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On domination-type invariants of Fibonacci cubes and hypercubes*

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Abstract

The Fibonacci cube Γ_n is the subgraph of the n-dimensional cube Q_n induced by the vertices that contain no two consecutive 1s. Using integer linear programming, exact values are obtained for $\gamma_t(\Gamma_n)$, $n \leq 12$. Consequently, $\gamma_t(\Gamma_n) \leq 2F_{n-10} + 21F_{n-8}$ holds for $n \geq 11$, where F_n are the Fibonacci numbers. It is proved that if $n \geq 9$, then $\gamma_t(\Gamma_n) \geq \lceil (F_{n+2} - 11)/(n-3) \rceil - 1$. Using integer linear programming exact values for the 2-packing number, connected domination number, paired domination number, and signed domination number of small Fibonacci cubes and hypercubes are obtained. A conjecture on the total domination number of hypercubes asserting that $\gamma_t(Q_n) = 2^{n-2}$ holds for $n \geq 6$ is also disproved in several ways.

Keywords: Total domination number, Fibonacci cube, hypercube, integer linear programming, covering codes.

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1 Introduction

Fibonacci cubes were introduced by Hsu [19] because of their appealing properties applicable to interconnection networks. Afterwards they have been extensively studied and found additional applications, see the survey [23]. The interest for Fibonacci cubes continues, recent research of them includes asymptotic properties [24], connectivity issues [7], the structure of their disjoint induced hypercubes [14, 30], the (non)-existence of perfect codes [5], and the q-cube enumerator polynomial [31]. From the algorithmic point of view, Ramras [29] investigated congestion-free routing of linear permutations on Fibonacci cubes, while Vesel [34] designed a linear time recognition algorithm for this class of graphs.

The domination number of Fibonacci cubes was investigated by now in two papers. Pike and Zou [28, Theorem 3.2] proved that $\gamma(\Gamma_n) \geq \lceil (F_{n+2} - 2)/(n-2) \rceil$ for $n \geq 9$, where F_n are the *Fibonacci numbers*: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Exact values of $\gamma(\Gamma_n)$ for $n \leq 8$ were also obtained in [28]. In the second related paper [9] the domination number of Fibonacci cubes was then compared with the domination number of Lucas cubes.

In this note we turn our attention to domination invariants of Fibonacci cubes and of hypercubes with a prime interest on the total domination. We proceed as follows. In the rest of this section we introduce concepts and notation needed. Then, in Section 2, we determine the exact value of the total domination number of Γ_n for $n \leq 12$, and obtain an upper bound and a lower bound on $\gamma_t(\Gamma_n)$. In Section 3 we use integer linear programming to either extend or obtain values for several domination-type invariants on Fibonacci cubes and hypercubes. In the final section we consider the total domination of hypercubes with respect to a recent conjecture from [22]. In particular, using known results from coding theory we show that the conjecture does not hold. It is also observed that for any $c > 0$ there exists $n_0 \in \mathbb{N}$, such that if $n \geq n_0$, then $\gamma_t(Q_n) \leq 2^{n-c}$.

The n-dimensional (hyper)cube Q_n, $n \geq 1$, is the graph with $V(Q_n) = \{0,1\}^n$, two vertices being adjacent if they differ in a single coordinate. For convenience we also set $Q_0 = K_1$. The vertices of Q_n will be briefly written as binary strings $b_1 \ldots b_n$. A *Fibonacci string* of length n is a binary string $b_1 \ldots b_n$ with $b_i \cdot b_{i+1} = 0$ for $1 \leq i < n$. Fibonacci strings are thus binary strings that contain no consecutive 1s. The *Fibonacci cube* Γ_n, $n \geq 1$, is the subgraph of Q_n induced by the Fibonacci strings of length n. It is well known that $|V(\Gamma_n)| = F_{n+2}$.

If u is a binary string, then the number of its bits equal to 1 is the *weight* of u. If u and v are binary strings, then uv denotes the usual concatenation of the two strings. If u is a binary string and X a set of binary strings, then $uX = \{ux : x \in X\}$.

Let G be a graph. Then $D \subseteq V(G)$ is a *dominating set* if every vertex from $V(G) \setminus D$ is adjacent to some vertex from D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. D is a *total dominating set* if every vertex from $V(G)$ is adjacent to some vertex from D. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. Note that the total domination number is not defined for graphs that contain isolated vertices, hence unless stated otherwise, all graphs in this paper are isolate-free. For more information on the total domination in graphs see the recent book [17] and papers [11, 12].

2 Total domination in Fibonacci cubes

In this section we present exact values of $\gamma_t(\Gamma_n)$ for $n \leq 12$, prove an upper bound on $\gamma_t(\Gamma_n)$, and a lower bound on $\gamma_t(\Gamma_n)$. The exact values were obtained by computer and are collected in Table 1, where the order of the cubes is also given so that the complexity of the problem is emphasized. In particular, $|V(\Gamma_{12})| = 377$.

Table 1: Exact total domination numbers of Fibonacci cubes up to dimension 12.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\|V(\Gamma_n)\|$	2	3	5	8	13	21	34	55	89	144	233	377
$\gamma_t(\Gamma_n)$	2	2	2	3	5	7	10	13	20	30	44	65

More precisely, the results from Table 1 were obtained using integer linear programming as follows. Suppose we associate to each vertex $v \in V(\Gamma_n)$ a binary variable x_v. The problem of determining $\gamma_t(\Gamma_n)$ can then be expressed as a problem of minimizing the objective function

$$\sum_{v \in V(\Gamma_n)} x_v,$$

subject to the condition that for every $v \in V(\Gamma_n)$ we have

$$\sum_{u \sim v} x_u \geq 1.$$

The value of the objective function is then $\gamma_t(\Gamma_n)$.

We have found out that the most efficient solver for the above problem is Gurobi$^{\text{TM}}$ Optimizer [15]. For example, it takes less than $9s$ to compute $\gamma_t(\Gamma_{12})$ on a standard desktop machine. On the other hand, we were not able to make the computation for $\gamma_t(\Gamma_{13})$ in real time (note that the order of Γ_{13} is 610), we could only get the estimates

$$97 \leq \gamma_t(\Gamma_{13}) \leq 101\,.$$

Using the above computations, the following result can be derived.

Theorem 2.1. *If $n \geq 11$, then $\gamma_t(\Gamma_n) \leq 2F_{n-10} + 21F_{n-8}$.*

Proof. Consider the so-called fundamental decomposition of Γ_n into the subgraphs induced by the vertices that start with 0 and 10, respectively (cf. [23]). These subgraphs are isomorphic to Γ_{n-1} and Γ_{n-2} respectively, hence we infer that $\gamma_t(\Gamma_n) \leq \gamma_t(\Gamma_{n-1}) + \gamma_t(\Gamma_{n-2})$. From the above computations we know that $\gamma_t(\Gamma_{11}) = 44$ and $\gamma_t(\Gamma_{12}) = 65$. Define the sequence (a_n), $n \geq 11$, with $a_{11} = 44$, $a_{12} = 65$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 13$. Then one can check by a simple induction argument that $a_n = 2F_{n-10} + 21F_{n-8}$ holds for any $n \geq 11$. Since $\gamma_t(\Gamma_n) \leq a_n$ the argument is complete. $\square$

Arnautov [3] and independently Payan [27] proved that

$$\gamma(G) \leq \frac{|V(G)|}{\delta + 1} \sum_{j=1}^{\delta+1} \frac{1}{j} \tag{2.1}$$

holds for any graph G of minimum degree δ. Since $\delta(\Gamma_n) = \lfloor (n+2)/3 \rfloor$, cf. [25, Corollary 3.5], and because $\gamma_t \le 2\gamma$, we get that

$$\gamma_t(\Gamma_n) \le \frac{2F_{n+2}}{\lfloor \frac{n+5}{3} \rfloor} \sum_{j=1}^{\lfloor \frac{n+5}{3} \rfloor} \frac{1}{j}\,. \tag{2.2}$$

Computing the values of the right-hand side of the bound of Theorem 2.1 and of (2.2) we find out that Theorem 2.1 is better than the bound of (2.2) for $n \le 33$.

By using the fact $\gamma_t(\Gamma_{13}) \le 101$ that was obtained by our computations, the bound of Theorem 2.1 can be further improved to give

$$\gamma_t(\Gamma_n) \le 601F_{n-1} - 371F_n,\ n \ge 12\,.$$

We continue by establishing a lower bound on $\gamma_t(\Gamma_n)$.

Theorem 2.2. *If $n \ge 9$, then*

$$\gamma_t(\Gamma_n) \ge \left\lceil \frac{F_{n+2} - 11}{n-3} \right\rceil - 1\,.$$

Proof. The proof mimics the proof of [28, Theorem 3.2] which gives a lower bound on the domination number of Fibonacci cubes, hence we will not give all the details.

For a graph G and its total dominating set D we introduce the *over-total-domination* of D in G as $\mathrm{OD}_G(D) = \sum_{v \in D} \deg(v) - |V(G)|$. Consider now Γ_n, $n \ge 9$, and let D be a total dominating set of Γ_n. In Γ_n, the vertex 0^n is the unique vertex of degree n, vertices 10^{n-1} and $0^{n-1}1$ have degree $n-1$, and all other vertices of weight 1 have degree $n-2$. In addition, the vertices 1010^{n-3}, $10^{n-2}1$, and $0^{n-3}101$ are of degree $n-2$, while all other vertices of Γ_n have degree at most $n-3$, cf. [25].

Let k be the number of vertices of weight 1 from $D \setminus \{10^{n-1}, 0^{n-1}1\}$. In addition, let $\ell = |D \cap \{1010^{n-3}, 10^{n-2}1, 0^{n-3}101\}|$. Note that $k + \ell$ is the number of vertices from D that have degree $n-2$. The proof now proceeds by considering the cases that happen based on the membership of the vertices 0^n, 10^{n-1}, and $0^{n-1}1$ in D. Here we consider only the case when $\{0^n, 10^{n-1}, 0^{n-1}1\} \subseteq D$. We have:

$$\mathrm{OD}_G(D) \le n + 2(n-1) + (k+\ell)(n-2) + (\gamma_t(\Gamma_n) - 3 - k - \ell)(n-3) - F_{n+2}\,.$$

Since clearly $\mathrm{OD}_G(D) \ge 0$, from the above inequality we derive that $\gamma_t(\Gamma_n)(n-3) \ge F_{n+2} - k - \ell - 7$. Because $k + \ell \le n+1$ we get

$$\begin{aligned} \gamma_t(\Gamma_n) &\ge \frac{F_{n+2} - k - \ell - 7}{n-3} \ge \frac{F_{n+2} - (n+1) - 7}{n-3} \\ &= \frac{F_{n+2} - 11 - (n-3)}{n-3} = \frac{F_{n+2} - 11}{n-3} - 1\,, \end{aligned}$$

and the stated inequality holds in this case. All the other cases are treated similarly. □

We conclude the section with Table 2 in which known values and current best bounds on $\gamma_t(\Gamma_n)$ for $n \le 33$ are collected. The values for $n \le 12$ were computed using the linear program explained above. The bounds for $\gamma_t(\Gamma_{13})$ were established by Gurobi, and we conjecture that in fact $\Gamma_t(\Gamma_{13}) = 101$. Finally, the remaining bound in Table 2 were obtained by the bounds given in Theorems 2.1 and 2.2. Recall that $n = 33$ is the last value for which Theorem 2.1 gives a better bound than the bound (2.2).

Table 2: Exact values and current best bounds on $\gamma_t(\Gamma_n)$, $n \leq 33$.

n	$\gamma_t(\Gamma_n)$	n	$\gamma_t(\Gamma_n)$	n	$\gamma_t(\Gamma_n)$
1	2	12	65	23	3749-13276
2	2	13	97-101	24	5779-21481
3	2	14	87-174	25	8926-34757
4	3	15	131-283	26	13816-56238
5	5	16	196-457	27	21424-90995
6	7	17	296-740	28	33280-147233
7	10	18	449-1197	29	51778-238228
8	13	19	682-1937	30	80676-385461
9	20	20	1040-3134	31	125876-623689
10	30	21	1590-5071	32	196649-1009150
11	44	22	2438-8205	33	307580-1632839

3 Additional invariants on small Fibonacci cubes and hypercubes

The integer linear programming approach can be used to compute several additional invariants of Fibonacci cubes (and other graphs). This has recently been done by Ilić and Milošević in [20], where they have computed the domination number, the 2-packing number, and the independent domination number of low dimensional Fibonacci cubes. In particular, they have used integer linear programming to confirm the conjecture from [9] stating that $\gamma(\Gamma_9) = 17$. In addition, an integer linear programming model for the connected domination number has been presented in [13]. In this section we add to the list of integer linear programming models paired domination and signed domination. The concepts mentioned in this paragraph that have not been introduced yet are defined next.

A set $X \subseteq V(G)$ is a 2-*packing* if $d(x, y) \geq 3$ holds for any $x, y \in X$, $x \neq y$. The maximum size of a 2-packing of G is the 2-*packing number* of G denoted $\rho(G)$. The *independence domination number* $i(G)$ of G is the minimum size of a dominating set that induces no edges [26]. The *connected domination number* $\gamma_c(G)$ of G is the order of a smallest dominating set that induces a connected graph [10]. The *paired domination number* $\gamma_p(G)$ is the order of a smallest dominating set $S \subseteq V(G)$ such that the graph induced by S contains a perfect matching [2]. Finally, we say that $f\colon V(G) \to \{-1, 1\}$ is a *signed dominating function* if $\sum_{u \in N[v]} f(u) \geq 1$ holds for every $v \in V(G)$, where $N[v]$ is the closed neighborhood of v, that is, $N[v] = \{v\} \cup \{u : vu \in E(G)\}$. The *signed domination number* $\gamma_s(G)$ is the minimum of $\sum_{v \in V(G)} f(v)$ taken over all signed dominating functions f of G, see [18].

We now present the problems to determine the paired domination number of a graph and the signed domination number of a graph as integer linear programs. To model the paired domination problem for a graph G we introduce a binary variable x_e indicating whether the edge $e \in E(G)$ is present in the graph induced by a paired dominating set of G. Then we can model the problem as follows:

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{e\in E(G)} x_e & \\
\text{subject to} & \displaystyle\sum_{u\sim v} x_{uv} \le 1, & v\in V(G) \\
& \displaystyle\sum_{u\sim v}\sum_{w\sim u} x_{uw} \ge 1, & v\in V(G)\,.
\end{array}
$$

Similarly, to model the signed domination number we introduce a binary variable x_v associated with every vertex $v \in V(G)$ indicating whether v is assigned weight 1 or -1, respectively. Then we have the following linear program.

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{v\in V(G)} (2x_v - 1) & \\
\text{subject to} & \displaystyle\sum_{u\in N[v]} (2x_u - 1) \ge 1, & v\in V(G)\,.
\end{array}
$$

Our computational results are collected in Tables 3 and 4. In the rows for $\gamma(\Gamma_n)$, $\rho(\Gamma_n)$, and $i(\Gamma_n)$, the results from [20] are in normal font, while the new values are in bold. We have thus extended the results from [20] for one additional dimension. It is interesting to observe that the gap between the independent domination number and the domination in dimension 9 is equal to 2, but then in dimensions 10 and 11 the difference goes down to 1.

Table 3: Additional invariants for small Fibonacci cubes and hypercubes.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\gamma(\Gamma_n)$	1	1	2	3	4	5	8	12	17	25	**39**	**54-61**
$\rho(\Gamma_n)$	1	1	2	2	3	5	6	9	14	20	29	**42**
$i(\Gamma_n)$	1	1	2	3	4	5	8	12	19	26	**40**	**?-?**

Table 4: Additional invariants for small Fibonacci cubes and hypercubes.

n	1	2	3	4	5	6	7	8	9	10
$\gamma_c(\Gamma_n)$	1	1	2	3	5	7	10	14	22	
$\gamma_c(Q_n)$	1	2	4	6	10	16	28			
$\gamma_p(\Gamma_n)$	2	2	2	4	6	8	10	14	20	30
$\gamma_p(Q_n)$	2	2	4	4	8	14	24	32		
$\gamma_s(\Gamma_n)$	2	3	3	2	5	9	10	17	25	40
$\gamma_s(Q_n)$	2	2	4	6	12	16	32			

4 On total domination in hypercubes

It has recently been conjectured in [22, Conjecture 4.6] that $\gamma_t(Q_n) = 2^{n-2}$ holds for $n \ge 6$. In [4] Arumugam and Kala first observed that $\gamma_t(Q_1) = \gamma_t(Q_2) = 2$ and $\gamma_t(Q_3) = \gamma_t(Q_4) = 4$, and then followed by proving that $\gamma_t(Q_5) = 8$ [4, Theorem 5.1] and

$\gamma_t(Q_6) = 14$ [4, Theorem 5.2]. The last result is then a sporadic counterexample to the conjecture. Actually, at this moment the exact value of $\gamma_t(Q_n)$ is known for $n \leq 10$: $\gamma_t(Q_7) = 24$, $\gamma_t(Q_8) = 32$, $\gamma_t(Q_9) = 64$, and $\gamma_t(Q_{10}) = 124$, see [33, Appendix B, p. 40]. Hence Q_7 and Q_{10} are additional sporadic counterexamples (and so are Q_8 and Q_9 since $\gamma_t(Q_8) = 32 \neq 2^6$ and $\gamma_t(Q_9) = 64 \neq 2^7$).

Total dominating sets of Q_n can be in coding theory equivalently described as *covering codes of empty spheres* (of length n and covering radius 1). The following result was first proved back in [21], see also [35, Theorem 1(b)]. Let us rephrase the result here in graph-theoretical terms and give a corresponding argument.

Proposition 4.1. *If $n = 2^k$, $k \geq 0$, then $\gamma_t(Q_n) = 2^{n-k}$.*

Proof. From [32] we know that if $n = 2^k$, then $\gamma(Q_n) = 2^{n-k}$ and from [16] that if $n = 2^k - 1$, then also $\gamma(Q_n) = 2^{n-k}$. Let $n = 2^k$ and consider Q_n. Let Q^L_{n-1} and Q^R_{n-1} be the subgraphs of Q_n induced by the sets of vertices $X_0 = \{0b_2 \dots b_n : b_i \in \{0,1\}\}$ and $X_1 = \{1b_2 \dots b_n : b_i \in \{0,1\}\}$, respectively. Clearly, $V(Q_n)$ partitions into X_0 and X_1, and in Q_n every vertex of X_0 has a unique neighbor in X_1. Moreover, Q^L_{n-1} and Q^R_{n-1} are both isomorphic to Q_{n-1}. Let C_L be a perfect code of Q^L_{n-1} and let C_R be its copy in Q^R_{n-1}. Then $C_L \cup C_R$ is a total dominating set of Q_n of order 2^{n-k}. Since on the other hand $\gamma_t(Q_n) \geq \gamma(Q_n) = 2^{n-k}$, the conclusion follows. $\square$

It follows from (2.1) that

$$\gamma(G) \leq |V(G)| \left(\frac{1 + \ln(\delta + 1)}{\delta + 1} \right) \tag{4.1}$$

holds for any graph G. Hence, again using the fact that $\gamma_t(G) \leq 2\gamma(G)$, we get for hypercubes that

$$\gamma_t(Q_n) \leq 2^{n+1} \left(\frac{1 + \ln(n+1)}{n+1} \right) .$$

Directly from this inequality we infer:

Remark 4.2. For any $c > 0$ there exists $n_0 \in \mathbb{N}$, such that if $n \geq n_0$, then

$$\gamma_t(Q_n) \leq 2^{n-c} .$$

Two remarks are in place here. First, (4.1) also follows from a more general result on transversals in hypergraphs due to Alon [1]. Second, the state of the art on the upper bounds on the domination number in terms of the minimum degree and the order of a given graph is given in [8].

It follows from the fact that $\gamma_t(Q_n) \leq 2\gamma(Q_{n-1})$ and from Proposition 4.1 that $\gamma_t(Q_{2^k+1}) \leq 2\gamma(Q_{2^k}) = 2^{2^k - k+1}$. As proved in [32], the equality actually holds here, that is, $\gamma_t(Q_{2^k+1}) = 2^{2^k-k+1}$. More generally, $\gamma_t(Q_{n+1}) = 2\gamma(Q_n)$ holds for any n, a result very recently proved in [6].

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