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Flag-transitive automorphism groups of 2-designs with $\lambda \geq (r, \lambda)^2$ and an application to symmetric designs*

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Abstract

Let $\mathcal{D}$ be a 2-(v, k, λ) design with $\lambda \geq (r, \lambda)^2$. If $G \leq \mathrm{Aut}(\mathcal{D})$ is flag-transitive, then G cannot be of simple diagonal or twisted wreath product type, and if G is product type then the socle of G has exactly two components and G has rank 3. Furthermore, we prove that if $\mathcal{D}$ is symmetric, then G must be an affine or almost simple group.

Keywords: 2*-design, automorphism group, primitivity, flag-transitivity.*

Math. Subj. Class.: 05B05, 05B25, 20B25

1 Introduction

A 2-(v, k, λ) design is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ where $\mathcal{P}$ is a set of v points and $\mathcal{B}$ is a set of b blocks with incidence relation such that every block is incident with exactly k points, and every 2-element subset of $\mathcal{P}$ is incident with exactly λ blocks. Let r be the number of blocks incident with a given point. The numbers v, b, r, k, and λ are the parameters of $\mathcal{D}$. A design $\mathcal{D}$ is called simple if it has no repeated blocks, and is called symmetric if $v = b$, and nontrivial if $2 < k < v - 1$. Here we always assume that $\mathcal{D}$ is simple and nontrivial. An automorphism of $\mathcal{D}$ is a permutation of the points which also permutes the blocks and preserves the incidence relation. The set of all automorphisms of $\mathcal{D}$ with the composition of maps is a group, denoted by $\mathrm{Aut}(\mathcal{D})$. Let $G \leq \mathrm{Aut}(\mathcal{D})$. If G is a primitive permutation group on the point set $\mathcal{P}$, then G is called point-primitive,

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otherwise point-imprimitive. A flag in a design is an incident point-block pair, and G is called flag-transitive if G is transitive on the set of flags.

There are many research works on flag-transitive 2-designs. It was shown in [1] that if a linear space admits a flag-transitive automorphism group G, then G is either of affine or almost simple type. Then the classification of flag-transitive linear spaces was announced in [2], and the complete proof was given by Liebeck [11] for affine type and Saxl [15] for almost simple type. In 1988, Zieschang [19] proved that if G is a flag-transitive automorphism group of a 2-design with $(r, \lambda) = 1$, then G is an affine or almost simple group. This paper study flag-transitive 2-(v, k, λ) designs under the condition that $\lambda \geq (r, \lambda)^2$. This condition has significance in design theory. On the one hand, the condition $\lambda \geq (r, \lambda)^2$ and the flag-transitivity of G implies that G is primitive [5, (2.3.7)] (also see Lemma 2.3 below), so we can use the O'Nan-Scott Theorem to analyze this type of designs. On the other hand, there exists many flag-transitive 2-designs satisfying the conditions $\lambda \geq (r, \lambda)^2$ and $(r, \lambda) > 1$. Before stating our main results, we give an example in the following.

Example 1.1. Let $\mathcal{P} = \{1, 2, 3, 4, 5, 6\}$, $G = \langle(3546), (162)(345)\rangle \cong S_5$ be a primitive group of degree 6 acting on $\mathcal{P}$. Let $B = \{1, 2, 4\}$. It is easily known that

$$\begin{aligned} B^G = \big\{&\{1,2,4\}, \{1,3,5\}, \{4,5,6\}, \{1,3,4\}, \{1,2,6\}, \{2,3,6\}, \{1,2,5\}, \{2,4,6\},\\ &\{1,4,6\}, \{3,5,6\}, \{2,3,4\}, \{1,2,3\}, \{1,5,6\}, \{3,4,5\}, \{1,4,5\}, \{1,3,6\},\\ &\{2,5,6\}, \{2,4,5\}, \{2,3,5\}, \{3,4,6\}\big\}, \end{aligned}$$

and $G_B = \langle(124)(356), (12)(56)\rangle \cong D_6$ is transitive on B. Let $\mathcal{B} = B^G$. Then $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a 2-$(6, 3, 4)$ design, and G acts flag-transitively on it.

More examples of flag-transitive 2-designs with $\lambda \geq (r, \lambda)^2$ and $(r, \lambda) > 1$ can be found in [18]. Our main theorem is the following partial improvement of Zieschang's result.

Theorem 1.2. *Let $\mathcal{D}$ be a 2-(v, k, λ) design with $\lambda \geq (r, \lambda)^2$. If G is a flag-transitive automorphism group of $\mathcal{D}$, then G is of affine, almost simple type, or product type with $\mathrm{Soc}(G) \cong T \times T$, where T is a nonabelian simple group and G has rank 3.*

Flag-transitive symmetric designs with λ small have been investigated by many researchers, including Kantor [10] for finite projective planes, Regueiro [13] for $\lambda \leq 3$, Fang et al. [8] and Regueiro [14] for $\lambda = 4$, Tian and Zhou [16] for $\lambda \leq 100$. In all these cases, it was proved that if a 2-(v, k, λ) symmetric design $\mathcal{D}$ admits a flag-transitive, point-primitive automorphism group G, then G must be of affine or almost simple type. As an application of Theorem 1.2, we get the following theorem on symmetric designs.

Theorem 1.3. *Let $\mathcal{D}$ be a 2-(v, k, λ) symmetric design with $\lambda \geq (r, \lambda)^2$, which admits a flag-transitive automorphism group G. Then G is an affine or almost simple group.*

The structure of the paper is organized as follows. Section 2 gives some preliminary lemmas on flag-transitive designs and permutation groups that will apply directly to our situation. In Section 3, we prove Theorem 1.2. Our strategy is based on the O'Nan-Scott Theorem [12] on finite primitive permutation groups, so we deal with the simple diagonal type, the twisted wreath product type, and the product type in Subsections 3.1, 3.2 and 3.3, respectively. In Section 4, we give a proof of Theorem 1.3.

2 Preliminaries

The following lemma is well known.

Lemma 2.1. *The parameters v, b, k, r, λ of a 2-(v, k, λ) design satisfy the following conditions:*

(i) $vr = bk$.

(ii) $\lambda(v-1) = r(k-1)$.

(iii) $b \geq v$ *and* $k \leq r$.

Lemma 2.2. *Let $\mathcal{D}$ be a 2-(v, k, λ) design, and G be a flag-transitive automorphism group of $\mathcal{D}$. Then*

(i) $v \leq \lambda v < r^2$.

(ii) $r \mid \lambda(v-1, |G_\alpha|)$, *where G_α is the stabilizer of a point α.*

(iii) $r \mid \lambda d$ *for all nontrivial subdegrees d of G, i.e., the lengths of the G_α-orbits.*

Proof. (i) By Lemma 2.1(ii), we have $\lambda v = r(k-1) + \lambda = rk - (r - \lambda)$, and the result follows by combining it with $k \leq r$ and $1 \leq \lambda < r$.

(ii) Since G is flag-transitive and $\lambda(v-1) = r(k-1)$, we have $r \mid |G_\alpha|$ and $r \mid \lambda(v-1)$. It follows that r divides $(\lambda(v-1), |G_\alpha|)$, and hence $r \mid \lambda(v-1, |G_\alpha|)$.

For (iii), $r \mid \lambda d$ was proved in [4] and [3, p. 91]. □

The following lemma first appears in [5, (2.3.7)].

Lemma 2.3. *Let $\mathcal{D}$ be a 2-(v, k, λ) design with $\lambda \geq (r, \lambda)^2$. If $G \leq \mathrm{Aut}(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$, then G is point-primitive.*

Proof. Suppose that $G \leq \mathrm{Aut}(\mathcal{D})$ is flag-transitive and $\{C_1, C_2, ..., C_t\}$ is a system of t sets of imprimitivity each of size s. Then $v = st$. The set of imprimitivity containing a point α is a union of G_α-orbits, one of which is $\{\alpha\}$, hence by Lemma 2.2(iii) we have $s \equiv 1 \pmod{\frac{r}{(r,\lambda)}}$. Then $v = st \equiv t \pmod{\frac{r}{(r,\lambda)}}$, which implies $t \equiv \frac{r(k-1)}{\lambda} + 1 \equiv 1 \pmod{\frac{r}{(r,\lambda)}}$. Now let $s = \sigma\frac{r}{(r,\lambda)} + 1$ and $t = \tau\frac{r}{(r,\lambda)} + 1$. Then

$$v = \frac{r(k-1)}{\lambda} + 1 = st = (\sigma\frac{r}{(r,\lambda)} + 1)(\tau\frac{r}{(r,\lambda)} + 1)$$

and thus

$$\sigma\tau\frac{r\lambda}{(r,\lambda)^2} + (\sigma+\tau)\frac{\lambda}{(r,\lambda)} = k - 1. \tag{2.1}$$

Since G is flag-transitive and imprimitive, we must have a solution of (2.1) with $\sigma\tau \neq 0$. Hence if $\lambda \geq (r,\lambda)^2$, then (2.1) implies $r \leq \sigma\tau r < k - 1 < k$, a contradiction. □

Lemma 2.4 ([6, Lemma 2.5])**.** *Let $\mathcal{D}$ be a symmetric design and assume that $G \leq \mathrm{Aut}(\mathcal{D})$ is a primitive rank 3 permutation group on points and blocks. If $N = \mathrm{Soc}(G)$ is non-abelian, then N is simple.*

3 Proof of Theorem 1.2

In this section, we will assume that $\mathcal{D}$ is a 2-(v, k, λ) design with $\lambda \geq (r, \lambda)^2$ and $G \leq \mathrm{Aut}(\mathcal{D})$ is flag-transitive. By Lemma 2.3, G is point-primitive. The O'Nan-Scott Theorem classifies primitive groups into five types: (i) Affine type; (ii) Almost simple type; (iii) Simple diagonal type; (iv) Product type; (v) Twisted wreath product type, see [12] for details. We will rely on the O'Nan-Scott Theorem to prove Theorem 1.2 by dealing with the cases of simple diagonal action, twisted wreath product action and product action separately.

3.1 Simple diagonal action

Proposition 3.1. *Let $\mathcal{D}$ be a 2-(v, k, λ) design with $\lambda \geq (r, \lambda)^2$. If G is a flag-transitive automorphism group of $\mathcal{D}$, then G is not of simple diagonal type.*

Proof. Suppose that G is of simple diagonal type. Then

$$G \leq W = \{(a_1, \ldots, a_m)\pi \mid a_i \in \mathrm{Aut}(T), \pi \in S_m, a_i \equiv a_j \bmod \mathrm{Inn}(T) \text{ for all } i, j\},$$

and there is $\alpha \in \mathcal{P}$ such that

$$G_\alpha \leq \{(a, \ldots, a)\pi \mid a \in \mathrm{Aut}(T), \pi \in S_m\} \cong \mathrm{Aut}(T) \times S_m,$$

and

$$M_\alpha = D = \{(a, \ldots, a) \mid a \in \mathrm{Inn}(T)\}$$

is a diagonal subgroup of $M = T_1 \times \cdots \times T_m \cong T^m$. Put $\Sigma = \{T_1, \ldots, T_m\}$, where T_i is identified with the group $\{(1, 1, \ldots, t, \ldots, 1) \mid t \in T\}$ where t is in the i-th position. Then G acts on Σ [12]. Moreover the set $\mathcal{P}$ of points can be identified with the set M/D of right cosets of D in M so that $\alpha = D(1, \ldots, 1)$, $v = |\mathcal{P}| = |T|^{m-1}$, and for $\beta = D(t_1, \ldots, t_m)$, $s = (s_1, \ldots, s_m) \in M$, $\sigma \in \mathrm{Aut}(T)$, $\pi \in S_m$, we have the actions

$$\beta^s = D(t_1 s_1, \ldots, t_m s_m), \ \beta^\sigma = D(t_1^\sigma, \ldots, t_m^\sigma), \ \beta^\pi = D(t_{1\pi^{-1}}, \ldots, t_{m\pi^{-1}}).$$

Since $M \trianglelefteq G$ and G is primitive on $\mathcal{P}$, M is transitive on $\mathcal{P}$. Since $T_1 \trianglelefteq M$ it follows that T_1 acts $\frac{1}{2}$-transitively on $\mathcal{P}$ ([17, Theorem 10.3]), and so all its orbits have equal length $c > 1$. Let Γ_1 be the orbit of T_1 containing the point α. For any $t_1 = (t, 1, \ldots, 1) \in T_1$, we have $\alpha^{t_1} = D(t, 1, \ldots, 1)$, so that

$$\Gamma_1 = \alpha^{T_1} = \{D(t, 1, \ldots, 1) \mid t \in T\}$$

and $|\Gamma_1| = |T| = c$. Similarly, define $\Gamma_i = \alpha^{T_i}$ for $1 < i \leq m$. Clearly $\Gamma_i \cap \Gamma_j = \{\alpha\}$ for $i \neq j$ provided that $m \geq 2$.

Choose an orbit Δ of G_α in $\mathcal{P} - \{\alpha\}$ such that $|\Delta \cap \Gamma_1| = d \neq 0$. Let $m_1 = |G_\alpha : N_{G_\alpha}(T_1)|$. Since $G_\alpha \lesssim \mathrm{Aut}(T) \times S_m$ and G^Σ is transitive on Σ, it follows that $m_1 \leq m$, and thus

$$|\Delta| = m_1 d \leq m|T|.$$

Lemma 2.2(iii) implies $r \mid \lambda m_1 d$, so $r \leq (r, \lambda) m_1 d \leq (r, \lambda) m |T|$. From $\lambda v < r^2$ and $\lambda \geq (r, \lambda)^2$ we have

$$\lambda |T|^{m-1} < r^2 \leq ((r, \lambda) m |T|)^2 \leq \lambda m^2 |T|^2.$$

As T is a nonabelian simple group, we have

$$60^{m-3} \leq |T|^{m-3} < m^2,$$

from which it follows that $m \leq 3$. Since $T \cong M_\alpha \trianglelefteq G_\alpha \lesssim \mathrm{Aut}(T) \times S_m$ and $r \mid |G_\alpha|$, r also divides $|T||\mathrm{Out}(T)|m!$. Let $a = (r,\lambda)$, so that $\frac{r}{a}(k-1) = \frac{\lambda}{a}(|T|^{m-1}-1)$. It follows that $\frac{r}{a}$ divides $|T|^{m-1}-1$, and so $(\frac{r}{a}, |T|) = 1$, which implies $\frac{r}{a} \mid |\mathrm{Out}(T)|m!$. Therefore,

$$|T|^{m-1} = v \leq \frac{\lambda v}{a^2} < \frac{r^2}{a^2} \leq (|\mathrm{Out}(T)|m!)^2.$$

It follows that $|T| < 4|\mathrm{Out}(T)|^2$ when $m = 2$, and $|T| < 6|\mathrm{Out}(T)|$ when $m = 3$. By [16, Lemma 2.3], T is isomorphic to one of following groups:

$$L_2(q) \text{ for } q = 5, 7, 8, 9, 11, 13, 16, 27, \text{ or } L_3(4).$$

However, from the facts $|\mathrm{Out}(L_3(4))| = 12$, $|\mathrm{Out}(L_2(q))| = 2$ for $q \in \{5, 7, 8, 11, 13, 16, 27\}$ and $|\mathrm{Out}(L_2(9))| = 4$ that $|T| > 4|\mathrm{Out}(T)|^2 > 6|\mathrm{Out}(T)|$, a contradiction. □

3.2 Twisted wreath product action

Proposition 3.2. *Let $\mathcal{D}$ be a 2-(v,k,λ) design with $\lambda \geq (r,\lambda)^2$. If G is a flag-transitive automorphism group of $\mathcal{D}$, then G is not of twisted wreath product type.*

Proof. By Lemma 2.3, G is primitive on $\mathcal{P}$. Suppose G has a twisted wreath product action. Then

$$G = T \,\mathrm{twr}_Q\, P = {}_QB \rtimes P$$

where P is a transitive permutation group on $\{1, \ldots, m\}$ with $m \geq 6$ (see [7, Theorem 4.7B(iv)]), $Q = P_1$ and $M = \mathrm{Soc}(G) = {}_QB = T_1 \times \cdots \times T_m \cong T^m$. Put $\Sigma = \{T_1, \ldots, T_m\}$, where T_i is identified with the group $\{(1, 1, \ldots, t, \ldots, 1; 1) \mid t \in T\} \cong T$ where t is in the i-th position. Then G acts on Σ (see [12]). Moreover, the set $\mathcal{P}$ of points can be identified with $G \backslash P$, the set of right cosets of P in G, so that G acts transitively on $\mathcal{P}$. Define $\alpha = P$, so that $G_\alpha = P$ and $v = |\mathcal{P}| = |T|^m$.

Similarly to the case of simple diagonal action, let $\Gamma_1 = \alpha^{T_1} = \{P(t, 1, \ldots, 1; 1) \mid t \in T\}$ so that $|\Gamma_1| = |T|$, and define $\Gamma_i = \alpha^{T_i}$ for $1 < i \leq m$. Clearly $\Gamma_i \cap \Gamma_j = \{\alpha\}$ for $i \neq j$.

Choose an orbit Δ of G_α in $\mathcal{P} - \{\alpha\}$ such that $|\Delta \cap \Gamma_1| = d \neq 0$. Let $m_1 = |G_\alpha : N_{G_\alpha}(T_1)|$. Since $G_\alpha = P$ and G^Σ is transitive on Σ, it follows that $m_1 \leq m$, and thus $|\Delta| = m_1 d \leq m|T|$. Lemma 2.2(iii) implies $\frac{r}{(r,\lambda)} \mid m_1 d$, and then $r \leq (r,\lambda)m_1 d \leq (r,\lambda)m|T|$. On the other hand, by $\lambda v < r^2$ and $\lambda \geq (r,\lambda)^2$, we have $\lambda |T|^m < r^2 \leq ((r,\lambda)m|T|)^2$. It follows that

$$60^{m-2} \leq |T|^{m-2} < m^2.$$

Thus, $m \leq 2$. However, this contradicts the fact that $m \geq 6$. □

3.3 Product action

Proposition 3.3. *Let $\mathcal{D}$ be a 2-(v,k,λ) design with $\lambda \geq (r,\lambda)^2$ admitting a flag-transitive automorphism group G and G is of product type. Then $\mathrm{Soc}(G) = T_1 \times T_2$ (where $T_i \cong T$ is a nonabelian simple group) and G has rank 3.*

Suppose that G has a product action on $\mathcal{P}$. Then there is a group K with a primitive action (of almost simple or diagonal type) on a set Γ of size $v_0 \geq 5$, where

$$\mathcal{P} = \Gamma^m, G \leq K^m \rtimes S_m = K \wr S_m \text{ and } m \geq 2.$$

The proof of Proposition 3.3 follows from the next two lemmas.

Lemma 3.4. *If G acts flag-transitively on a 2-(v, k, λ) design with $\lambda \geq (r, \lambda)^2$ and G is of product type, then $m = 2$.*

Proof. Let $H = K \wr S_m$, and let S_m act on $M = \{1, 2, \ldots, m\}$. As G is flag-transitive, by Lemma 2.2(iii) we get $[G_\alpha : G_{\alpha\beta}] \geq \frac{r}{(r,\lambda)}$ for any two distinct points α, β. Since $H \geq G$, it follows that

$$[H_\alpha : H_{\alpha\beta}] \geq [G_\alpha : G_{\alpha\beta}] \geq \frac{r}{(r, \lambda)} = \frac{\lambda(v-1)}{(r, \lambda)(k-1)}. \tag{3.1}$$

Let $\alpha = (\gamma, \gamma, \ldots, \gamma)$, $\gamma \in \Gamma$, $\beta = (\delta, \gamma, \ldots, \gamma)$, $\gamma \neq \delta \in \Gamma$ and let $B \cong K^m$ be the base group of H. Then $B_\alpha = K_\gamma \times \cdots \times K_\gamma$, $B_{\alpha\beta} = K_{\gamma\delta} \times K_\gamma \times \cdots \times K_\gamma$. Now $H_\alpha = K_\gamma \wr S_m$, and $H_{\alpha\beta} \geq K_{\gamma\delta} \times (K_\gamma \wr S_{m-1})$. Suppose K has rank s on Γ with $s \geq 2$. We can choose a δ satisfying $[K_\gamma : K_{\gamma\delta}] \leq \frac{v_0-1}{s-1}$, so that

$$[H_\alpha : H_{\alpha\beta}] = \frac{|H_\alpha|}{|H_{\alpha\beta}|} \leq \frac{|K_\gamma|^m \cdot m!}{|K_{\gamma\delta}||K_\gamma|^{m-1} \cdot (m-1)!} \leq m\frac{v_0 - 1}{s-1}$$

and hence by Equation (3.1),

$$\frac{\lambda(v-1)}{(r, \lambda)(k-1)} \leq [G_\alpha : G_{\alpha\beta}] \leq m\frac{v_0-1}{s-1}. \tag{3.2}$$

So

$$\frac{v_0^m - 1}{v_0 - 1} \leq m\frac{(r, \lambda)(k-1)}{\lambda(s-1)}. \tag{3.3}$$

Now $(k-1)^2 \leq (r-1)(k-1) < r(k-1) = \lambda v$. Thus

$$v_0^{m-1} < m v_0^{\frac{m}{2}} \frac{(r, \lambda)}{\lambda^{\frac{1}{2}}} \leq m v_0^{\frac{m}{2}}.$$

Hence $m \leq 2$, or $m = 3$ and $v_0 < 9$.

If $m = 3$, from Equation (3.3) we have $v_0^2 + v_0 + 1 < \frac{3v_0^{\frac{3}{2}}}{s-1}$, so that $v_0 = 5$ or 6 and $s = 2$. Now, from $(k-1)^2 \leq \lambda v$ we have $\frac{(k-1)^2}{v_0^3} \leq \lambda$. On the other hand, Equation (3.3) and $\lambda \geq (r, \lambda)^2$ imply $v_0^2 + v_0 + 1 \leq \frac{3(k-1)}{\lambda^{\frac{1}{2}}}$, so that $\lambda \leq \frac{9(k-1)^2}{(v_0^2+v_0+1)^2}$. It follows that

$$\frac{(k-1)^2}{v_0^3} \leq \lambda \leq \frac{9(k-1)^2}{(v_0^2 + v_0 + 1)^2},$$

where $v_0 = 5$ or 6. Now $G \leq K \wr S_3 \leq S_{v_0} \wr S_3$ implies that G is a $\{2, 3, 5\}$-group, so by flag-transitivity, k divides $|G_B|$, and hence the only primes dividing k are $2, 3$ or 5. The only integers v_0, k, λ satisfying these conditions are $v_0 = 5$, $k = 32$, $\lambda = 8$ or 9, by using the software package GAP [9]. Then $r = 32$ or 36 which contradicts the condition $\lambda \geq (r, \lambda)^2$. Hence $m = 2$. □

Lemma 3.5. *If G acts flag-transitively on 2-(v,k,λ) design with $\lambda \geq (r,\lambda)^2$ and G is of product type, then G is a point-primitive rank 3 group and v is an odd number.*

Proof. Since G is of product type, then $m = 2$ by Lemma 3.4. From Equation (3.3), we have

$$v_0 + 1 \leq \frac{2(r,\lambda)(k-1)}{\lambda(s-1)} < \frac{2(r,\lambda)v_0}{\lambda^{\frac{1}{2}}(s-1)} \leq \frac{2v_0}{s-1}.$$

This implies $s = 2$. It follows that K acts 2-transitively on Γ, and $H = K \wr S_2$ has rank 3 with subdegrees $1, 2(v_0-1), (v_0-1)^2$.

Now $G \leq H$, so each subdegree of H is the sum of some subdegrees of G and so $\frac{r}{(r,\lambda)} \mid 2(v_0-1)$. If $\frac{r}{(r,\lambda)} \neq 2(v_0-1)$, then $\frac{r}{(r,\lambda)} \leq v_0 - 1$, so that

$$r^2 \leq (r,\lambda)^2(v_0-1)^2 < (r,\lambda)^2 v_0^2 \leq \lambda v$$

which is a contradiction. Thus $\frac{r}{(r,\lambda)} = 2(v_0-1)$, by Equation (3.2), we obtain that G must have a subdegree $2(v_0-1)$ and it follows that G induces a 2-transitive group $\overline{G} \leq K$ on Γ. We conclude that G itself has rank 3 on $\mathcal{P}$ with subdegrees: 1, $2(v_0-1)$, $(v_0-1)^2$. Therefore, $\frac{r}{(r,\lambda)} \mid (v_0-1)^2$, i.e., $2 \mid v_0 - 1$. So $v = v_0^2$ is an odd number. □

Proof of Theorem 1.2. Follows immediately from Propositions 3.1, 3.2 and 3.3. □

4 Proof of Theorem 1.3

In this section, we will apply Theorem 1.2 to symmetric designs. For this purpose, we first give some basic facts on rank 3 permutation groups and symmetric designs. Lemma 4.1 first appears in [16, Lemma 1.5] with a sketch of proof. Since it is an important result for symmetric designs, we provide its proof here for completeness.

Lemma 4.1. *Let G be a finite imprimitive rank 3 permutation group on a set P. Let $P = \{\alpha\} \cup X \cup Y$ be the decomposition into G_α-orbits. Assume $|X| \leq |Y|$. Then $Q = \{\alpha\} \cup X$ is a block of the action of G. Set $\Omega = \{Q^g \mid g \in G\}$, then G acts 2-transitively on Ω.*

Proof. Let M be a maximal subgroup of G containing G_α. Then $G_\alpha < M$ and M is not transitive on P (otherwise, we have $G = MG_\alpha = M$ a contradiction). From $G_\alpha < M$, we have $M_\alpha = G_\alpha \cap M = G_\alpha$ and

$$1 + |X| + |Y| = |G : G_\alpha| = |G : M_\alpha| = |G : M||M : M_\alpha| \geq 2|M : M_\alpha|. \qquad (4.1)$$

Let $R = \{\alpha^x \mid x \in M\}$. Since both X and Y are orbits of G_α, and $G_\alpha < M$, there exists $m \in M \backslash G_\alpha$ such that $\alpha^m \in X$ or Y. If $\alpha^m \in X$, then $X = \alpha^{mG_\alpha} \subseteq \alpha^M = R$, so that $Q \subseteq R$. We argue that $Q = R$. For if $Q \subsetneq R$, then there exists $m' \in M \backslash G_\alpha$ such that $y = \alpha^{m'} \in R \backslash Q \subseteq Y$, from which it follows that $Y \subsetneq R$. Hence $\{\alpha\} \cup X \cup Y = \mathcal{P} \subseteq R$, and thus $P = R$ which contradicts the fact that M is intransitive. Therefore, $Q = R = \{\alpha\} \cup X$. Similarly, if $\alpha^m \in Y$, we have $R = \{\alpha\} \cup Y$. Next, we prove that $R \neq \{\alpha\} \cup Y$, and $R = Q$ is a block.

If $R = \{\alpha\} \cup Y$, since M is transitive on R, we have $|R| = |M : M_\alpha| = 1 + |Y|$. Equation (4.1) implies that $1 + |X| + |Y| \geq 2(1 + |Y|)$, so that $|X| \geq 1 + |Y| > |Y|$ which contradicts the assumption. Thus we must have $Q = \{\alpha\} \cup X = R$. Since $Q = R =$

$\{\alpha^x \mid x \in M\}$, if $Q^g \cap Q \neq \emptyset$ for some $g \in G$, then there exist $x, y \in M$ such that $\alpha^{xg} = \alpha^y$, so that $xgy^{-1} \in G_\alpha < M$ and $g \in M$. Hence $Q^g = Q$ and Q is a block.

Since $Q = \{\alpha\} \cup X$ is a block and $\alpha \in Q$, then $G_\alpha \leq G_Q$, where G_Q is the stabilizer of the block Q. Let $\Omega = \{Q^g \mid g \in G\}$ be a block system of imprimitivity of G. As G_α is transitive on $Y = P - Q$, it follows that G_Q acts transitively on $\Omega \backslash \{Q\}$, and thus G acts 2-transitively on Ω. □

Lemma 4.2 ([16, Lemma 1.6]). *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a symmetric 2-(v, k, λ) design and $G \leq \mathrm{Aut}(\mathcal{D})$ be a point-primitive rank 3 group. Then G is also a block-primitive rank 3 group if one of the following holds for $(G, \mathcal{P})$:*

(a) The permutation group is of product or affine type.

(b) The group G is almost simple and G has no 2-transitive representation of degree d, such that d properly divides v.

Now we begin the proof of Theorem 1.3.

Proof of Theorem 1.3. Assume $\mathcal{D}$ is a 2-(v, k, λ) symmetric design with $\lambda \geq (r, \lambda)^2$, which admits a flag-transitive automorphism group G. By Theorem 1.2, G is one of the following: (i) affine type, (ii) almost simple type, or (iii) product type with $\mathrm{Soc}(G) \cong T \times T$ where T is a nonabelian simple group and G is a primitive rank 3 group. So we only need to prove that case (iii) cannot occur.

Suppose for the contrary that G has a product action on the set of points. Here G is a point-primitive rank 3 group, so we know from Lemma 4.2 that G is also a block-primitive rank 3 group. By Lemma 2.4, we have $m = 1$. This contradicts the fact that $m = 2$ (see Lemma 3.4). Hence G is of affine or almost simple type. □

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