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Examples of computer experimentation in algebraic combinatorics

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Abstract

We introduce certain paradigms for procuring computer-free explanations from data acquired via computer algebra experimentation. Our established context is the field of algebraic combinatorics, with special focus on coherent configurations and association schemes. All results presented here were obtained by the authors with the aid of computer algebra systems, especially COCO and GAP. A number of examples are explored, in particular of objects on 28, 50, 63, and 210 points. In a few cases, initial experimental data pointed to appropriate theoretical generalizations that yielded an infinite class of related combinatorial structures. Special attention is paid to algebraic automorphisms (of a coherent algebra), a fairly new concept that has already proved to have far-reaching consequences. Finally, we focus on the Doyle-Holt graph on 27 vertices, and some of its related structures.

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1 Introduction

Coherent configurations, and in particular association schemes, form one of the central concepts in algebraic combinatorics, as defined in the seminal book of E. Bannai and T. Ito [3]. The aim of this paper is to elaborate to a wide mathematical audience with interdisciplinary interests how computer algebra tools can be applied effectively for the enumeration and investigation of various structures that arise in algebraic combinatorics.

Special attention is paid to the spirit of computer-aided activity, specifically to computer packages and their algorithmic tools, to technical data, successful *ad hoc* tricks, and computer-free interpretations of obtained results. Our methodology can be traced to traditions developed by the Moscow school, a group led by V.L. Arlazarov and I.A. Faradžev during the latter part of the twentieth century. (See [25] and [26] for first early attempts to apply this methodology in a reasonably systematic manner.)

Our paper is intentionally written in the genre of a digest, amalgamating a number of our previous publications with reports on several works in progress. Our aim here is to expose the reader to numerous prototypes through the use of striking and relatively self-contained examples, which we supplement with detailed references. Each such example has been chosen to illustrate a particular facet of computer-aided manipulation with combinatorial and algebraic objects, as well as subsequent theoretical computer-free interpretation of the achieved results.

An additional goal, by no means secondary, is to share with the reader a sense of the pleasure and fulfillment we experienced in transforming routine computer output into suitable combinatorial models. Indeed, it is this procedure that sheds light on the connection between structural properties of the investigated objects and their various abstract algebraic manifestations.

Before beginning, we detail our vision of the principal objectives of computer experimentation:

- constructing new objects (distance regular graphs, incidence structures, association schemes);
- exhaustively enumerating such objects with given parameters;
- understanding and formulating an explanation of their symmetry properties;
- formulating theoretical generalizations of such objects and their properties.

Finally, we stress that esthetic criteria play an essential role in our investigations.

2 Preliminaries

Let (G, Ω) be a permutation group. Interpreting each element of G as a permutation matrix X of order $n = |\Omega|$, we define

$$V(G, \Omega) := \{A \in M^{n \times n}(\mathbb{C}) \mid AX = XA \text{ for all } X \in G\},$$

where $M^{n\times n}(\mathbb{C})$ denotes the algebra of all matrices of order $n = |\Omega|$ over $\mathbb{C}$. It is easy to see that $V(G,\Omega)$ is a matrix algebra having a standard basis consisting of $(0,1)$-matrices. In addition, it contains the unit matrix I, the all-ones matrix J, and is closed with respect to complex conjugation and Schur-Hadamard (elementwise) multiplication. We call $V(G,\Omega)$ the *centralizer algebra* of (G,Ω).

The centralizer algebra has a nice relational formulation in terms of binary relations (arc sets of graphs) that are invariant with respect to (G,Ω). An axiomatic formulation of the most significant properties of centralizer algebras leads to the definition of *coherent algebras* (*coherent configurations*, in relational terminology), which is one of the central concepts in algebraic combinatorics. A particular case of coherent configurations, called association schemes, arises in this manner when (G,Ω) is assumed to be transitive.

A most important class of association schemes is comprised of those which have the form $(\Omega, 2\text{-orb}(G,\Omega))$ where $2\text{-orb}(G,\Omega)$ is the set of all 2-orbits of a suitable transitive permutation group (G,Ω) (that is, orbits of the induced group (G,Ω^2)). Such schemes are called *Schurian* [66, 26]. Schurian schemes may be described in purely group theoretic terms via the language of double cosets.

It is easy to see that the intersection of any number of coherent algebras is again a coherent algebra. Thus, for a given set M of matrices of order n there exists a unique smallest coherent algebra $\langle\langle M\rangle\rangle$ which contains M, commonly referred to as the *coherent closure* of M. Many significant coherent configurations and association schemes appear as coherent closures of certain matrices that represent graphs or incidence structures, see e.g. [36, 37]. Another robust source of such structures arises if we consider the coherent subalgebras of a given coherent algebra. Here an equivalent relational name is *merging configuration*, in particular *merging association scheme*, or simply *fusion*.

We assume a certain familiarity of the reader with the jargon in this area. We further mention that it is very convenient to be able to jump back and forth between the algebraic and relational languages of such objects. If, for example, $|\Omega| = n$, we may speak either of a coherent configuration on n points or a coherent algebra of order n. The number of basic relations is referred to as the *rank* of the configuration, which equals the dimension of the algebra as a vector space. In the case of an association scheme, we refer to the basic non-reflexive relations as *classes*, in which case an association scheme of rank $d+1$ has d classes. A *fiber* of a coherent configuration is the combinatorial analogue of an orbit of a permutation group.

Metric association schemes with d classes are canonically generated by distance regular graphs (DRGs) of diameter d. The seminal book [14] is an encyclopedic source of information about such objects. Distance regular graphs of diameter 2 are called strongly regular graphs (SRGs). Interest in SRGs stems from various links between algebraic combinatorics and such diverse areas as permutation groups, design of statistical experiments, coding theory, and finite geometries.

To each coherent configuration $\mathfrak{M}$ we may attribute a number of groups: $\mathrm{Aut}(\mathfrak{M})$, which consists of usual automorphisms, $\mathrm{CAut}(\mathfrak{M})$, consisting of color automorphisms, and $\mathrm{AAut}(\mathfrak{M})$, comprised of algebraic automorphisms, see e.g. [45]. Note that $\mathrm{CAut}(\mathfrak{M})/\mathrm{Aut}(\mathfrak{M})$ embeds canonically in $\mathrm{AAut}(\mathfrak{M})$. Those elements of $\mathrm{AAut}(\mathfrak{M})$ that do not arise via this embedding are called *proper algebraic automorphisms* of $\mathfrak{M}$. (See also [45], for details on the concepts of algebraic twins and algebraic fusion.) The full significance of proper algebraic automorphisms became obvious only recently by virtue of a number of computer algebra experiments fulfilled by the authors.

We refer the reader to [35, 3, 14, 26, 45] as sources of additional information about all combinatorial structures we consider in this paper.

3 Computer tools

Nowadays, the computer is an extremely important tool in algebraic combinatorics. We use it in order to enumerate all combinatorial objects with prescribed properties, to identify those objects up to isomorphism, to describe the automorphism groups of such objects, and to investigate various algebraic properties of the obtained groups. In this manner, we obtain additional knowledge about the structure of a given combinatorial object.

Below, we briefly discuss the most significant computer packages we use.

3.1 COCO

COCO is a system of programs designed to deal with coherent configurations. It was developed in 1990-2, Moscow, USSR, mainly by Faradžev and Klin [25, 26]. A UNIX version, developed by A.E. Brouwer, is available from Brouwer's homepage [12].

The COCO system includes:

ind: a program for calculating induced action of a permutation group on a combinatorial structure;

cgr: a program to calculate the centralizer algebra of a permutation group;

inm: a program to calculate the intersection numbers (also known as structure constants) of a coherent configuration;

sub: a program to find fusions (*aka* merging association schemes) of a coherent configuration given by its structure constants;

aut: a program to calculate the automorphism group of a coherent configuration, as well as the automorphism groups of its fusions.

Usually, these programs are fulfilled in the above order. This provides a computerized way to find all association schemes invariant under a given permutation group, plus their automorphism groups.

One of the significant methodological advantages of COCO is related to the program **ind**. Suppose we are given a permutation group (G, X), and assume $\mathcal{S}$ is a combinatorial structure defined on the subset $X' \subseteq X$. Let H be the stabilizer in (G, X) of $\mathcal{S}$ (in most simple cases, one has $H = Aut(\mathcal{S})$). Let Ω be the set of cosets of H in G. Then (G, Ω) is isomorphic to $(G, \widetilde{\Omega})$, where $\widetilde{\Omega}$ is the set of all images of $\mathcal{S}$ obtained with the aid of permutations from G. Typically, we do not distinguish between Ω and $\widetilde{\Omega}$ in our considerations.

3.2 WL-stabilization

The polynomial time Weisfeiler-Leman algorithm for the computation of the coherent closure of a given set of matrices (briefly WL-stabilization) was suggested in [70, 69]. The first efficient implementations of this algorithm were presented in [2].

In exceptional cases, such as when the order or rank of a coherent closure turns out to be too large, it may be more efficient to use certain *ad hoc* computational tricks, for example those based on the Schur-Wielandt principle, see [52]. The use of such tricks, in conjunction with theoretically obtained bounds for the rank, may allow one to reach the desired closure after only a few iterations.

Note that coherent closure typically applies to sets of matrices, however we may equally well apply it to the corresponding sets of graphs (i.e., relations). In such case, our output is a coherent configuration as opposed to a coherent algebra.

3.3 GAP

GAP [28, 65] is an acronym for "Groups, Algorithms and Programming." It is a system for computation in discrete abstract algebra. The system supports easy addition of extensions ("packages" in GAP nomenclature) that are written in the GAP programming language, and thus can add new features to the GAP system.

One such package that is very useful in algebraic combinatorics is GRAPE [67]. It is designed for the construction and analysis of finite graphs. GRAPE itself is dependent on an external program, nauty [60], which is used to calculate the automorphism group of a graph.

In the course of investigations in algebraic combinatorics, one uses GAP to:

- construct incidence structures (graphs, block designs, geometries, coherent configurations, etc);
- calculate automorphism groups of such structures;
- check regularity properties and parameters of structures;
- find cliques in graphs, and substructures of given structures in general;
- find the abstract structure of a group, as well as identifying it as a permutation group;
- find conjugacy classes and subgroups of a group.

3.4 DISCRETA

The package DISCRETA was created in Bayreuth, see [7]. As a rule, the main function of DISCRETA is to obtain computer-aided proof of the existence of new t-designs with a prescribed set of parameters. The input of DISCRETA consists of such a prescribed parameter set, together with a permutation group (H, Ω). The output is the complete set of all t-designs that have said parameters, and are invariant with respect to (H, Ω).

3.5 COCO v.2

The COCO v.2 initiative aims to reimplement the algorithms in COCO, together with other packages such as WL-stabilization and DISCRETA, as a GAP package. In addition, a number of new functions are being developed that are based on new theoretical results obtained since the original COCO package had been written. Kernel steps in this development have been fulfilled by the author SR. A number of his colleagues continue to contribute to this activity.

4 From computer experiment to computer-free interpretation

New striking examples are the main goal of each round of computer algebra experimentation. As a rule, an initial description of an object or property being investigated is limited solely to available computer output. From here, one performs *a posteriori* reasoning in an effort to obtain descriptions of greater clarity and simplicity. We distinguish between two such levels of description as follows.

Suppose we obtain a computer-generated description of, say, an incidence structure $\mathfrak{S} = (P, B)$. By an *explanation* of $\mathfrak{S}$, we mean a lucid computer-free description of P, B, and the incidence between them. Essential use of a computer, or of additional hand calculations, is not required in this case.

By an *interpretation* of $\mathfrak{S}$, we mean that in addition to an explanation we have a self-contained proof that $\mathfrak{S}$ indeed has its purported structure or properties. Ideally, an interpretation should be reasonably short and methodologically clear.

The following two examples are chosen to illustrate these notions. In each case, the relevant incidence structure is an SRG.

Example 4.1. We recall an investigation initiated by I.A. Faradžev (e.g., see [25]). Consider the intransitive action of $G = \mathrm{PGL}(3, 3)$ on the set $\Omega = \Omega_1 \cup \Omega_2$ of cardinality 247, where Ω_1 is the point set of the projective plane Π of order 3, and Ω_2 is the set of all ovals in Π.

With the aid of COCO, it was determined that (G, Ω) has rank 26. Additionally, COCO revealed an SRG Γ with parameter set $(247, 54, 21, 9)$ as a fusion in the coherent configuration $(\Omega, 2\text{-}\mathrm{orb}(G, \Omega))$. As this parameter set is pseudo-geometric (more explicitly, corresponding to that of the block graph of a Steiner triple system $\mathrm{STS}(39)$), it is natural to ask whether or not it is geometric, that is, isomorphic to the point graph of a suitable partial geometry.

This problem remained open for several years until SR, while testing his new program for the construction of partial geometries with a given point graph, reexamined Γ. The program confirmed that Γ is indeed geometric. Below we give an interpretation as suggested in [63]. (Note that we use language dual to that in [25].)

Start with the projective plane $\Pi = (\Omega_1, \mathcal{L})$ of order 3. Consider a new incidence structure $\mathfrak{I} = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ consists of 39 partitions of a suitable line in $\mathcal{L}$ to two point subsets of size 2. Define $\mathcal{B} = \mathcal{L} \cup S$, where S consists of all quadrangles (dual ovals) in Π. There are 234 quadrangles, each consisting of four lines, no three of which are concurrent. Pairs of lines in a given quadrangle Q intersect in six distinct points which we call *intersection points*. There are three additional lines which join two intersection points of Q, not collinear in Q. Each such additional line may be partitioned into two parts: its two intersection points and two remaining points. Such a partition forms an element of $\mathcal{P}$. This partition is incident to quadrangle Q in the above consideration. Also each line in $\mathcal{L}$ is incident to its three partitions in $\mathcal{P}$.

A short proof that $\mathfrak{I}$ is indeed an $\mathrm{STS}(39)$ is presented in [63]. □

The distinction between explanation and interpretation can be both subtle and subjective, cf. [50]. Typically the word explanation is used. One of the most fruitful approaches to elaborate a reasonably good explanation is to start from a suitable auxiliary structure – the plane Π of order 3 was used in this role in the above example. We refer to [32, 33], where a few helpful explanations of graphs with 36 and 196 vertices allowed us to describe a new infinite series of proper loops of order $2p$, p a prime, having a regular collineation group of order $4p^2$ (see also [44]).

During the last decade, a number of new interesting SRGs discovered with the aid of COCO were described by us in the literature, e.g. see [27, 42, 15, 49].

R_i	valency	g
R_5	24	$(0,1,3,4,5,6)$
R_7	24	$(0,3,4,5,6,2)$
R_9	12	$(0,1,4,3,5,6)$
R_{10}	24	$(0,3,1,4,5,6)$
R_{12}	6	$(0,2,4,3,5,6)$
R_{14}	8	$(0,3,2,4,5,1)$
R_{15}	1	$(0,2,4,3,5,1)$

Table 1: Representatives of 2-orbits merged to Γ

In the following example, we consider one such SRG which was the subject of a talk given by the author MK at the 1995 conference BCC15. This SRG has imprimitive automorphism group S_7 of degree 210 and the parameters $(210, 99, 48, 45)$. Most significantly, it is the first discovered graph with these parameters, see [13]. At present, a description of this graph does not appear in the literature. We now fill this gap, describing below the entire pattern of our methodology based on the use of COCO.

Example 4.2. We considered all transitive actions of the group S_7 of degree 210. As a first step, we described up to conjugacy all subgroups of order 24 in S_7. It turns out that there are 14 conjugacy classes of such subgroups. For each subgroup H, COCO was employed to construct the action of S_7 on the set Ω of cosets of H, compute the centralizer algebra of each corresponding permutation group (S_7, Ω), and enumerate all fusions of $(\Omega, 2\text{-orb}(S_7, \Omega))$. Our goal was to produce a primitive rank 3 association scheme. For all classes but one the result was negative.

Let us now direct our attention to a particular choice of H, namely $H = \langle(0, 1, 2, 3, 4, 5), (0, 3)\rangle$, which is isomorphic to both $\mathbb{Z}_3 \wr \mathbb{Z}_2$ and $\mathbb{Z}_2 \times A_4$. In the internal framework of COCO, it is convenient to represent H as the automorphism group of a directed graph Σ with vertex set $[0, 6]$ and two connected components: the isolated vertex 6, and the Cayley graph $\Delta = \text{Cay}(\langle g_1\rangle, \{1, 4\})$, where $g_1 = (0, 1, 2, 3, 4, 5)$. (Note that here we are using COCO's literal coding for a circulant graph: $\{1, 4\}$ replaces $\{g_1, g_1^4\}$ for the connection set of the Cayley graph Δ over the cyclic group $\langle g_1\rangle$ of order 6.) Then we may consider a new transitive permutation group $(S_7, \widetilde{\Omega})$, where $\widetilde{\Omega}$ is the orbit of Σ with respect to S_7. Identifying $\widetilde{\Omega}$ with Ω, COCO returns that the group (S_7, Ω) has rank 16 with subdegrees $1^2, 6^2, 8^2, 12^5, 24^5$.

There are 14 fusions, two of which are primitive. Both primitive fusions are non-Schurian schemes with 2 classes, which define two isomorphic strongly regular graphs with the parameters $(210, 99, 48, 45)$.

We describe one of these graphs Γ as an explicit fusion of certain 2-orbits of (S_7, Ω). Each such 2-orbit R_i is represented with the aid of a representative $(0, r_i)$, where 0 is notation for the canonical graph Σ and $r_i \in \Omega$ is a label of an element of Ω. Note that each r_i is nothing more than a suitable isomorphic copy of the Cayley graph Δ, together with an isolated vertex. To describe this copy, it suffices to indicate a cycle g which replaces g_1 in our definition of Δ. All required information is presented in Table 1. □

Remark: Our description of graph Γ in Example 4.2 is largely predicated on the form of available computer data, and falls short of even being an explanation in our terminology, as it would be difficult to construct this graph via hand computations. In fact, some nice geometrical ideas were expressed a long time ago by G. Jones and K. Lloyd as to how to produce an interpretation of the constructed graph Γ. We intend to realize such an interpretation in the context of a future project.

The construction of new examples of SRGs was one of our original goals, even prior to conception of the computer package COCO.

The paper [26] contains detailed information about a number of SRGs constructed by MK, in particular ones on 120, 126, 330, 495, 1716 vertices invariant with respect to a suitable action of the symmetric group S_n ($n = 10, 9, 11, 12, 13$, respectively). It seems that some interesting links between these graphs still require special clarification. Another family of striking examples was described in [41], see also [26]. These are graphs on 144, 280, 280, 280, 560 vertices with primitive automorphism groups $\mathrm{PSL}(3, 3)$, $\mathrm{Aut}(J_2)$, S_9, $\mathrm{Aut}(J_2)$, $\mathrm{Aut}(Sz(8))$, respectively. The graph on 280 vertices with group S_9 was also independently discovered by R. Mathon and A. Rosa (see [59]).

All these results were originally presented in preprints in Russian, not readily accessible to a Western audience. As a consequence, proper attribution of authorship was never indicated in catalogues of known SRGs. This is why we take this opportunity to recall once more our old discoveries.

5 Total graph coherent configurations

Let $\Sigma = (V, E)$ be a graph. The total graph $T(\Sigma)$ is the graph with vertex set $V \cup E$, in which two vertices are adjacent in $T(\Sigma)$ if they are either adjacent or incident in graph Σ. (Edges of Σ are said to be adjacent if they have a common vertex.) The concept of a total graph was suggested in [6], see also [5].

Observe that in some exceptional cases the total graph $T(\Sigma)$ may have a more robust automorphism group than the original graph Σ. For example, for $n \geq 2$ one has $\mathrm{Aut}(K_n) = S_n$ and $\mathrm{Aut}(T(K_n)) \cong S_{n+1}$, see [6].

We call the coherent closure $W(T(\Sigma))$ of the total graph $T(\Sigma)$ the *total graph coherent configuration* of Σ. We also wish to consider the Schurian coherent closure $S(T(\Sigma))$ of $T(\Sigma)$. Here $S(T(\Sigma)) = (\Omega, \text{2-orb}(G, \Omega))$, where Ω is the vertex set of $T(\Sigma)$ and $G = \mathrm{Aut}(T(\Sigma))$. It is clear that in general $W(T(\Sigma))$ is a suitable subalgebra (merging, in relational language) of $S(T(\Sigma))$. In our eyes, the question as to when $S(T(\Sigma))$ coincides with $W(T(\Sigma))$ is one of significant theoretical interest. This question was considered in [53, 74] for two classes of classical strongly regular graphs, namely the triangular graphs T_n and the lattice square graphs $L_2(n)$. For both classes of graphs it was proved that $W(T(\Sigma)) = S(T(\Sigma))$.

Another motivation of our interest in total graph configurations is of a more concrete nature. In the course of proving $W(T(T_n)) = S(T(T_n))$, we investigated all association schemes which appear as mergings of $\mathfrak{I}(n) = W(T(T_n))$. It turns out that for sufficiently large n, the configuration $\mathfrak{I}(n)$ has just two easily predictable imprimitive mergings of respective ranks 3 and 4. Pleasant surprises appear only for $n = 5, 7$.

In the case $n = 5$, we get a very interesting Schurian rank 5 association scheme of order 40 with automorphism group of order 1920, see [47]. For $n = 7$, we get a nontrivial merging which is of an independent interest. It corresponds to an embedding of the symmetric group S_7 in the group $\mathrm{U}(4, 3).2^2$ of order 13063680, and provides a new model for the unique Zara graph on 126 vertices. This case will be considered in a forthcoming paper by MK and MZ-A, along with L. Jørgensen.

Remark: The motivation behind the introduction of WL-stabilization in [70] was its apparent link to the graph isomorphism problem. Indeed, the coherent closure $\langle\langle\Gamma\rangle\rangle$ of a given graph Γ may serve as a source of various algebraic invariants of Γ. Moreover, $\langle\langle\Gamma\rangle\rangle$ may be computed in polynomial time on the number of vertices of Γ. This is why the notion of coherent algebra is especially useful in the theory of complexity of algorithms, e.g. see [52, 24].

A new attempt to solve the graph isomorphism problem was initiated in [64]. The invariants formulated in that paper seemed to indicate that the total graph coherent configuration would play a promising role in systematically identifying the major difficulties of the problem. However, the paper [4] appears to have suppressed part of such hopes.

6 Coherent configurations and the Hoffman-Singleton graph

It is well known (e.g., see [18]) that if a Moore graph of diameter 2 has valency k then $k \in \{2, 3, 7, 57\}$. The unique examples with $k = 2$ and $k = 3$ are the pentagon and the Petersen graph, respectively. Below we consider from diverse points of view the unique Moore graph of valency 7 on 50 vertices, that is, the Hoffman-Singleton graph $HoSi$ [39]. The existence of a Moore graph of valency 57 is an open problem.

Our first goal is to provide a new model of $HoSi$. First we need an auxiliary proposition, cf. [18].

Proposition 6.1. *There are six distinct, pairwise isomorphic, 1-factorizations of the graph K_6. Each of these has automorphism group S_5, acting 3-transitively on six points.*

In what follows we consider a representative 1-factorization $\mathcal{F}$ of K_6 with vertex set $[0, 5]$, namely

$$\mathcal{F} = \{\{\{0,1\},\{2,4\},\{3,5\}\},\{\{0,2\},\{1,5\},\{3,4\}\},\{\{0,3\},\{1,2\},\{4,5\}\},\\ \{\{0,4\},\{1,3\},\{2,5\}\},\{\{0,5\},\{1,4\},\{2,3\}\}\}.$$

It is convenient to regard the considered copy of K_6 as a subgraph of K_7 with isolated vertex 6.

Example 6.2. Let $\Omega_1 = \{\emptyset\}$, $\Omega_2 = [0, 6]$ and $\Omega_3 = \mathcal{F}^{S_7}$, where $\mathcal{F}^{S_7}$ is the orbit of $\mathcal{F}$ under action of $S_7 = \mathrm{Aut}(K_7)$. Denote $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. Clearly, the symmetric group $S_7 = S([0, 6])$ acts naturally on Ω with orbits Ω_1, Ω_2, Ω_3. Thus we may consider the coherent configuration $\mathcal{H} = (\Omega, 2\text{-orb}(S_7, \Omega))$. Using COCO, we obtain that:

a) $\mathcal{H}$ is a rank 15 configuration with three fibers of size $1, 7, 42$. Its type is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{pmatrix}$ with valencies $\begin{pmatrix} 1 & 7 & 42 \\ 1 & (1,6) & (36,3) \\ 1 & (6,1) & (1,30,5,6) \end{pmatrix}$.

b) Merging of the relations $1, 3, 7, 10, 14$ of respective valencies $7, 1, 6, 1, 6$ provides a copy of the graph $HoSi$. □

In (a) above, the labeling of relations is lexicographic: first down columns, then across rows. For example, the second column in the matrix of valencies indicates that the relations R_3, R_4, R_5, R_6, R_7 have respective valencies $7, 1, 6, 6, 1$.

We mention that we were able to achieve a rather nice computer-free interpretation of $HoSi$, however the exposition is quite lengthy. Two additional models of $HoSi$, which we briefly describe in Example 6.3 below, will also be presented in detail elsewhere. In a sense, these latter two models may be interpreted as information accumulated from diverse sources (in particular, due to N. Robertson, P. Hafner, L. O. James, G. Fan & A.J. Schwenk, R. H. Jeurissen, and others) transformed into the language of coherent configurations.

Example 6.3. *Model A:* Consider the group $D = D_5 \times \mathrm{AGL}(1,5)$ of order 200 acting intransitively on a set of cardinality 50. It defines a coherent configuration $\mathcal{X}_D$ of rank 29 with three fibers of size $5, 25, 20$. Two Schurian fusions of $\mathcal{X}_D$ correspond to the rank 3 association scheme coming from the graph $HoSi$. In fact, configuration $\mathcal{X}_D$ corresponds to the stabilizer of an arbitrary pentagon in $HoSi$.

Model B: Consider the stabilizer G in $\mathrm{Aut}(HoSi)$ of a Petersen subgraph P in $HoSi$. There are 525 copies of P in $HoSi$, all belonging to the same orbit of $\mathrm{Aut}(HoSi)$. We obtain that G is a group of order 480, specifically an extension of $\mathbb{Z}_4$ with the aid of S_5 (see [47] for details). We associate to G a coherent configuration $\mathcal{X}_G$ of rank 16 with two fibers of size 40 and 10. Configuration $\mathcal{X}_G$ has a unique rank 3 fusion which corresponds to $HoSi$. □

We conclude this section with an example which shows how the notion of total graph coherent configuration may be applied to shed additional light on the Hoffman-Singleton graph.

Example 6.4. Consider the complement graph $\overline{HoSi}$ of $HoSi$, and let $T(\overline{HoSi})$ denote its total graph. With the aid of COCO we obtained the total graph coherent configuration, and detected in it a very interesting primitive fusion with four classes on 1100 points. As the automorphism group of $T(\overline{HoSi})$ coincides with the automorphism group $HS:2$ of the Higman-Sims graph, this fusion provides new insight into the classical embedding of $\mathrm{Aut}(HoSi)$ into $HS{:}2$. Again, details will appear elsewhere. □

Remarks:

1. The second constituent of $HoSi$ is a distance transitive graph of valency 6 on 42 vertices which is an antipodal cover of K_7, see [14]. One of the additional functions of Example 6.2 is its direct construction in terms of 42 1-factorizations of K_6 inside of K_7.

2. It is well known that if there exists a Moore graph of valency 57, then it cannot have a transitive automorphism group (e.g., see [55], and the references therein). Thus, the resulting association scheme may arise as a fusion within a suitable coherent configuration with at least two fibers. In this context, the coherent configurations of Example 6.3 may provide an excellent training ground for deciding which structures would be promising initial candidates from which a Moore graph on 3250 vertices may be constructed.

3. Example 6.4 illustrates one of the most successful paradigms in algebraic combinatorics for attaining a better understanding of a given combinatorial structure $\mathfrak{I}$. Namely, we embed $\mathfrak{I}$ into a larger structure $\widehat{\mathfrak{I}}$ for which $\mathrm{Aut}(\widehat{\mathfrak{I}})$ is a proper overgroup of $\mathrm{Aut}(\mathfrak{I})$. Here the roles of $\mathfrak{I}$ and $\widehat{\mathfrak{I}}$ were played by $\overline{HoSi}$ and $T(\overline{HoSi})$, respectively, and the resulting overgroup turned out to be the automorphism group of the Higman-Sims graph. Speculating once more on possible ways to attempt a construction of a Moore graph Γ of valency 57, one may expect that it is possible to embed $\overline{\Gamma}$ into a suitable merging of $W(T(\overline{\Gamma}))$ which has larger group than $\mathrm{Aut}(\Gamma)$.

7 Some association schemes on 28 points

In [45], a number of interesting combinatorial objects on 28 points were investigated. Starting with the regular group E_8, the authors considered its induced intransitive action on the set $\Omega = \left\{ {[0,7] \atop 2} \right\}$ of all 2-element subsets of $[0,7]$. It turns out that the corresponding coherent configuration W provides an example of a so-called Wallis-Fon-Der-Flaas (or briefly WFDF) coherent configuration. In terms of W, we were able to give a new uniform interpretation of a number of classical objects on 28 points, as well as other association schemes that had formerly been presented only at the level of strict computer output. Below we provide a representative case, in which we also add new detail to the presentation given in [45].

Example 7.1. Recall from Section 2 the notion of a proper algebraic automorphism. We call two association schemes $\mathcal{A}$, $\mathcal{A}'$ *twins* with respect to the coherent configuration W if they are fusions in W, and there exists a proper $\rho \in \mathrm{AAut}(W)$ for which $\mathcal{A}^\rho = \mathcal{A}'$. One of many pairs of such twins found in [45] is described presently. Here the role of W is fulfilled by the WFDF configuration described above.

Preserving the notation of [45], #109 and #110 form a pair of twins. In fact, #109 and #110 are not combinatorially isomorphic. Indeed, scheme #109 is Schurian with automorphism group isomorphic to $\mathrm{A\Gamma L}(1,8)$, while #110 is non-Schurian with group H of index 7 inside $\mathrm{A\Gamma L}(1,8)$. As an abstract group, H is isomorphic to the group considered in Example 4.2, however in our current context it behooves us to represent this group in terms of the set $[0,7]$. For this purpose, an old construction due to L. E. Dickson and F. H. Safford [20] becomes quite relevant. Namely, we consider a copy of the 3-cube Q_3 with vertex set $V = [0,7]$, and having 12 edges as follows: $\{0,2\}$, $\{0,6\}$, $\{0,7\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{3,7\}$, $\{4,6\}$, $\{5,6\}$, $\{5,7\}$.

We now produce seven disjoint 1-factors of K_8. The first of these is the set of space diagonals of Q_3: $d = \{\{0,1\},\{2,5\},\{3,6\},\{4,7\}\}$. Our six remaining 1-factors are:

$$\begin{aligned} a &= \{\{0,4\},\{2,6\},\{1,5\},\{3,4\}\}, & b &= \{\{0,7\},\{2,3\},\{1,6\},\{4,5\}\},\\ c &= \{\{0,2\},\{1,7\},\{3,5\},\{4,6\}\}, & e &= \{\{0,6\},\{1,2\},\{3,4\},\{5,7\}\},\\ f &= \{\{0,5\},\{1,3\},\{2,4\},\{6,7\}\}, & g &= \{\{0,3\},\{1,4\},\{2,7\},\{5,6\}\}. \end{aligned}$$

It is easy to check by hand that

$$H = \langle (1,2,6)(3,0,5), (2,6,7)(3,4,5), (1,3)(2,4)(5,7)(0,6) \rangle$$

is the full automorphism group of this 1-factorization, here denoted by $\mathcal{F}$.

The advantage of our model is that each element of Ω appears exactly once in $\mathcal{F}$. Thus the induced action of $H = \mathrm{Aut}(\mathcal{F})$ on Ω, as well as its 2-orbits, are quite visible in terms of $\mathcal{F}$. In fact, H has rank 38 with three orbits on Ω of lengths 12, 12 and 4. These orbits correspond to unordered pairs of vertices of respective distance 1, 2 and 3 in the graph Q_3. The coherent configuration $\mathfrak{M} = (\Omega, \text{2-orb}(H,\Omega))$ has very interesting properties. Indeed, $\mathrm{CAut}(\mathfrak{M})/\mathrm{Aut}(\mathfrak{M}) \cong \mathbb{Z}_2 \times D_4$ is a group of order 16, while $\mathrm{AAut}(\mathfrak{M}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times D_4$ has order 32. Thus $\mathfrak{M}$ has many algebraic fusions as well as twins. In particular, the above schemes #109 and #110 appear as twins in $\mathfrak{M}$. □

Remark: As discussed in [45], a striking feature of WFDF configurations is that they tend to have a large number of proper algebraic automorphisms. Moreover, it is rare to find this feature in coherent configurations of non-WFDF type. From this point of view, configuration $\mathfrak{M}$ is of definite independent interest.

8 The simple group of order 504

The simple group $L = \mathrm{PSL}(2, 8)$ of order 504 has many intriguing links to diverse combinatorial and geometrical structures. Here we mention only a few: overlarge sets of Fano planes, overlarge sets of affine designs $S(3, 4, 8)$, partial geometries $\mathrm{pg}(8, 9, 4)$, partial geometries $\mathrm{pg}(5, 7, 3)$, a number of coherent configurations and association schemes. All these structures will be described in a unified manner in [51], based on a lecture given by the author AW at the 2004 conference ICIG, Belgium.

As usual, our initial investigations made extensive use of computer algebra tools, followed by computer-free interpretations in most cases. We present only a small fragment of these *sans* details.

Example 8.1. Denote by N the group $\mathrm{Aut}(L) \cong \mathrm{P\Gamma L}(2, 8)$ of order 1512. We consider the natural 3-transitive action of N on the points of $\Omega = \mathrm{PG}(1, 8)$. Group N is not geometric in the sense of D. Betten [8], that is, one cannot realize N as the full automorphism group of a suitable incidence structure with $\mathrm{PG}(1, 8)$ as point set. For this reason, we first show that N is geometrical of second order, in other words N is the automorphism group of a suitable set of incidence structures, namely in our case an overlarge set $\mathcal{O}$ of affine designs (cf. [11]).

Starting with the set $\mathcal{O}$, we provide a interpretation of one of two partial geometries $\mathrm{pg}(5, 7, 3)$ discovered by R. Mathon [58]. For this purpose, we define an incidence structure $\mathfrak{M} = (\mathcal{P}, \mathcal{L})$ with point set $\mathcal{P} = \Omega \cup \binom{\Omega}{2}$ (i.e., all singletons and unordered pairs of elements from Ω) and line set $\mathcal{L}$ consisting of all partitions of Ω into four classes of size 2 and one class of size 1 which satisfy a special requirement formulated in terms of $\mathcal{O}$. Finally, incidence is ordinary containment. In this manner, we obtain an incidence structure $\mathfrak{M}$ with 45 points and 63 lines which we were able to prove is a $\mathrm{pg}(5, 7, 3)$. Moreover, we showed that $\mathrm{Aut}(\mathfrak{M}) = \mathrm{Aut}(\mathcal{O}) \cong \mathrm{P\Gamma L}(2, 8)$.

An alternative interpretation may be arranged in terms of the set X of 63 cosets of the subgroup $H = \mathbb{Z}_2 \times A_4$ in N. The association scheme $\mathcal{H} = (X, 2\text{-orb}(N, X))$ has rank 6 with subdegrees $1, 3, 3, 8, 24, 24$. All 2-orbits have a natural geometrical explanation. It turns out that $\mathrm{CAut}(\mathcal{H}) = \mathrm{Aut}(\mathcal{H}) = N$, while $\mathrm{AAut}(\mathcal{H})$ has order 2. The unique proper algebraic automorphism defines two pairs of twins: two non-Schurian association schemes with 3 classes and two strongly regular graphs with the parameters $(63, 30, 13, 15)$. One SRG is a rank 3 graph, while the second is non-Schurian with automorphism group N. The latter graph turns out to be the block graph of the constructed geometry $\mathrm{pg}(5, 7, 3)$.

Remark: A nice feature of our presentation is that we are able to consider in parallel two non-isomorphic overlarge sets in a unified manner. From each we obtain a $\mathrm{pg}(5, 7, 3)$. The second geometry has smaller automorphism group, specifically $\mathrm{ASL}(2, 3)$ of order 216.

9 Doyle-Holt graph and related structures

We come now to our final example of computer experimentation in service of algebraic combinatorics. Our investigation begins with the Doyle-Holt graph (*aka* Doyle graph, *aka*

Holt graph), which has an interesting history. In [68] W. Tutte posed a question about the existence of a graph that is vertex-transitive, edge-transitive but not arc-transitive. The first actual examples were provided by I.Z. Bouwer in [10], specifically an infinite series of such graphs of valency $2n$, $n \geq 2$, with smallest member being on 54 vertices. In 1981, D.F. Holt [40] provided an example on 27 vertices, however it was later discovered that Holt's example had appeared earlier in the 1976 senior thesis of P.G. Doyle, e.g., see [21]. In [73] M.Y. Xu proved that the Doyle-Holt graph is the unique graph on 27 vertices having the requested properties, while in [1] it was established that no graph on fewer vertices could be of such type.

It turns out that the Doyle-Holt graph is related to many other diverse combinatorial structures such as the Gray configuration, generalized quadrangle on 27 points, generalized octagons on 80 and 160 points, etc. In what follows we provide a fresh context for this famous graph, based on strict use of group theoretic arguments framed in the language of association schemes. We start with a description of its automorphism group.

9.1 Automorphism group

In [40], Holt computed the automorphism group H by hand, divulging that it has order 54 and possesses a regular nonabelian subgroup. In fact, group H is abstractly the holomorph $\mathrm{Hol}(\mathbb{Z}_9)$, and one concrete realization of it is given by $H = \langle g_1, g_2 \rangle$ where $g_1 = (0,1,2,3,4,5,6,7,8)$ and $g_2 = (1,2,4,8,7,5)(3,6)$. Thus H is the full normalizer of $\langle g_1 \rangle$ in S_9.

Consider the three undirected 9-cycles with common vertex set $[0,8]$ given by consecutive vertices as follows:

$$C_{9,1}: \ 012345678 \qquad C_{9,2}: \ 024681357 \qquad C_{9,3}: \ 048372615$$

It is immediate that $\mathrm{Aut}(C_{9,i}) = \langle g_1, g_2^3 \rangle \cong D_9$ for all i, while g_2^2 permutes $C_{9,1}$, $C_{9,2}$, $C_{9,3}$ cyclically.

Form the sum graph $\Sigma = C_{9,1} + C_{9,2} + C_{9,3}$, that is $V(\Sigma) = [0,8]$ and $E(\Sigma) = E(C_{9,1}) \cup E(C_{9,2}) \cup E(C_{9,3})$. Note that H acts transitively on vertices, edges and arcs of Σ, and that $\overline{\Sigma} = 3 \circ K_3$. Denote by Ω and $\overline{\Omega}$ the edge sets of graphs Σ, $\overline{\Sigma}$ respectively.

9.2 Induced action and association scheme

Consider now the induced action of H on $\Omega = E(\Sigma)$, as well as the resulting association scheme $\mathfrak{X} = (\Omega, 2\text{-orb}(H, \Omega))$. The following facts about $\mathfrak{X}$ are easily obtained through the aid of COCO.

Proposition 9.1. (i) $\mathfrak{X}$ *has 14 classes, exactly four of which are symmetric;*
(ii) (H, Ω) *is 2-closed;*
(iii) *the color group of* $\mathfrak{X}$ *has order 162;*
(iv) $\mathfrak{X}$ *has exactly 49 fusion schemes;*
(v) *three of its fusions are isomorphic, and correspond to a strongly regular graph with parameters* $(\nu, k, \lambda, \mu) = (27, 10, 1, 5)$ *with group of order 51840;*
(vi) *among its 17 rank 4 fusions, there are three isomorphic non-Schurian fusions with group of order 324, and three isomorphic Schurian fusions with group of order 1296;*
(vii) *all six rank 4 fusions from* (vi) (*both Schurian and non-Schurian*) *are metric association schemes generated by an antipodal DRG of valency 8.*

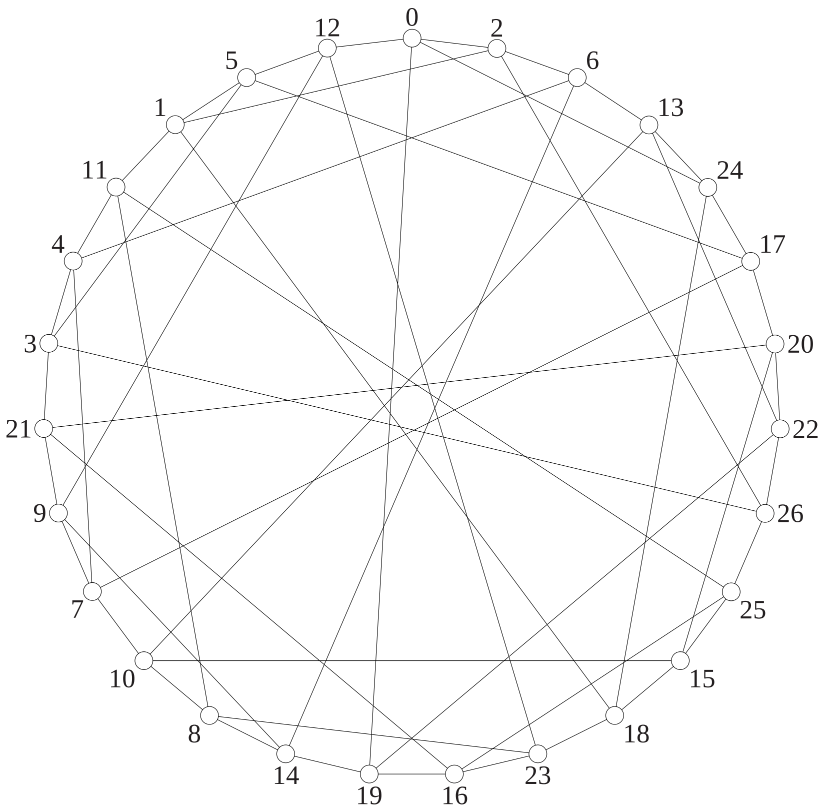

Figure 1: Factorization of Δ into two Hamiltonian cycles. Note: The second cycle is $(0, 24, 18, 1, 2, 26, 3, 5, 17, 7, 4, 6, 14, 9, 12, 23, 8, 11, 25, 16, 21, 20, 15, 10, 13, 22, 19)$.

9.3 Model of Doyle-Holt graph

Analyzing the lattice of fusions of $\mathfrak{X}$, one immediately identifies three copies of an undirected regular graph of valency 4 whose coherent closure coincides with $\mathfrak{X}$. Each such graph is obtained as the union of two paired non-symmetric (connected) 2-orbits of (H, Ω). This implies the existence of a graph on 27 vertices that is both vertex- and edge-transitive but not arc-transitive.

Such considerations led us to a new interpretative model of the Doyle-Holt graph Δ, as well as a computer-free independent confirmation that its automorphism group is H. (We alert the reader that $Aut(\Delta) = H$ is already a simple consequence of Proposition 9.1. Our text below is aimed at providing a nice alternative justification of this fact.)

Our graphical representation was greatly aided by certain bits of established information about the Doyle-Holt graph (e.g., see [71]), in particular, that it can be factorized into two disjoint Hamiltonian cycles each invariant under a cyclic subgroup of H of order 9, see Figure 1. In fact, there are exactly three such factorizations possible.

Let us write $\widetilde{x}$ to denote the induced action of $x \in H$ on the edge set $E = E(\Delta)$. The following result is easily proved.

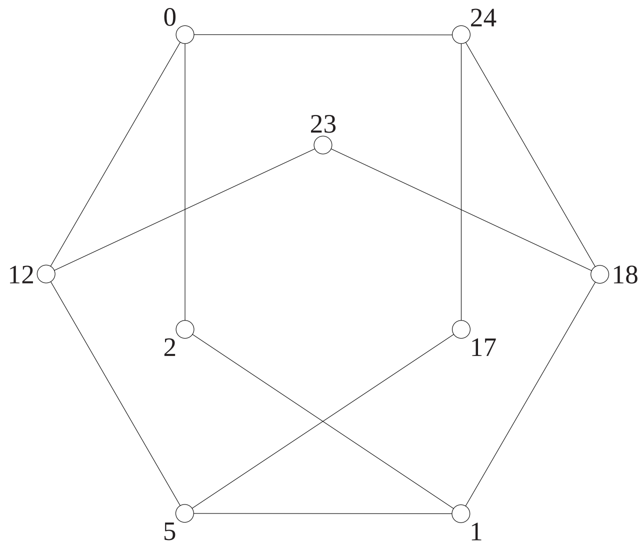

Figure 2: Three pairs of pentagons providing the same noble hexagon.

Proposition 9.2. (i) *The element $\widetilde{g_1} \in H$ of order 9 preserves both Hamiltonian cycles C_1, C_2 in a given factorization;*
(ii) $\langle \widetilde{g_1} \rangle$ *has six orbits on E, three comprising C_1 and the others C_2;*
(iii) *Element $\widetilde{g_2} \in H$ of order 6 cyclically permutes these six orbits on E;*
(iv) $D_9 := \langle \widetilde{g_1}, \widetilde{g_2}^3 \rangle$ *stabilizes the Hamiltonian factorization $\{C_1, C_2\}$;*
(v) $\langle \widetilde{g_2}^2 \rangle$ *acts regularly on the three Hamiltonian factorizations of Δ;*
(vi) $H = \langle \widetilde{g_1}, \widetilde{g_2} \rangle$ *is a subgroup of* $\mathrm{Aut}(\Delta)$.

The new proof that $|\,\mathrm{Aut}(\Delta)| = 54$ (and hence that $H = \mathrm{Aut}(\Delta)$) proceeds roughly as follows. Graph Δ is easily seen to contain 54 pentagons (use edge-transitivity and the fact that there are five pentagons on a fixed edge). Next observe that two distinct pentagons can intersect in at most two edges (since Δ has girth 5). Moreover, given a fixed pentagon P there is exactly one pentagon P' for which $|E(P) \cap E(P')| = 2$. Call P, P' *mates* if they have this property. Clearly, the symmetric difference of any pair of mates yields a hexagon, called by us a *noble hexagon*. It is not hard to see that every noble hexagon arises in this manner from exactly three distinct pairs of mates, e.g., see Figure 2.

Proposition 9.3. (i) *The edge set E of Δ partitions into nine noble hexagons;*
(ii) *this partition is invariant with respect to* $\mathrm{Aut}(\Delta)$ *and forms a single orbit under the action of H;*
(iii) $\mathrm{Aut}(\Delta)$ *acts faithfully on the set of noble hexagons;*
(iv) *the stabilizer in H of a noble hexagon is a cyclic group of order 6.*

So it remains only to show that the stabilzer in $\mathrm{Aut}(\Delta)$ of a noble hexagon has order 6. Clearly, the only other possibility is that this order is 12, which arises if the stabilizer is the full automorphism group of the hexagon, i.e., the dihedral group D_6. To refute this it suffices to exhibit a single reflection in a single noble hexagon that cannot be extended to an automorphism of Δ. This gives the following.

Corollary 9.4. *(i) Graph Δ has automorphism group H;*
(ii) Δ is vertex-transitive and edge-transitive, but not arc-transitive;
(iii) Δ is isomorphic to the Doyle-Holt graph.

Remarks:

1. Alternatively, one may prove Corollary 9.4 by WL-stabilization, as discussed in Section 3.2. Here, one inputs the graph Δ and obtains as output the association scheme $\mathfrak{X}$ introduced in Section 9.2. Scheme $\mathfrak{X}$ is quasithin in the sense of [38], whence it is implied by Theorem 3.7(iv) of that paper that a point stabilizer in $\mathrm{Aut}(\Delta)$ has order 2. Thus $|\mathrm{Aut}(\Delta)| = 2 \cdot |\Omega| = 54$.

2. We may also establish Corollary 9.4 by using results from the catalogue [17]. Indeed, from the list of transitive permutation groups of degree 9 we are able to quickly rule out any candidate for $\mathrm{Aut}(\Delta)$ as a prospective overgroup of H in S_9. Note also that part (iii) of Corollary 9.4 follows at once from the properties of the partition of Δ into noble hexagons.

3. The smallest Bouwer graph on 54 vertices (cf. [10]) may be obtained from the Doyle-Holt graph via the process of standard bipartite doubling. This immediately identifies the automorphism group of this Bouwer graph as $H \times \mathbb{Z}_2$.

4. Implicit in our investigations is a realization of the Doyle-Holt graph as a Cayley graph over the non-abelian group $R = \langle g_1, g_2^2 \rangle$ with connection set $\{g_1, g_1^{-1}, g_2^2, g_2^{-2}\}$.

5. Last but not least, we wish to emphasize an evident split of the edge set E into two pairs of non-symmetric relations of valency 2. In fact, these relations are both H-orbits, specifically $\overrightarrow{E} = (0,2)^H$ and $\overleftarrow{E} = (2,0)^H$. Figure 1 provides an especially nice way to view each such relation depending on the orientation chosen. For example, consider the clockwise orientation of the “external” Hamiltonian cycle starting from the arc $(0, 2)$, and similarly do this for the clockwise “internal” Hamiltonian cycle starting from $(0, 24)$. In total, the 54 arcs so transversed comprise the relation $\overrightarrow{E}$.

9.4 From Doyle-Holt graph to Gray graph

Let us return to the auxiliary graph Σ of Section 9.1. As before, let Ω denote its edge set, and let $\mathcal{B}$ be the set of triangles in Σ. Recalling that $\overline{\Sigma} = 3 \circ K_3$, it is easy to deduce that $|\mathcal{B}| = 27$, and that each edge of Σ is contained in exactly three triangles. Now form the design $\mathfrak{S} = (\Omega, \mathcal{B})$ where incidence is given by inclusion. Clearly $\mathfrak{S}$ is a *configuration*, that is a uniform, regular partial linear space. Further, $\mathfrak{S}$ is symmetric (because $|\Omega| = |\mathcal{B}|$), though it is not self-dual.

The incidence graph $I(\mathfrak{S})$ of this configuration first appeared in [9], though Bouwer accredits its discovery to M.C. Gray (unpublished work, 1932). Thus one now refers to $\mathfrak{S}$ as the *Gray configuration*, and to $I(\mathfrak{S})$ as the *Gray graph*.

Let us start with the same group H (automorphism group of the Doyle-Holt graph) acting on the initial set $[0, 8]$, and consider the 2-closure $(H^{(2)}, [0, 8])$. This latter group turns out to be the automorphism group of each of the graphs Σ and $\overline{\Sigma} = 3 \circ K_3$. As an abstract group, $H^{(2)}$ is the familiar wreath product $S_3 \wr S_3$, however its concrete action on the set $\mathcal{B}$ coincides with the slightly less familiar product action of $S_3 \wr S_3$ (coined “exponentiation” by F. Harary, and denoted by $S_3 \uparrow S_3$). In fact, as an abstract group $H^{(2)}$ is isomorphic to the authomorphism group of the Gray graph. Note further, that while the action $(H^{(2)}, \mathcal{B})$ is well known to be primitive, the action $(H^{(2)}, \Omega)$ is imprimitive.

The Gray graph has received much attention over the years, and has been the subject of diverse investigations, e.g., see [56, 57, 62]. Especially noteworthy is the fact that it is regular and edge-transitive but not vertex-transitive. Following [43] we call such a graph *semisymmetric*. See [54, 22, 72, 61] for constructions of infinite series of semisymmetric graphs in which the Gray graph appears as an initial member.

9.5 From Gray configuration to generalized quadrangle

The generalized quadrangle $\mathrm{GQ}(2,4)$ has 27 points and 45 lines. Its point graph is the unique SRG with parameters $(27,10,1,5)$, hence we may identify it as the graph which arises in connection with three separate fusions of $\mathfrak{X}$, see Proposition 9.1(v). Traditionally, one refers to this SRG (though more often to its complement) as the Schläfli graph, e.g., see [18, 29]. Thus we shall denote it by Sch.

In a sense, Sch is the most symmetric of all nontrivial graphs. More precisely, it is *4-homogeneous*, which means that each isomorphism between any two of its k-vertex induced subgraphs, $k \leq 4$, may be extended to an automorphism of the entire graph, see [30, 48]. In fact, Sch is the only *strictly* 4-homogeneous graph (i.e., 4- but not 5-homogeneous), as can be verified by applying the major classification theorem CFSG.

Natural constructions of $\mathrm{GQ}(2,4)$ (and hence, Sch) most often occur in the context of finite geometries and Coxeter systems. However, the model we wish to here describe has the advantage of visibly extending the Gray configuration. Currently, this model appears only in the Ph.D. thesis of A. Heinze [31], so this marks its first presentation in the formal literature.

Once again we start from the auxiliary graph Σ with edge set Ω. Denote by V_1, V_2, V_3 the vertex sets of the three disjoint triangles which form the graph $\overline{\Sigma}$. For each $1 \leq i < j \leq 3$, denote by $\mathcal{F}_{ij}$ the set of all perfect 1-factors of the induced subgraph $\Sigma[V_i \cup V_j]$ of Σ. Finally set $\mathcal{F} = \mathcal{F}_{12} \cup \mathcal{F}_{13} \cup \mathcal{F}_{23}$. Clearly, $\mathcal{F}$ consists of $18 = 3 \cdot 3!$ partial 1-factors of the graph Σ.

Now set $\mathcal{L} = \mathcal{B} \cup \mathcal{F}$, where $\mathcal{B}$ is the earlier defined block set of the Gray configuration. (Note that $|\mathcal{L}| = |\mathcal{B}| + |\mathcal{F}| = 27 + 18 = 45$.) We consider the incidence system $\Pi = (\Omega, \mathcal{L})$, where once again the incidence relation is inclusion.

We leave to the reader verification that Π is indeed a model of $\mathrm{GQ}(2,4)$. One clear advantage of our model is that it exploits the Gray configuration in evident form.

Recall that a *spread* of a generalized quadrangle is a subset of lines that partitions its point set. It turns out that up to isomorphism $\mathrm{GQ}(2,4)$ has two classes of spreads, e.g., see [16]. An additional advantage of our model is that both types of spreads are quite visible in the developed framework. Indeed, one may choose suitable non-isomorphic spreads $\mathcal{S}_1, \mathcal{S}_2$ for which $\mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{F}$. Note that the stabilizers in $\mathrm{Aut}(\mathrm{GQ}(2,4)) \cong \mathrm{P\Gamma U}(4,2)$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ have respective orders 324 and 1296, and are in fact the groups appearing in Proposition 9.1(vi). The group of order 324 is the normalizer in S_9 of the semiregular group $\langle g_1^3 \rangle$ of order 3, while the group of order 1296 is a non-split extension of N by S_4, where N is the transitive group #12 (so denoted in [17]) of degree 9 and order 54.

Recall that deletion of a spread from the point graph of a generalized quadrangle yields an antipodal DRG of diameter 3, see [14]. In this context, the DRGs so arising from the spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ are non-isomorphic, in fact only one such graph is distance transitive. The non-distance transitive graph has a celebrated property however: it satisfies the 5-vertex condition for DRGs (a natural generalization of the t-vertex condition for SRGs, see [34] for precise definitions). This newly described property of the latter DRG complements quite nicely the discovery in [23] that Levi graphs of projective planes satisfy the 6-vertex condition for DRGs.

9.6 On an observation of Pisanski

Let $\mathrm{W}(3)$ denote the generalized quadrangle of order (3,3) (symplectic type), having 40 points and 40 lines. Denote its incidence graph by Θ. Note that Θ is semisymmetric because $\mathrm{W}(3)$ is not self-dual.

We refer now to the closing portion of [62], where one learns that the Gray graph can be detected as an induced subgraph of Θ (e.g., induce on the 54 vertices at distance 4 from a fixed edge of Θ). It is further mentioned in [62], though without details, that there are exactly 160 embeddings of the Gray graph in Θ. Our goal is to shed some additional light on these embeddings.

It is known (though not widely, cf. p. 305 of [19]) that the incidence graph Θ is simultaneously the point graph of the generalized octagon $\mathrm{GO}(1,3)$. As $\mathrm{W}(3)$ and $\mathrm{GQ}(2,4)$ have isomorphic automorphism groups, namely $\mathrm{P\Gamma U}(4,2)$, we see that the stabilizer of an edge in Θ has order $\frac{51840}{160} = 324$. We summarize our findings below.

Proposition 9.5. (i) *The stabilizer of an edge in* Θ *coincides with the earlier encountered stabilizer of a representative spread* $\mathcal{S}_1$ *of* $\mathrm{GQ}(2,4)$*;*
(ii) *the transitive action of* $\mathrm{Aut}(\mathrm{GQ}(2,4))$ *on the 160 spreads of type* $\mathcal{S}_1$ *is permutation isomorphic to the action of* $\mathrm{Aut}(\mathrm{GO}(1,3))$ *on the lines of the octagon;*
(iii) *the 160 embeddings of the Gray graph in the incidence graph of* $\mathrm{W}(3)$ *correspond to points of the dual octagon* $\mathrm{GO}(3,1)$.

All results in Proposition 9.5 were initially obtained and/or confirmed with the aid of GAP and COCO. Independent computer-free proofs and interpretations have since been supplied by us. However, being technically cumbersome they have been relegated to a future paper.

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