridiagonal pair; this was introduced in [13] to describe the irreducible modules for the subconstituent algebra of a Q -polynomial distance-regular graph. Recently some authors have connected the two concepts. In [1], [3] Pascal Baseilhac and Kozo Koizumi use L -operators for the quantum loop algebra $U_q(L(\mathfrak{sl}_2))$ to construct a family of finite-dimensional modules for the q -Onsager algebra $\mathcal{O}_q$; see [2, 4, 5, 6] for related work. A finite-dimensional irreducible $\mathcal{O}_q$ -module is essentially the same thing as a tridiagonal pair of q -Racah type [9, Section 12], [23, Section 3]. In [22, Section 9], Kei Miki uses similar L -operators to describe how $U_q(L(\mathfrak{sl}_2))$ is related to the q -tetrahedron algebra $\boxtimes_q$. A finite-dimensional irreducible $\boxtimes_q$ -module is essentially the same thing as a tridiagonal pair of q -geometric type [15, Theorem 2.7], [16, Theorems 10.3, 10.4]. Following Baseilhac, Koizumi, and Miki, in the present paper we use L -operators for $U_q(L(\mathfrak{sl}_2))$ to describe the Bockting operator ψ associated with a tridiagonal pair of q -Racah type. Before going into detail, we recall some notation and basic concepts. Throughout this paper

E-mail address: terwilli@math.wisc.edu (Paul Terwilliger)

$\mathbb{F}$ denotes a field. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. For an $\mathbb{F}$ -linear map $A : V \rightarrow V$ and a subspace $W \subseteq V$, we say that W is an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{F}$ such that $W = \{v \in V \mid Av = \theta v\}$; in this case θ is called the *eigenvalue* of A associated with W . We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A .

Definition 1.1. (See [13, Definition 1.1].) Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on V we mean an ordered pair of $\mathbb{F}$ -linear maps $A : V \rightarrow V$ and $A^* : V \rightarrow V$ that satisfy the following four conditions:

- (i) Each of A, A^* is diagonalizable.
- (ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d), \quad (1.1)$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

- (iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta), \quad (1.2)$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

- (iv) There does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

We refer the reader to [12, 13, 17] for background on TD pairs, and here mention only a few essential points. Let A, A^* denote a TD pair on V , as in Definition 1.1. By [13, Lemma 4.5] the integers d and δ from (1.1) and (1.2) are equal; we call this common value the *diameter* of A, A^* . An ordering of the eigenspaces for A (resp. A^*) is called *standard* whenever it satisfies (1.1) (resp. (1.2)). Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of A . By [13, Lemma 2.4] the ordering $\{V_{d-i}\}_{i=0}^d$ is standard and no further ordering is standard. A similar result holds for the eigenspaces of A^* . Until the end of this section fix a standard ordering $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) of the eigenspaces for A (resp. A^*). For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) for the eigenspace V_i (resp. V_i^*). By construction $\{\theta_i\}_{i=0}^d$ are mutually distinct and contained in $\mathbb{F}$. Moreover $\{\theta_i^*\}_{i=0}^d$ are mutually distinct and contained in $\mathbb{F}$. By [13, Theorem 11.1] the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d-1$. For this constraint the solutions can be given in closed form [13, Theorem 11.2]. The “most general” solution is called q -Racah, and will be described shortly.

We now recall the split decomposition [13, Section 4]. For $0 \leq i \leq d$ define

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i}).$$

For notational convenience define $U_{-1} = 0$ and $U_{d+1} = 0$. By [13, Theorem 4.6] the sum $V = \sum_{i=0}^d U_i$ is direct. By [13, Theorem 4.6] both

$$\begin{aligned} U_0 + U_1 + \cdots + U_i &= V_0^* + V_1^* + \cdots + V_i^*, \\ U_i + U_{i+1} + \cdots + U_d &= V_0 + V_1 + \cdots + V_{d-i} \end{aligned}$$

for $0 \leq i \leq d$. Let $I : V \rightarrow V$ denote the identity map. By [13, Theorem 4.6] both

$$(A - \theta_{d-i}I)U_i \subseteq U_{i+1}, \quad (A^* - \theta_i^*I)U_i \subseteq U_{i-1} \quad (1.3)$$

for $0 \leq i \leq d$.

We now describe the q -Racah case. Pick a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. We say that A, A^* has q -Racah type whenever there exist nonzero $a, b \in \mathbb{F}$ such that both

$$\theta_i = aq^{2i-d} + a^{-1}q^{d-2i}, \quad \theta_i^* = bq^{2i-d} + b^{-1}q^{d-2i} \quad (1.4)$$

for $0 \leq i \leq d$. For the rest of this section assume that A, A^* has q -Racah type. For $1 \leq i \leq d$ we have $q^{2i} \neq 1$; otherwise $\theta_i = \theta_0$. Define an $\mathbb{F}$ -linear map $K : V \rightarrow V$ such that for $0 \leq i \leq d$, U_i is an eigenspace of K with eigenvalue q^{d-2i} . Thus

$$(K - q^{d-2i}I)U_i = 0 \quad (0 \leq i \leq d). \quad (1.5)$$

Note that K is invertible. For $0 \leq i \leq d$ the following holds on U_i :

$$aK + a^{-1}K^{-1} = \theta_{d-i}I. \quad (1.6)$$

Define an $\mathbb{F}$ -linear map $R : V \rightarrow V$ such that for $0 \leq i \leq d$, R acts on U_i as $A - \theta_{d-i}I$. By (1.6),

$$A = aK + a^{-1}K^{-1} + R. \quad (1.7)$$

By the equation on the left in (1.3),

$$RU_i \subseteq U_{i+1} \quad (0 \leq i \leq d). \quad (1.8)$$

We now recall the Bockting operator ψ . By [8, Lemma 5.7] there exists a unique $\mathbb{F}$ -linear map $\psi : V \rightarrow V$ such that both

$$\psi U_i \subseteq U_{i-1} \quad (0 \leq i \leq d), \quad (1.9)$$

$$\psi R - R\psi = (q - q^{-1})(K - K^{-1}). \quad (1.10)$$

The known properties of ψ are described in [7, 8, 9]. Suppose we are given A, A^*, R, K in matrix form, and wish to obtain ψ in matrix form. This can be done using (1.8), (1.9), (1.10) and induction on i . The calculation can be tedious, so one desires a more explicit description of ψ . In the present paper we give an explicit description of ψ , in terms of a certain L -operator for $U_q(L(\mathfrak{sl}_2))$. According to this description, ψ is equal to $-a$ times the ratio of two components for the L -operator. Theorem 5.4 is our main result.

The paper is organized as follows. In Section 2 we review the algebra $U_q(L(\mathfrak{sl}_2))$ in its Chevalley presentation. In Section 3 we recall the equitable presentation for $U_q(L(\mathfrak{sl}_2))$. In Section 4 we discuss some L -operators for $U_q(L(\mathfrak{sl}_2))$. In Section 5 we use these L -operators to describe ψ .

2 The quantum loop algebra $U_q(L(\mathfrak{sl}_2))$

Recall the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. We will be discussing algebras. An algebra is meant to be associative and have a 1. Recall the field $\mathbb{F}$. Until the end of Section 4, fix a nonzero $q \in \mathbb{F}$ such that $q^2 \neq 1$. Define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{Z}.$$

All tensor products are meant to be over $\mathbb{F}$.

Definition 2.1. (See [10, Section 3.3].) Let $U_q(L(\mathfrak{sl}_2))$ denote the $\mathbb{F}$ -algebra with generators $E_i, F_i, K_i^{\pm 1}$ ($i \in \{0, 1\}$) and relations

$$\begin{aligned} K_i K_i^{-1} &= 1, & K_i^{-1} K_i &= 1, \\ K_0 K_1 &= 1, & K_1 K_0 &= 1, \\ K_i E_i &= q^2 E_i K_i, & K_i F_i &= q^{-2} F_i K_i, \\ K_i E_j &= q^{-2} E_j K_i, & K_i F_j &= q^2 F_j K_i, & i \neq j, \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ E_i^3 E_j - [3]_q E_i^2 E_j E_i + [3]_q E_i E_j E_i^2 - E_j E_i^3 &= 0, & i \neq j, \\ F_i^3 F_j - [3]_q F_i^2 F_j F_i + [3]_q F_i F_j F_i^2 - F_j F_i^3 &= 0, & i \neq j. \end{aligned}$$

We call $E_i, F_i, K_i^{\pm 1}$ the Chevalley generators for $U_q(L(\mathfrak{sl}_2))$.

Lemma 2.2. (See [18, p. 35].) We turn $U_q(L(\mathfrak{sl}_2))$ into a Hopf algebra as follows. The coproduct Δ satisfies

$$\begin{aligned} \Delta(K_i) &= K_i \otimes K_i, & \Delta(K_i^{-1}) &= K_i^{-1} \otimes K_i^{-1}, \\ \Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & \Delta(F_i) &= 1 \otimes F_i + F_i \otimes K_i^{-1}. \end{aligned}$$

The counit ε satisfies

$$\varepsilon(K_i) = 1, \quad \varepsilon(K_i^{-1}) = 1, \quad \varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0.$$

The antipode S satisfies

$$S(K_i) = K_i^{-1}, \quad S(K_i^{-1}) = K_i, \quad S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i.$$

We now discuss the $U_q(L(\mathfrak{sl}_2))$ -modules.

Lemma 2.3. (See [10, Section 4].) There exists a family of $U_q(L(\mathfrak{sl}_2))$ -modules

$$\mathbf{V}(d, t) \quad 0 \neq d \in \mathbb{N}, \quad 0 \neq t \in \mathbb{F} \quad (2.1)$$

with this property: $\mathbf{V}(d, t)$ has a basis $\{v_i\}_{i=0}^d$ such that

$$\begin{aligned} K_1 v_i &= q^{d-2i} v_i & (0 \leq i \leq d), \\ E_1 v_i &= [d-i+1]_q v_{i-1} & (1 \leq i \leq d), \quad E_1 v_0 = 0, \\ F_1 v_i &= [i+1]_q v_{i+1} & (0 \leq i \leq d-1), \quad F_1 v_d = 0, \\ K_0 v_i &= q^{2i-d} v_i & (0 \leq i \leq d), \\ E_0 v_i &= t[i+1]_q v_{i+1} & (0 \leq i \leq d-1), \quad E_0 v_d = 0, \\ F_0 v_i &= t^{-1}[d-i+1]_q v_{i-1} & (1 \leq i \leq d), \quad F_0 v_0 = 0. \end{aligned}$$

The module $\mathbf{V}(d, t)$ is irreducible provided that $q^{2i} \neq 1$ for $1 \leq i \leq d$.

Definition 2.4. Referring to Lemma 2.3, we call $\mathbf{V}(d, t)$ an *evaluation module* for $U_q(L(\mathfrak{sl}_2))$. We call d the *diameter*. We call t the *evaluation parameter*.

Example 2.5. For $0 \neq t \in \mathbb{F}$ the $U_q(L(\mathfrak{sl}_2))$ -module $\mathbf{V}(1, t)$ is described as follows. With respect to the basis v_0, v_1 from Lemma 2.3, the matrices representing the Chevalley generators are

$$\begin{aligned} E_1 &: \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & F_1 &: \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & K_1 &: \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \\ E_0 &: \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, & F_0 &: \begin{pmatrix} 0 & t^{-1} \\ 0 & 0 \end{pmatrix}, & K_0 &: \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}. \end{aligned}$$

Lemma 2.6. (See [19, p. 58].) Let U and V denote $U_q(L(\mathfrak{sl}_2))$ -modules. Then $U \otimes V$ becomes a $U_q(L(\mathfrak{sl}_2))$ -module as follows. For $u \in U$ and $v \in V$,

$$\begin{aligned} K_i(u \otimes v) &= K_i(u) \otimes K_i(v), \\ K_i^{-1}(u \otimes v) &= K_i^{-1}(u) \otimes K_i^{-1}(v), \\ E_i(u \otimes v) &= E_i(u) \otimes v + K_i(u) \otimes E_i(v), \\ F_i(u \otimes v) &= u \otimes F_i(v) + F_i(u) \otimes K_i^{-1}(v). \end{aligned}$$

Definition 2.7. (See [11, p. 110].) Up to isomorphism, there exists a unique $U_q(L(\mathfrak{sl}_2))$ -module of dimension 1 on which each $u \in U_q(L(\mathfrak{sl}_2))$ acts as $\varepsilon(u)I$, where ε is from Lemma 2.2. This $U_q(L(\mathfrak{sl}_2))$ -module is said to be *trivial*.

Proposition 2.8. (See [22, Theorem 3.2].) Assume that $\mathbb{F}$ is algebraically closed with characteristic zero, and q is not a root of unity. Let V denote a nontrivial finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$ -module on which each eigenvalue of K_1 is an integral power of q . Then V is isomorphic to a tensor product of evaluation $U_q(L(\mathfrak{sl}_2))$ -modules.

3 The equitable presentation for $U_q(L(\mathfrak{sl}_2))$

In this section we recall the equitable presentation for $U_q(L(\mathfrak{sl}_2))$. Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4. In a moment we will discuss some objects X_{ij} . The subscripts i, j are meant to be in $\mathbb{Z}_4$.

Lemma 3.1. (See [14, Theorem 2.1], [22, Proposition 4.2].) The algebra $U_q(L(\mathfrak{sl}_2))$ has a presentation by generators

$$X_{01}, \quad X_{12}, \quad X_{23}, \quad X_{30}, \quad X_{13}, \quad X_{31} \tag{3.1}$$

and the following relations:

$$\begin{aligned} X_{13}X_{31} &= 1, \quad X_{31}X_{13} = 1, \quad \frac{qX_{01}X_{12} - q^{-1}X_{12}X_{01}}{q - q^{-1}} = 1, \quad \frac{qX_{12}X_{23} - q^{-1}X_{23}X_{12}}{q - q^{-1}} = 1, \\ \frac{qX_{23}X_{30} - q^{-1}X_{30}X_{23}}{q - q^{-1}} &= 1, \quad \frac{qX_{30}X_{01} - q^{-1}X_{01}X_{30}}{q - q^{-1}} = 1, \quad \frac{qX_{01}X_{13} - q^{-1}X_{13}X_{01}}{q - q^{-1}} = 1, \\ \frac{qX_{31}X_{12} - q^{-1}X_{12}X_{31}}{q - q^{-1}} &= 1, \quad \frac{qX_{23}X_{31} - q^{-1}X_{31}X_{23}}{q - q^{-1}} = 1, \quad \frac{qX_{13}X_{30} - q^{-1}X_{30}X_{13}}{q - q^{-1}} = 1, \\ X_{i,i+1}^3 X_{i+2,i+3} - [3]_q X_{i,i+1}^2 X_{i+2,i+3} X_{i,i+1} + [3]_q X_{i,i+1} X_{i+2,i+3} X_{i,i+1}^2 - X_{i+2,i+3} X_{i,i+1}^3 &= 0. \end{aligned}$$

An isomorphism with the presentation in Definition 2.1 sends

$$\begin{aligned} X_{01} &\mapsto K_0 + q(q - q^{-1})K_0F_0, & X_{12} &\mapsto K_1 - (q - q^{-1})E_1, \\ X_{23} &\mapsto K_1 + q(q - q^{-1})K_1F_1, & X_{30} &\mapsto K_0 - (q - q^{-1})E_0, \\ X_{13} &\mapsto K_1, & X_{31} &\mapsto K_0. \end{aligned}$$

The inverse isomorphism sends

$$\begin{aligned} E_1 &\mapsto (X_{13} - X_{12})(q - q^{-1})^{-1}, & E_0 &\mapsto (X_{31} - X_{30})(q - q^{-1})^{-1}, \\ F_1 &\mapsto (X_{31}X_{23} - 1)q^{-1}(q - q^{-1})^{-1}, & F_0 &\mapsto (X_{13}X_{01} - 1)q^{-1}(q - q^{-1})^{-1}, \\ K_1 &\mapsto X_{13}, & K_0 &\mapsto X_{31}. \end{aligned}$$

Note 3.2. For notational convenience, we identify the copy of $U_q(L(\mathfrak{sl}_2))$ given in Definition 2.1 with the copy given in Lemma 3.1, via the isomorphism given in Lemma 3.1.

Definition 3.3. Referring to Lemma 3.1, we call the generators (3.1) the *equitable generators* for $U_q(L(\mathfrak{sl}_2))$.

Lemma 3.4. (See [24, Theorem 3.4].) *From the equitable point of view the Hopf algebra $U_q(L(\mathfrak{sl}_2))$ looks as follows. The coproduct Δ satisfies*

$$\begin{aligned} \Delta(X_{13}) &= X_{13} \otimes X_{13}, & \Delta(X_{31}) &= X_{31} \otimes X_{31}, \\ \Delta(X_{01}) &= (X_{01} - X_{31}) \otimes 1 + X_{31} \otimes X_{01}, & \Delta(X_{12}) &= (X_{12} - X_{13}) \otimes 1 + X_{13} \otimes X_{12}, \\ \Delta(X_{23}) &= (X_{23} - X_{13}) \otimes 1 + X_{13} \otimes X_{23}, & \Delta(X_{30}) &= (X_{30} - X_{31}) \otimes 1 + X_{31} \otimes X_{30}. \end{aligned}$$

The counit ε satisfies

$$\begin{aligned} \varepsilon(X_{13}) &= 1, & \varepsilon(X_{31}) &= 1, & \varepsilon(X_{01}) &= 1, \\ \varepsilon(X_{12}) &= 1, & \varepsilon(X_{23}) &= 1, & \varepsilon(X_{30}) &= 1. \end{aligned}$$

The antipode S satisfies

$$\begin{aligned} S(X_{31}) &= X_{13}, & S(X_{13}) &= X_{31}, \\ S(X_{01}) &= 1 + X_{13} - X_{13}X_{01}, & S(X_{12}) &= 1 + X_{31} - X_{31}X_{12}, \\ S(X_{23}) &= 1 + X_{31} - X_{31}X_{23}, & S(X_{30}) &= 1 + X_{13} - X_{13}X_{30}. \end{aligned}$$

4 Some L -operators for $U_q(L(\mathfrak{sl}_2))$

In this section we recall some L -operators for $U_q(L(\mathfrak{sl}_2))$, and describe their basic properties.

We recall some notation. Let Δ denote the coproduct for a Hopf algebra H . Then the opposite coproduct Δ^{op} is the composition

$$\Delta^{\text{op}} : H \xrightarrow[\Delta]{} H \otimes H \xrightarrow[r \otimes s \mapsto s \otimes r]{} H \otimes H.$$

Definition 4.1. (See [22, Section 9.1].) Let V denote a $U_q(L(\mathfrak{sl}_2))$ -module and $0 \neq t \in \mathbb{F}$. Consider an $\mathbb{F}$ -linear map

$$L : V \otimes \mathbf{V}(1, t) \rightarrow V \otimes \mathbf{V}(1, t).$$

We call this map an L -operator for V with parameter t whenever the following diagram commutes for all $u \in U_q(L(\mathfrak{sl}_2))$:

$$\begin{array}{ccc} V \otimes \mathbf{V}(1, t) & \xrightarrow{\Delta(u)} & V \otimes \mathbf{V}(1, t) \\ L \downarrow & & \downarrow L \\ V \otimes \mathbf{V}(1, t) & \xrightarrow{\Delta^{\text{op}}(u)} & V \otimes \mathbf{V}(1, t) \end{array}$$

Definition 4.2. (See [22, Section 9.1].) Let V denote a $U_q(L(\mathfrak{sl}_2))$ -module and $0 \neq t \in \mathbb{F}$. Consider any $\mathbb{F}$ -linear map

$$L : V \otimes \mathbf{V}(1, t) \rightarrow V \otimes \mathbf{V}(1, t). \quad (4.1)$$

For $r, s \in \{0, 1\}$ define an $\mathbb{F}$ -linear map $L_{rs} : V \rightarrow V$ such that for $v \in V$,

$$L(v \otimes v_0) = L_{00}(v) \otimes v_0 + L_{10}(v) \otimes v_1, \quad (4.2)$$

$$L(v \otimes v_1) = L_{01}(v) \otimes v_0 + L_{11}(v) \otimes v_1. \quad (4.3)$$

Here v_0, v_1 is the basis for $\mathbf{V}(1, t)$ from Lemma 2.3.

Lemma 4.3. Referring to Definition 4.2, the map (4.1) is an L -operator for V with parameter t if and only if the following equations hold on V :

$$\begin{aligned} K_1 L_{00} &= L_{00} K_1, & K_1 L_{01} &= q^{-2} L_{01} K_1, \\ K_1 L_{10} &= q^2 L_{10} K_1, & K_1 L_{11} &= L_{11} K_1; \end{aligned}$$

$$\begin{aligned} L_{00} E_1 - q E_1 L_{00} &= L_{10}, & L_{01} E_1 - q E_1 L_{01} &= L_{11} - L_{00} K_1, \\ L_{10} E_1 - q^{-1} E_1 L_{10} &= 0, & L_{11} E_1 - q^{-1} E_1 L_{11} &= -L_{10} K_1; \end{aligned}$$

$$\begin{aligned} F_1 L_{00} - q^{-1} L_{00} F_1 &= L_{01}, & F_1 L_{01} - q L_{01} F_1 &= 0, \\ F_1 L_{10} - q^{-1} L_{10} F_1 &= L_{11} - K_0 L_{00}, & F_1 L_{11} - q L_{11} F_1 &= -K_0 L_{01}; \end{aligned}$$

$$\begin{aligned} K_0 L_{00} &= L_{00} K_0, & K_0 L_{01} &= q^2 L_{01} K_0, \\ K_0 L_{10} &= q^{-2} L_{10} K_0, & K_0 L_{11} &= L_{11} K_0; \end{aligned}$$

$$\begin{aligned} L_{00} E_0 - q^{-1} E_0 L_{00} &= -t L_{01} K_0, & L_{01} E_0 - q^{-1} E_0 L_{01} &= 0, \\ L_{10} E_0 - q E_0 L_{10} &= t L_{00} - t L_{11} K_0, & L_{11} E_0 - q E_0 L_{11} &= t L_{01}; \end{aligned}$$

$$\begin{aligned} F_0 L_{00} - q L_{00} F_0 &= -t^{-1} K_1 L_{10}, & F_0 L_{01} - q^{-1} L_{01} F_0 &= t^{-1} L_{00} - t^{-1} K_1 L_{11}, \\ F_0 L_{10} - q L_{10} F_0 &= 0, & F_0 L_{11} - q^{-1} L_{11} F_0 &= t^{-1} L_{10}. \end{aligned}$$

Proof. This is routinely checked. $\square$

Example 4.4. (See [21, Appendix], [22, Proposition 9.2].) Referring to Definition 4.2, assume that V is an evaluation module $\mathbf{V}(d, \mu)$ such that $q^{2i} \neq 1$ for $1 \leq i \leq d$. Consider the matrices that represent the L_{rs} with respect to the basis $\{v_i\}_{i=0}^d$ for $\mathbf{V}(d, \mu)$ from Lemma 2.3. Then the following are equivalent:

- (i) the map (4.1) is an L -operator for V with parameter t ;
- (ii) the matrix entries are given in the table below (all matrix entries not shown are zero):

operator	$(i, i-1)$ -entry	(i, i) -entry	$(i-1, i)$ -entry
L_{00}	0	$\frac{q^{1-i}-\mu^{-1}tq^{i-d}}{q-q^{-1}}\xi$	0
L_{01}	$[i]_q q^{1-i}\xi$	0	0
L_{10}	0	0	$[d-i+1]_q q^{i-d}\mu^{-1}t\xi$
L_{11}	0	$\frac{q^{i-d+1}-\mu^{-1}tq^{-i}}{q-q^{-1}}\xi$	0

Here $\xi \in \mathbb{F}$.

Lemma 4.5. (See [22, Proposition 9.3].) *Let U and V denote $U_q(L(\mathfrak{sl}_2))$ -modules, and consider the $U_q(L(\mathfrak{sl}_2))$ -module $U \otimes V$ from Lemma 2.6. Let $0 \neq t \in \mathbb{F}$. Suppose we are given L -operators for U and V with parameter t . Then there exists an L -operator for $U \otimes V$ with parameter t such that for $r, s \in \{0, 1\}$,*

$$L_{rs}(u \otimes v) = L_{r0}(u) \otimes L_{0s}(v) + L_{r1}(u) \otimes L_{1s}(v) \quad u \in U, \quad v \in V. \quad (4.4)$$

Proof. For $r, s \in \{0, 1\}$ define an $\mathbb{F}$ -linear map $L_{rs} : U \otimes V \rightarrow U \otimes V$ that satisfies (4.4). Using (4.4) and Lemma 2.6 one checks that the L_{rs} satisfy the equations in Lemma 4.3. The result follows by Lemma 4.3. $\square$

Corollary 4.6. *Adopt the notation and assumptions of Proposition 2.8. Then for $0 \neq t \in \mathbb{F}$ there exists a nonzero L -operator for V with parameter t .*

Proof. By Proposition 2.8 along with Example 4.4 and Lemma 4.5. $\square$

5 TD pairs and L -operators

In Section 1 we discussed a TD pair A, A^* on V . We now return to this discussion, adopting the notation and assumptions that were in force at the end of Section 1. Recall the scalars q, a, b from (1.4). Recall the map K from above (1.5).

Proposition 5.1. (See [17, p. 103].) *Assume that $\mathbb{F}$ is algebraically closed with characteristic zero, and q is not a root of unity. Then the vector space V becomes a $U_q(L(\mathfrak{sl}_2))$ -module on which $K = X_{31}$, $K^{-1} = X_{13}$ and*

$$A = aX_{01} + a^{-1}X_{12}, \quad A^* = bX_{23} + b^{-1}X_{30}.$$

Proof. This is how [17, p. 103] looks from the equitable point of view. $\square$

Note 5.2. The $U_q(L(\mathfrak{sl}_2))$ -module structure from Proposition 5.1 is not unique in general.

We now investigate the $U_q(L(\mathfrak{sl}_2))$ -module structure from Proposition 5.1. Recall the map R from above (1.7).

Lemma 5.3. *Assume that the vector space V becomes a $U_q(L(\mathfrak{sl}_2))$ -module on which $K = X_{31}$, $K^{-1} = X_{13}$ and*

$$A = aX_{01} + a^{-1}X_{12}, \quad A^* = bX_{23} + b^{-1}X_{30}.$$

On this module,

(i) R looks as follows in the equitable presentation:

$$R = a(X_{01} - X_{31}) + a^{-1}(X_{12} - X_{13}). \quad (5.1)$$

(ii) R looks as follows in the Chevalley presentation:

$$R = (q - q^{-1})(aqK_0F_0 - a^{-1}E_1). \quad (5.2)$$

Proof. (i) In line (1.7) eliminate A, K, K^{-1} using the assumptions of the present lemma.
(ii) Evaluate the right-hand side of (5.1) using the identifications from Lemma 3.1 and Note 3.2. $\square$

We now present our main result. Recall the Bockting operator ψ from (1.9), (1.10).

Theorem 5.4. *Assume that the vector space V becomes a $U_q(L(\mathfrak{sl}_2))$ -module on which $K = X_{31}$, $K^{-1} = X_{13}$ and*

$$A = aX_{01} + a^{-1}X_{12}, \quad A^* = bX_{23} + b^{-1}X_{30}.$$

Consider an L -operator for V with parameter a^2 . Then on V ,

$$\psi = -a(L_{00})^{-1}L_{01} \quad (5.3)$$

provided that L_{00} is invertible.

Proof. Let $\widehat{\psi}$ denote the expression on the right in (5.3). We show $\psi = \widehat{\psi}$. To do this, we show that $\widehat{\psi}$ satisfies (1.9), (1.10). Concerning (1.9), by Lemma 4.3 the equation $K_0\widehat{\psi} = q^2\widehat{\psi}K_0$ holds on V . By Lemma 3.1, Note 3.2, and the construction, we obtain $K_0 = X_{31} = K$ on V . By these comments $K\widehat{\psi} = q^2\widehat{\psi}K$ on V . By this and (1.5) we obtain $\widehat{\psi}U_i \subseteq U_{i-1}$ for $0 \leq i \leq d$. So $\widehat{\psi}$ satisfies (1.9). Next we show that $\widehat{\psi}$ satisfies (1.10). Since L_{00} is invertible and $K_0K_1 = 1$ it suffices to show that on V ,

$$L_{00}(\widehat{\psi}R - R\widehat{\psi}) = (q - q^{-1})L_{00}(K_0 - K_1). \quad (5.4)$$

By this and (5.2) it suffices to show that on V ,

$$aqL_{00}(\widehat{\psi}K_0F_0 - K_0F_0\widehat{\psi}) - a^{-1}L_{00}(\widehat{\psi}E_1 - E_1\widehat{\psi}) + L_{00}(K_1 - K_0) = 0. \quad (5.5)$$

We examine the terms in (5.5). By Lemma 4.3 and the construction, the following hold on V :

$$\begin{aligned} L_{00}\widehat{\psi}K_0F_0 &= -aL_{01}K_0F_0 \\ &= -aq^{-2}K_0L_{01}F_0 \\ &= -aq^{-1}K_0(F_0L_{01} - a^{-2}L_{00} + a^{-2}K_1L_{11}) \end{aligned}$$

and

$$\begin{aligned} L_{00}K_0F_0\widehat{\psi} &= K_0L_{00}F_0\widehat{\psi} \\ &= q^{-1}K_0(a^{-2}K_1L_{10} + F_0L_{00})\widehat{\psi} \\ &= q^{-1}K_0(a^{-2}K_1L_{10}\widehat{\psi} - aF_0L_{01}) \end{aligned}$$

and

$$\begin{aligned} L_{00}\widehat{\psi}E_1 &= -aL_{01}E_1 \\ &= -a(qE_1L_{01} + L_{11} - L_{00}K_1) \\ &= -a(qE_1L_{01} + L_{11} - K_1L_{00}) \end{aligned}$$

and

$$\begin{aligned} L_{00}E_1\widehat{\psi} &= (L_{10} + qE_1L_{00})\widehat{\psi} \\ &= L_{10}\widehat{\psi} - qaE_1L_{01} \end{aligned}$$

and

$$L_{00}K_1 = K_1L_{00}, \quad L_{00}K_0 = K_0L_{00}.$$

To verify (5.5), evaluate its left-hand side using the above comments and simplify the result using $K_0K_1 = 1$. The computation is routine, and omitted. We have shown that $\widehat{\psi}$ satisfies (1.10). The result follows. $\square$

6 Acknowledgment

The author thanks Sarah Bockting-Conrad and Edward Hanson for giving this paper a close reading and offering valuable suggestions. The author also thanks Pascal Baseilhac for many conversations concerning quantum groups, L -operators, and tridiagonal pairs.

References

- [1] P. Baseilhac, An integrable structure related with tridiagonal algebras, *Nuclear Phys. B* **705** (2005), 605–619, [arXiv:math-ph/0408025](#), doi: 10.1016/j.nuclphysb.2004.11.014.
- [2] P. Baseilhac and S. Belliard, The half-infinite XXZ chain in Onsager’s approach. *Nuclear Phys. B* **873** (2013), 550–584, doi: 10.1016/j.nuclphysb.2013.05.003.
- [3] P. Baseilhac and K. Koizumi, A new (in)finite dimensional algebra for quantum integrable models. *Nuclear Phys. B* **720** (2005), 325–347, [arXiv:math-ph/0503036](#), doi: 10.1016/j.nuclphysb.2005.05.021
- [4] P. Baseilhac and K. Koizumi, A deformed analogue of Onsager’s symmetry in the XXZ open spin chain, *J. Stat. Mech. Theory Exp.* **10** (2005), P10005, 15 pp. (electronic), [arXiv:hep-th/0507053](#), doi: doi.org/10.1088/1742-5468/2005/10/p10005.
- [5] P. Baseilhac and K. Koizumi, Exact spectrum of the XXZ open spin chain from the q -Onsager algebra representation theory, *J. Stat. Mech. Theory Exp.* **9** (2007), P09006, 27 pp. (electronic), [arXiv:hep-th/0703106](#), doi: 10.1088/1742-5468/2007/09/p09006.
- [6] P. Baseilhac and K. Shigechi, A new current algebra and the reflection equation, *Lett. Math. Phys.* **92** (2010), 47–65, doi: 10.1007/s11005-010-0380-x.
- [7] S. Bockting-Conrad, Two commuting operators associated with a tridiagonal pair, *Linear Algebra Appl.* **437** (2012), 242–270, doi: 10.1016/j.laa.2012.02.007.
- [8] S. Bockting-Conrad, Tridiagonal pairs of q -Racah type, the double lowering operator ψ , and the quantum algebra $U_q(\mathfrak{sl}_2)$, *Linear Algebra Appl.* **445** (2014), 256–279, doi: 10.1016/j.laa.2013.12.007.

- [9] S. Bockting-Conrad and P. Terwilliger, The algebra $U_q(\mathfrak{sl}_2)$ in disguise, *Linear Algebra Appl.* **459** (2014), 548–585, doi: 10.1016/j.laa.2014.07.022.
- [10] V. Chari and A. Pressley, Quantum affine algebras, *Commun. Math. Phys.* **142** (1991) 261–283.
- [11] V. Chari and A. Pressley. *A guide to quantum groups*. Cambridge University Press, Cambridge, 1994.
- [12] T. Ito, K. Nomura and P. Terwilliger. A classification of sharp tridiagonal pairs, *Linear Algebra Appl.* **435** (2011), 1857–1884, arXiv:1001.1812, doi: 10.1016/j.laa.2011.03.032.
- [13] T. Ito, K. Tanabe and P. Terwilliger, Some algebra related to P - and Q -polynomial association schemes, in: *Codes and Association Schemes (Piscataway NJ, 1999)*, Amer. Math. Soc., Providence RI, 2001, pp. 167–192, arXiv:math.CO/0406556.
- [14] T. Ito and P. Terwilliger, Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$, *Ramanujan J.* **13** (2007), 39–62, arXiv:math/0310042, doi: 10.1007/s11139-006-0242-4.
- [15] T. Ito and P. Terwilliger, Two non-nilpotent linear transformations that satisfy the cubic q -Serre relations, *J. Algebra Appl.* **6** (2007), 477–503, arXiv:math/0508398, doi: 10.1142/s021949880700234x.
- [16] T. Ito and P. Terwilliger, The q -tetrahedron algebra and its finite-dimensional irreducible modules, *Comm. Algebra* **35** (2007), 3415–3439, arXiv:math/0602199, doi: 10.1080/00927870701509180.
- [17] T. Ito and P. Terwilliger, The augmented tridiagonal algebra, *Kyushu J. Math.* **64** (2010), 81–144, doi: 10.2206/kyushujm.64.81.
- [18] J. Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics, 6. Amer. Math. Soc., Providence, RI, 1996.
- [19] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics 155, Springer, New York, 1995.
- [20] P.P. Kulish and E.K. Sklyanin, Solutions of the Yang-Baxter equations, *J. Sov. Math.* **19** (1982), 1596–1620.
- [21] P.P. Kulish and N. Yu. Reshetikhin, Quantum linear problem for the Sine-Gordon equation and higher representations, *J. Sov. Math.* **23** (1983), 2435–2441.
- [22] K. Miki, Finite dimensional modules for the q -tetrahedron algebra, *Osaka J. Math.* **47** (2010), 559–589.
- [23] P. Terwilliger, Two relations that generalize the q -Serre relations and the Dolan-Grady relations, *Physics and combinatorics 1999 (Nagoya)*, 377–398, World Sci. Publ., River Edge, NJ, 2001.
- [24] P. Terwilliger, The equitable presentation for the quantum group $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody algebra $\mathfrak{g}$, *J. Algebra* **298** (2006), 302–319, arXiv:math/0507478, doi: 10.1016/j.jalgebra.2005.11.013.