

# Order-chain polytopes\*

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## Abstract

Given two families  $X$  and  $Y$  of integral polytopes with nice combinatorial and algebraic properties, a natural way to generate a new class of polytopes is to take the intersection  $\mathcal{P} = \mathcal{P}_1 \cap \mathcal{P}_2$ , where  $\mathcal{P}_1 \in X$ ,  $\mathcal{P}_2 \in Y$ . Two basic questions then arise: 1) when  $\mathcal{P}$  is integral and 2) whether  $\mathcal{P}$  inherits the “old type” from  $\mathcal{P}_1, \mathcal{P}_2$  or has a “new type”, that is, whether  $\mathcal{P}$  is unimodularly equivalent to a polytope in  $X \cup Y$  or not. In this paper, we focus on the families of order polytopes and chain polytopes. Following the above framework, we create a new class of polytopes which are named order-chain polytopes. When studying

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their volumes, we discover a natural relation with Ehrenborg and Mahajan’s results on maximizing descent statistics.

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## 1 Introduction

This paper was motivated by the following two questions about intersecting two integral polytopes  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , which come from two given families  $X$  and  $Y$  of polytopes respectively:

- 1) when the intersection  $\mathcal{P} = \mathcal{P}_1 \cap \mathcal{P}_2$  is integral and
- 2) whether  $\mathcal{P}$  inherits the “old type” from  $\mathcal{P}_1, \mathcal{P}_2$  or has a “new type”, that is, whether  $\mathcal{P}$  is unimodularly equivalent to a polytope in  $X \cup Y$  or not.

Usually, we shall start with those families  $X$  and  $Y$  of polytopes which have nice combinatorial and algebraic properties. In this paper, we focus on the families of order polytopes and chain polytopes. Instead of considering the intersection of an arbitrary  $d$ -dimensional order polytope and an arbitrary  $d$ -dimensional chain polytope, we will consider the intersection of an order polytope  $\mathcal{O}(P')$  and a chain polytope  $\mathcal{C}(P'')$ , both of which arise from weak subposets  $P', P''$  of a given poset. The resulting polytope is called an order-chain polytope, which generalizes both order polytope and chain polytope.

The order polytope  $\mathcal{O}(P)$  as well as the chain polytope  $\mathcal{C}(P)$  arising from a finite partially ordered set  $P$  has been studied by many authors from viewpoints of both combinatorics and commutative algebra. Especially, in [16], the combinatorial structures of order polytopes and chain polytopes are explicitly discussed. Furthermore, in [9], the natural question when the order polytope  $\mathcal{O}(P)$  and the chain polytope  $\mathcal{C}(P)$  are unimodularly equivalent is solved completely. It follows from [5] and [8] that the toric ring ([7, p. 37]) of  $\mathcal{O}(P)$  and that of  $\mathcal{C}(P)$  are algebras with straightening laws ([6, p. 124]) on finite distributive lattices. Thus in particular the toric ideal ([7, p. 35]) of each of  $\mathcal{O}(P)$  and  $\mathcal{C}(P)$  possesses a squarefree quadratic initial ideal ([7, p. 10]) and possesses a regular unimodular triangulation ([7, p. 254]) arising from a flag complex. Furthermore, toric rings of order polytopes naturally appear in algebraic geometry (e.g., [2]) and in representation theory (e.g., [18]).

We begin by introducing some basic notation and terminology. Given a convex polytope  $\mathcal{P} \subset \mathbb{R}^d$ , a *facet hyperplane* of  $\mathcal{P} \subset \mathbb{R}^d$  is defined to be a hyperplane in  $\mathbb{R}^d$  which contains a facet of  $\mathcal{P}$ . If

$$H = \{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d : a_1x_1 + a_2x_2 + \dots + a_dx_d - b = 0\},$$

where each  $a_i$  and  $b$  belong to  $\mathbb{R}$ , is a hyperplane of  $\mathbb{R}^d$  and  $v = (y_1, y_2, \dots, y_d) \in \mathbb{R}^d$ , then we set

$$H(v) = a_1y_1 + a_2y_2 + \dots + a_dy_d - b.$$

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Let  $(P, \preceq)$  be a finite partially ordered set (*poset*, for short) on  $[d] = \{1, \dots, d\}$ . For each subset  $S \subseteq P$ , we define  $\rho(S) = \sum_{i \in S} \mathbf{e}_i$ , where  $\mathbf{e}_1, \dots, \mathbf{e}_d$  are the canonical unit coordinate vectors of  $\mathbb{R}^d$ . In particular  $\rho(\emptyset) = (0, 0, \dots, 0)$ , the origin of  $\mathbb{R}^d$ . A subset  $I$  of  $P$  is an *order ideal* of  $P$  if  $i \in I, j \in [d]$  together with  $j \preceq i$  in  $P$  imply  $j \in I$ . An *antichain* of  $P$  is a subset  $A$  of  $P$  such that any two elements in  $A$  are incomparable. We say that  $j$  *covers*  $i$  if  $i \prec j$  and there is no  $k \in P$  such that  $i \prec k \prec j$ . A chain  $j_1 \prec j_2 \prec \dots \prec j_s$  is *saturated* if  $j_q$  covers  $j_{q-1}$  for  $1 < q \leq s$ , and it is called a *maximal chain* if, moreover,  $j_1$  is a minimal element and  $j_s$  is a maximal element of  $P$ . A poset can be represented with its Hasse diagram, in which each cover relation  $i \prec j$  corresponds to an edge denoted by  $e = \{i, j\}$ . For a finite poset  $P$ , we let  $c(P), m_*(P)$  and  $m^*(P)$  denote the number of maximal chains, the number of minimal elements and the number of maximal elements of  $P$ , respectively. We denote by  $E(P)$  the set of edges in the Hasse diagram of  $P$ .

In [16], Stanley introduced two convex polytopes arising from a finite poset, the order polytope and the chain polytope. Following [9], we employ slightly different definitions. Given a finite poset  $(P, \preceq)$  on  $[d]$ , the *order polytope*  $\mathcal{O}(P)$  is defined to be the convex polytope consisting of those  $(x_1, \dots, x_d) \in \mathbb{R}^d$  such that

- (1)  $0 \leq x_i \leq 1$  for  $1 \leq i \leq d$ ;
- (2)  $x_i \geq x_j$  if  $i \preceq j$  in  $P$ .

The *chain polytope*  $\mathcal{C}(P)$  of  $P$  is defined to be the convex polytope consisting of those  $(x_1, \dots, x_d) \in \mathbb{R}^d$  such that

- (1)  $x_i \geq 0$  for  $1 \leq i \leq d$ ;
- (2)  $x_{i_1} + \dots + x_{i_k} \leq 1$  for every maximal chain  $i_1 \prec \dots \prec i_k$  of  $P$ .

Recall (see [16] for details) that there is a close connection between the combinatorial structure of  $P$  and the geometric structures of  $\mathcal{O}(P)$  and  $\mathcal{C}(P)$ . For instance, the following connections are not hard to prove:

- The number  $f_{d-1}(\mathcal{O}(P))$  of facets of  $\mathcal{O}(P)$  is equal to  $m_*(P) + m^*(P) + |E(P)|$ . Equivalently, if we let  $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$  be the poset obtained from  $P$  by adjoining a minimum element  $\hat{0}$  and a maximum element  $\hat{1}$ , then we have  $f_{d-1}(\mathcal{O}(P)) = |E(\hat{P})|$ .
- The number  $f_{d-1}(\mathcal{C}(P))$  of facets of  $\mathcal{C}(P)$  is equal to  $d + c(P)$ .
- The vertices of  $\mathcal{O}(P)$  are exactly those  $\rho(I)$  for which  $I$  is an order ideal of  $P$ , and the vertices of  $\mathcal{C}(P)$  are exactly those  $\rho(A)$  for which  $A$  is an antichain of  $P$ . Since it is well known that order ideals of  $P$  are in one-to-one correspondence with antichains of  $P$ , the order polytope  $\mathcal{O}(P)$  and the chain polytope  $\mathcal{C}(P)$  have the same number of vertices.

Let  $P$  be a finite poset, we define an *edge partition* of  $P$  to be a map

$$\ell: E(P) \longrightarrow \{o, c\}.$$

Equivalently, an edge partition of  $P$  is an ordered pair

$$(oE(P), cE(P))$$

of subsets of  $E(P)$  such that  $oE(P) \cup cE(P) = E(P)$  and  $oE(P) \cap cE(P) = \emptyset$ . An edge partition  $\ell$  is called *proper* if  $oE(P) \neq \emptyset$  and  $cE(P) \neq \emptyset$ .

Suppose that  $(P, \preccurlyeq)$  is a poset on  $[d]$  with an edge partition  $\ell = (oE(P), cE(P))$ . Let  $P'_\ell$  and  $P''_\ell$  denote the  $d$ -element weak subposets of  $P$  with cover relations given by the edge sets  $oE(P)$  and  $cE(P)$  respectively. Here by a weak subposet of  $P$ , we mean a subset  $Q$  of elements of  $P$  and a partial ordering  $\preccurlyeq^*$  of  $Q$  such that if  $x \preccurlyeq^* y$  in  $Q$ , then  $x \preccurlyeq y$  in  $P$ . The *order-chain polytope*  $\mathcal{OC}_\ell(P)$  with respect to the edge partition  $\ell$  of  $P$  is defined to be the convex polytope

$$\mathcal{O}(P'_\ell) \cap \mathcal{C}(P''_\ell)$$

in  $\mathbb{R}^d$ . Clearly the notion of order-chain polytope is a natural generalization of both order polytope and chain polytope of a finite poset.

For example, let  $P$  be the chain  $1 \prec 2 \prec \dots \prec 7$  with

$$oE(P) = \{\{1, 2\}, \{4, 5\}, \{5, 6\}\}, \quad cE(P) = \{\{2, 3\}, \{3, 4\}, \{6, 7\}\}.$$

Then  $P'_\ell$  is the disjoint union of the following four chains:

$$1 \prec 2, \quad 3, \quad 4 \prec 5 \prec 6, \quad 7$$

and  $P''_\ell$  is the disjoint union of

$$1, \quad 2 \prec 3 \prec 4, \quad 5 \quad \text{and} \quad 6 \prec 7.$$

Hence the order-chain polytope  $\mathcal{OC}_\ell(P)$  is the convex polytope consisting of those  $(x_1, \dots, x_7) \in \mathbb{R}^7$  such that

- (1)  $0 \leq x_i \leq 1$  for  $1 \leq i \leq 7$ ;
- (2)  $x_1 \geq x_2, x_4 \geq x_5 \geq x_6$ ;
- (3)  $x_2 + x_3 + x_4 \leq 1, x_6 + x_7 \leq 1$ .

It should be noted that, for any poset  $P$  on  $[d]$  and any edge partition  $\ell$  of  $P$ , the dimension of the order-chain polytope  $\mathcal{OC}_\ell(P)$  is equal to  $d$ . In fact, let  $x = (1/d, \dots, 1/d) \in \mathbb{R}^d$ , clearly, we have  $x \in \mathcal{OC}_\ell(P)$ . If  $P'_\ell$  is an antichain, then  $\mathcal{O}(P'_\ell)$  is the  $d$ -cube  $[0, 1]^d$ . In this case,  $\mathcal{OC}_\ell(P)$  is exactly the same as the chain polytope  $\mathcal{C}(P)$  and so is  $d$ -dimensional. If  $P'_\ell$  is not an antichain, then  $P''_\ell$  is not a  $d$ -element chain. In this case,  $x \in \partial\mathcal{O}(P'_\ell)$  and  $x \in \mathcal{C}(P''_\ell) \setminus \partial\mathcal{C}(P''_\ell)$ , since no facet hyperplane of  $\mathcal{C}(P''_\ell)$  contains  $x$ . In this case, we can find a ball  $B_d(x)$  centered at  $x$  such that  $B_d(x) \subset \mathcal{C}(P''_\ell) \setminus \partial\mathcal{C}(P''_\ell)$ . Keeping in mind that  $x$  belongs to the boundary of  $\mathcal{O}(P'_\ell)$ , we deduce that  $B_d(x) \cap (\mathcal{O}(P'_\ell) \setminus \partial\mathcal{O}(P'_\ell)) \neq \emptyset$ . It follows that  $(\mathcal{O}(P'_\ell) \setminus \partial\mathcal{O}(P'_\ell)) \cap (\mathcal{C}(P''_\ell) \setminus \partial\mathcal{C}(P''_\ell)) \neq \emptyset$ , as desired.

Recall that an integral convex polytope (a convex polytope is *integral* if all of its vertices have integer coordinates) is called *compressed* ([15]) if all of its “pulling triangulations” are unimodular. Equivalently, a compressed polytope is an integral convex polytope any of whose reverse lexicographic initial ideals are squarefree ([17]). It follows from [13, Theorem 1.1] that all order polytopes and all chain polytopes are compressed. Hence the intersection of an order polytope and a chain polytope is compressed if it is integral. In particular every integral order-chain polytope is compressed. It then follows that every integral order-chain polytope possesses a unimodular triangulation and is normal ([12]).

Then one of the natural question, which we study in Section 2, is when an order-chain polytope is integral. We call an edge partition  $\ell$  of a finite poset  $P$  *integral* if the order-chain polytope  $\mathcal{OC}_\ell(P)$  is integral. We show that every edge partition of a finite poset  $P$  is integral if and only if  $P$  is cycle-free. Here by a cycle-free poset  $P$  we mean that the Hasse diagram of  $P$  is a cycle-free graph (i.e., an unoriented graph that does not have cycles). Furthermore, we prove that every poset  $P$  with  $|E(P)| \geq 2$  possesses at least one proper integral edge partition.

In Section 3, we consider the problem when an integral order-chain polytope is unimodularly equivalent to either an order polytope or a chain polytope. This problem is related to the work [9], in which the authors characterize all finite posets  $P$  such that  $\mathcal{O}(P)$  and  $\mathcal{C}(P)$  are unimodularly equivalent. We show that if  $P$  is either a disjoint union of chains or a zigzag poset, then the order-chain polytope  $\mathcal{OC}_\ell(P)$ , with respect to each edge partition  $\ell$  of  $P$ , is unimodularly equivalent to the chain polytope of some poset (Theorem 3.3 and Theorem 3.4). On the other hand, for each positive integer  $d \geq 6$ , we find a  $d$ -dimensional integral order-chain polytope which is not unimodularly equivalent to any chain polytope nor order polytope. This means that the notion of order-chain polytope is a nontrivial generalization of order polytope or chain polytope.

We conclude the present paper with an observation on the volume of order-chain polytopes in Section 4. An interesting question is to find an edge partition  $\ell$  of a poset  $P$  which maximizes the volume of  $\mathcal{OC}_\ell(P)$ . In general, it seems to be very difficult to find a complete answer. We shall discuss the case when  $P$  is a chain on  $[d]$ , which involves Ehrenborg and Mahajan’s problem (see [3]) of maximizing the descent statistics over certain family of subsets.

## 2 Integral order-chain polytopes

In this section, we consider the problem when an order-chain polytope is integral. We shall prove that every edge partition of a poset  $P$  is integral if and only if  $P$  is cycle-free. We also prove that every finite poset  $P$  with  $|E(P)| \geq 2$  has at least one proper integral edge partition.

**Theorem 2.1.** *Let  $P$  be a finite poset. Then every edge partition of  $P$  is integral if and only if  $P$  is a cycle-free poset.*

*Proof.* Suppose that each edge partition  $\ell$  of  $P$  is integral. If the Hasse diagram of  $P$  has a cycle  $C$ , then it is easy to find a non-integral edge partition. In fact, let  $e = \{i, j\}$  be an arbitrary edge from  $C$  and  $\ell = (E(P) \setminus \{e\}, \{e\})$ . We now show that  $\ell$  is not integral. To this end, let  $I$  be the connected component of the Hasse diagram of  $P'_\ell$  which contains  $i$  and  $j$  and let  $v = (v_1, v_2, \dots, v_d) \in \mathbb{R}^d$  with

$$v_k = \begin{cases} \frac{1}{2}, & \text{if } k \in I \\ 0, & \text{otherwise.} \end{cases}$$

Then it is easy to see that

$$v = \bigcap_{\{p,q\} \in E(I)} H_{pq} \bigcap_{t \notin I} H_t \bigcap H_{ij},$$

where

$$\begin{aligned} H_{pq} &= \{(x_1, x_2, \dots, x_d) \mid x_p = x_q\} \text{ for } e = \{p, q\} \in E(I) \\ H_t &= \{(x_1, x_2, \dots, x_d) \mid x_t = 0\} \text{ for } t \notin I \\ H_{ij} &= \{(x_1, x_2, \dots, x_d) \mid x_i + x_j = 1\} \end{aligned}$$

are all facet hyperplanes of  $\mathcal{OC}_\ell(P)$ . So we deduce that  $v$  is a vertex of  $\mathcal{OC}_\ell(P)$ , and  $\ell$  is not integral.

Conversely, suppose that  $P$  is a cycle-free poset on  $[d]$  and  $\ell$  is an edge partition of  $P$ . If  $v = (a_1, a_2, \dots, a_d)$  is a vertex of  $\mathcal{OC}_\ell(P)$ , then we can find  $d$  independent facet hyperplanes of  $\mathcal{OC}_\ell(P)$  such that

$$v = \left( \bigcap_{i=1}^{d-m} H'_i \right) \cap \left( \bigcap_{j=1}^m H''_j \right), \tag{2.1}$$

where  $m = \dim \left( \bigcap_{i=1}^{d-m} H'_i \right)$ , each  $H'_i$  is a facet hyperplane of  $\mathcal{O}(P'_\ell)$  and each  $H''_j$  is a facet hyperplane of  $\mathcal{C}(P''_\ell)$  which corresponds to a chain  $C_j$  of length  $\geq 2$  in  $P''_\ell$ . By [16, Theorem 2.1], there is a set partition  $\pi = \{B_1, B_2, \dots, B_{m+1}\}$  of  $[d]$  such that  $B_1, B_2, \dots, B_m$  are connected as subposets of  $P'_\ell$ ,  $B_{m+1} = \{i \in [d] : a_i = 0 \text{ or } 1\}$  and

$$\bigcap_{i=1}^{d-m} H'_i = \{(x_1, x_2, \dots, x_d) \mid x_i = x_j \text{ if } \{i, j\} \subseteq B_k \text{ for some } 1 \leq k \leq m, \text{ and } x_r = a_r \text{ if } r \in B_{m+1}\}.$$

Let  $B_{m+1} = \{r_1, r_2, \dots, r_s\}$  and for  $1 \leq k \leq m$ , let  $b_k$  denote the same values of all  $a'_i$ 's,  $i \in B_k$ . Then it suffices to show that each  $b_k$  is an integer. Keeping in mind the assumption that the Hasse diagram of  $P$  is cycle-free, we find that  $|C_i \cap B_j| \leq 1$  for  $1 \leq i, j \leq m$ . For  $1 \leq i, j \leq m$ , let

$$c_{ij} = \begin{cases} 1, & \text{if } |C_i \cap B_j| = 1 \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

and for  $1 \leq i \leq m, 1 \leq j \leq s$ , let

$$d_{i,m+j} = \begin{cases} 1, & \text{if } r_j \in C_i \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

By (2.1),  $(b_1, b_2, \dots, b_m, a_{r_1}, a_{r_2}, \dots, a_{r_s})$  must be the unique solution of the following linear system:

$$\begin{cases} \sum_{j=1}^m c_{ij}y_j + \sum_{j=m+1}^{m+s} d_{ij}y_j = 1, & 1 \leq i \leq m \\ y_{m+1} = a_{r_1} \\ y_{m+2} = a_{r_2} \\ \vdots \\ y_{m+s} = a_{r_s}. \end{cases} \tag{2.4}$$

Now it suffices to show that the determinant of the coefficient matrix

$$A = \begin{pmatrix} c_{11} & \cdots & c_{1m} & d_{1,m+1} & \cdots & d_{1,m+s} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mm} & d_{m,m+1} & \cdots & d_{m,m+s} \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{2.5}$$

is equal to 1 or  $-1$ . Now construct a bipartite graph  $G$  with vertex set

$$\{B_1, B_2, \dots, B_m, C_1, C_2, \dots, C_m\},$$

and edge set

$$\{\{B_i, C_j\} \mid 1 \leq i, j \leq m, |B_i \cap C_j| = 1\}.$$

Let

$$C = \begin{pmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \vdots \\ c_{m1} & \cdots & c_{mm} \end{pmatrix}.$$

Then we have

$$\det(C) = \sum_{\sigma \in \mathfrak{S}_m} \text{sign}(\sigma) c_{1\sigma_1} \cdots c_{m\sigma_m}. \tag{2.6}$$

Clearly, each nonzero term in (2.6) corresponds to a perfect matching in the graph  $G$ . Since the Hasse diagram of  $P$  is cycle-free, the graph  $G$  must be a cycle-free bipartite graph, which means that there is at most one perfect matching in  $G$ . So we have  $\det(C) = 0, 1$  or  $-1$ . Note that the linear equations (2.4) has unique solution  $(b_1, b_2, \dots, b_m, a_{r_1}, \dots, a_{r_s})$ . Then we find that  $\det(C) = \pm 1$ . It follows that each  $b_i$  is an integer. So the vertex  $v$  of  $\mathcal{OC}_\ell(P)$  is integral.  $\square$

For a general finite poset  $P$  with  $|E(P)| \geq 2$ , the following theorem indicates that there exists at least one proper integral edge partition.

**Theorem 2.2.** *Suppose that  $P$  is a finite poset. Let  $\text{Min}(P)$  denote the set of all minimal elements in  $P$ . For  $S \subseteq \text{Min}(P)$ , let  $E_S(P)$  denote the set of all edges in  $E(P)$  which are incident to some elements in  $S$ . Then the edge partition*

$$\ell = (E(P) \setminus E_S(P), E_S(P))$$

*is integral.*

*Proof.* Suppose that  $v$  is a vertex of  $\mathcal{OC}_\ell(P)$ . Then  $v$  can be represented as intersection of  $d$  independent facet hyperplanes, as in (2.1). Keeping the notation in the proof of Theorem 2.1, we can deduce that  $|C_i| = 2$  and  $|B_i \cap C_j| \leq 1$  for  $1 \leq i, j \leq m$ . So we can construct in the same way two matrices  $A$  and  $C$  as those in the proof of Theorem 2.1. Then, we can construct a graph  $G$  with vertex set  $\{B_1, B_2, \dots, B_m, r_1, r_2, \dots, r_s\}$  and

edge set determined by  $C_1, C_2, \dots, C_m$ . More precisely,  $\{B_i, B_j\}$  is an edge of  $G$  if and only if there exists  $1 \leq k \leq m$  such that  $C_k = \{i', j'\}$  for some  $i' \in B_i, j' \in B_j$ , and  $\{B_i, r_j\}$  is an edge of  $G$  if and only if there exists  $1 \leq k \leq m$  such that  $C_k = \{r_j, i'\}$  for some  $i' \in B_i$ . Obviously,  $G$  is a bipartite graph with bipartition  $(\mathcal{B}_1, \mathcal{B}_2)$ , where

$$\mathcal{B}_2 = \{B_j : 1 \leq j \leq m, B_j = \{k\} \text{ for some } k \in S\} \cup \{r_t : 1 \leq t \leq s, r_t \in S\},$$

and

$$\mathcal{B}_1 = \{B_1, B_2, \dots, B_m, r_1, r_2, \dots, r_s\} \setminus \mathcal{B}_2.$$

Moreover, by the construction of the graph  $G$ , its incidence matrix is

$$\begin{pmatrix} c_{11} & \cdots & c_{1m} & d_{1,m+1} & \cdots & d_{1,m+s} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mm} & d_{m,m+1} & \cdots & d_{m,m+s} \end{pmatrix}$$

where  $c_{ij}, d_{i,m+j}$  are defined in (2.2) and in (2.3) respectively. A well known fact shows that the incidence matrix of any bipartite graph is totally unimodular (a matrix  $A$  is *totally unimodular* if every square submatrix has determinant 0, +1, or -1). So the submatrix  $C$  has determinant 0, 1 or -1. This completes the proof.  $\square$

**Example 2.3.** By Theorem 2.1, if the Hasse diagram of  $P$  has a cycle, then there exists at least one non-integral edge partition  $\ell$ .

- (1) For example, let  $P$  denote the poset whose Hasse diagram is a 4-cycle (see Figure 1) and let  $E_1 = \{\{1, 2\}, \{2, 4\}, \{3, 4\}\}$ . Then the edge partition  $\ell_1 = (E_1, \{1, 3\})$  is non-integral, since  $v = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  is a vertex of  $\mathcal{OC}_{\ell_1}(P)$  given by

$$\begin{cases} x_1 = x_2 = x_4 = x_3 \\ x_1 + x_3 = 1. \end{cases}$$

However, it is easy to see that the edge partition  $\ell_2 = (\{1, 3\}, E_1)$  is integral. So we find that the complementary edge partition  $\ell^c = (cE(P), oE(P))$  of an integral edge partition  $\ell = (oE(P), cE(P))$  is not necessarily integral.

- (2) For any poset  $P$  whose Hasse diagram is a cycle and any edge partition  $\ell$  of  $P$ , it is not hard to show that all coordinates of each vertex of  $\mathcal{OC}_{\ell}(P)$  are 0, 1 or  $\frac{1}{2}$ .

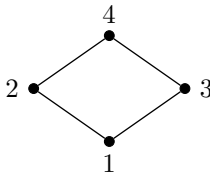


Figure 1: Hasse diagram of poset  $P$  from Example 2.3.



### 3 Unimodular equivalence

In this section, we shall compare the newly constructed order-chain polytopes with some known polytopes. Specifically, we will focus on integral order-chain polytopes and consider their unimodular equivalence relation with order polytopes or chain polytopes.

Recall (see, for example, [9]) that a  $d \times d$  integral matrix  $U$  is *unimodular* if  $\det(U) = \pm 1$ . A map  $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a *unimodular transformation* if there exist a  $d \times d$  unimodular matrix  $U$  and an integral vector  $w \in \mathbb{Z}^d$  such that  $\varphi(v) = vU + w$ . Two integral polytopes  $\mathcal{P}$  and  $\mathcal{Q}$  in  $\mathbb{R}^d$  are *unimodularly equivalent* if there exists a unimodular transformation  $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that  $\mathcal{Q} = \varphi(\mathcal{P})$ . Much of the importance of unimodular equivalence arises from the fact that combinatorial type and Ehrhart polynomial of an integral polytope are invariant modulo unimodular equivalence. For instance, classification of polytopes with certain properties (modulo unimodular equivalence) has gained some attentions recently (see, for example, [1, 10, 11]).

We shall use the ideas in the proof of the following theorem due to Hibi and Li [9].

**Theorem 3.1** ([9, Theorem 1.3]). *The order polytope  $\mathcal{O}(P)$  and the chain polytope  $\mathcal{C}(P)$  of a finite poset  $P$  are unimodularly equivalent if and only if the poset shown in Figure 2 does not appear as a subposet of  $P$ .*

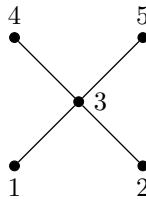


Figure 2: The “forbidden” poset from Theorem 3.1.

**Definition 3.2.** A poset  $P$  on  $[d]$  is said to be a zigzag poset if its cover relations are given by

$$1 \prec \cdots \prec i_1 \succ i_1 + 1 \succ \cdots \succ i_2 \prec i_2 + 1 \prec \cdots \prec i_3 \succ \cdots \succ i_k \prec i_k + 1 \prec \cdots \prec d$$

for some  $0 \leq i_1 < i_2 < \cdots < i_k \leq d$ .

**Theorem 3.3.** *Suppose that  $P$  is a disjoint union of chains. Then for any edge partition  $\ell$ , the order-chain polytope  $\mathcal{OC}_\ell(P)$  is unimodularly equivalent to a chain polytope  $\mathcal{C}(Q)$ , where  $Q$  is a disjoint union of zigzag posets.*

*Proof.* We firstly assume that  $P$  is a chain:

$$1 \prec 2 \prec 3 \prec \cdots \prec d.$$

and  $\ell$  is an edge partition of  $P$  given by:

$$\begin{aligned}
 o: & \quad 1 \prec 2 \prec \cdots \prec i_1 \\
 c: & \quad i_1 \prec i_1 + 1 \prec \cdots \prec i_2 \\
 o: & \quad i_2 \prec i_2 + 1 \prec \cdots \prec i_3 \\
 & \quad \vdots \\
 c: & \quad i_{t-1} \prec i_{t-1} + 1 \prec \cdots \prec i_t \\
 o: & \quad i_t \prec i_t + 1 \prec \cdots \prec i_{t+1} \\
 & \quad \vdots \\
 c: & \quad i_{k-1} \prec i_{k-1} + 1 \prec \cdots \prec i_k = d,
 \end{aligned}$$

where  $1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq i_k = d$ . Then the order-chain polytope  $\mathcal{OC}_\ell(P)$  is given by

$$\left\{ \begin{array}{l}
 x_1 \geq x_2 \geq \cdots \geq x_{i_1} \\
 x_{i_1} + x_{i_1+1} + \cdots + x_{i_2} \leq 1 \\
 x_{i_2} \geq x_{i_2+1} \geq \cdots \geq x_{i_3} \\
 \vdots \\
 x_{i_{t-1}} + x_{i_{t-1}+1} + \cdots + x_{i_t} \leq 1 \\
 x_{i_t} \geq x_{i_t+1} \geq \cdots \geq x_{i_{t+1}} \\
 \vdots \\
 x_{i_{k-1}} + x_{i_{k-1}+1} + \cdots + x_d \leq 1 \\
 0 \leq x_i \leq 1, \quad 1 \leq i \leq d.
 \end{array} \right. \tag{3.1}$$

Now define a map  $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}^d$  as follows:

- (1) if  $i$  is a maximal element in  $P'_\ell$ , then let  $x'_i = x_i$ ;
- (2) if  $i$  is not a maximal element in  $P'_\ell$ , then  $\{i, i + 1\}$  must be an edge in the Hasse diagram of  $P'_\ell$ . Let  $x'_i = x_i - x_{i+1}$ .

Let  $\varphi(x_1, x_2, \dots, x_d) = (x'_1, x'_2, \dots, x'_d)$ .

Now it is easy to show that  $\varphi$  is a unimodular transformation. Moreover, the system (3.1) is transformed into:

$$\left\{ \begin{array}{l}
 x'_1 + x'_2 + \cdots + x'_{i_1} \leq 1 \\
 x'_{i_1} + x'_{i_1+1} + \cdots + x'_{i_2} + x'_{i_2+1} + \cdots + x'_{i_3} \leq 1 \\
 \vdots \\
 x'_{i_{t-1}} + x'_{i_{t-1}+1} + \cdots + x'_{i_t} + x'_{i_t+1} + \cdots + x'_{i_{t+1}} \leq 1 \\
 \vdots \\
 x'_{i_{k-1}} + x'_{i_{k-1}+1} + \cdots + x'_d \leq 1 \\
 0 \leq x'_i \leq 1, \quad 1 \leq i \leq d.
 \end{array} \right.$$

Obviously, this system corresponds to the chain polytope  $\mathcal{C}(Q)$  for the zigzag poset  $Q$ :

$$1 \prec 2 \prec \cdots \prec i_1 \succ i_1 + 1 \succ \cdots \succ i_2 \succ i_2 + 1 \succ \cdots \succ i_3 \prec \cdots$$

or the dual zigzag poset  $Q^*$ :

$$1 \succ 2 \succ \cdots \succ i_1 \prec i_1 + 1 \prec \cdots \prec i_2 \prec i_2 + 1 \prec \cdots \prec i_3 \succ \cdots$$

So we deduce that  $\mathcal{OC}_\ell(P)$  is unimodularly equivalent to the chain polytope of some zigzag poset.

Now we continue to prove the general case that  $P$  is a disjoint union of  $k$  chains:

$$P = C_1 \uplus C_2 \uplus \cdots \uplus C_k.$$

Since

$$\mathcal{O}(P \uplus Q) = \mathcal{O}(P) \times \mathcal{O}(Q) \quad \text{and} \quad \mathcal{C}(P \uplus Q) = \mathcal{C}(P) \times \mathcal{C}(Q),$$

we have

$$\begin{aligned} \mathcal{OC}_\ell(P \uplus Q) &= \mathcal{O}((P \uplus Q)'_\ell) \cap \mathcal{C}((P \uplus Q)''_\ell) \\ &= \mathcal{O}(P'_\ell \uplus Q'_\ell) \cap \mathcal{C}(P''_\ell \uplus Q''_\ell) \\ &= [\mathcal{O}(P'_\ell) \times \mathcal{O}(Q'_\ell)] \cap [\mathcal{C}(P''_\ell) \times \mathcal{C}(Q''_\ell)] \\ &= [\mathcal{O}(P'_\ell) \cap \mathcal{C}(P''_\ell)] \times [\mathcal{O}(Q'_\ell) \cap \mathcal{C}(Q''_\ell)] \\ &= \mathcal{OC}_\ell(P) \times \mathcal{OC}_\ell(Q). \end{aligned} \tag{3.2}$$

Hence we conclude that

$$\begin{aligned} \mathcal{OC}_\ell(C_1 \uplus \cdots \uplus C_k) &= \mathcal{OC}_\ell(C_1) \times \cdots \times \mathcal{OC}_\ell(C_k) \\ &\cong^{\varphi_1 \times \cdots \times \varphi_k} \mathcal{C}(Q_1) \times \cdots \times \mathcal{C}(Q_k) \\ &= \mathcal{C}(Q_1 \uplus \cdots \uplus Q_k), \end{aligned}$$

where  $Q_i$  are zigzag posets. □

Similarly, we can modify the proof of Theorem 3.3 slightly to get the following result:

**Theorem 3.4.** *Suppose that  $P$  is a finite zigzag poset. Then for any edge partition  $\ell$ , the order-chain polytope  $\mathcal{OC}_\ell(P)$  is unimodularly equivalent to a chain polytope  $\mathcal{C}(Q)$  for some zigzag poset  $Q$ .*

*Proof.* Suppose that  $P$  is a zigzag poset on  $[d]$  and  $\ell$  is an edge partition of  $P$ . Define a map  $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}^d$  as follows:

- (1) if  $i$  is covered by at most one element in  $P'_\ell$ , let

$$x'_i = \begin{cases} x_i, & \text{if } i \text{ is a maximal element in } P'_\ell \\ x_i - x_j, & \text{if } i \text{ is covered by } j \text{ in } P'_\ell \quad (j = i - 1 \text{ or } i + 1). \end{cases}$$

- (2) if  $i$  is covered by both  $i - 1$  and  $i + 1$  in  $P'_\ell$ , let

$$x'_i = 1 - x_i.$$

Let

$$\varphi(x_1, x_2, \dots, x_d) = (x'_1, x'_2, \dots, x'_d).$$

It is not hard to show that  $\varphi$  is the desired unimodular transformation. □

The following example shows that not every order-chain polytope  $\mathcal{OC}_\ell(P)$  of a cycle-free poset  $P$  is unimodularly equivalent to some chain polytope.

**Example 3.5.** Let  $P$  be the poset shown in Figure 2 with an edge partition

$$\ell = (\{\{1, 3\}, \{3, 4\}, \{3, 5\}\}, \{2, 3\}).$$

Let

$$\varphi(x_1, x_2, x_3, x_4, x_5) = (x_1, 1 - x_2, x_3, x_4, x_5).$$

It is obvious that  $\varphi$  is a unimodular transformation and  $\varphi(\mathcal{OC}_\ell(P)) = \mathcal{O}(P)$ . However, by checking all 63 different non-isomorphic posets with 5 elements, we find that  $\mathcal{O}(P)$  is not equivalent to any chain polytope.

Furthermore, for any  $d \geq 6$ , we shall find an integral order-chain polytope in  $\mathbb{R}^d$  which is not unimodularly equivalent to any chain polytope or order polytope. To this end, we need the following lemma.

**Lemma 3.6.**

- (1) None of the chain polytopes of finite posets on  $[d]$  possesses  $d + 4$  vertices and  $d + 7$  facets.
- (2) None of the order polytopes of finite posets on  $[d]$  possesses  $d + 4$  vertices and  $d + 7$  facets.

*Proof.* (1) Assume, by contradiction, that  $P$  is a finite poset on  $[d]$  such that  $\mathcal{C}(P)$  has  $d + 4$  vertices and  $d + 7$  facets. Since the vertices of  $\mathcal{C}(P)$  are those  $\rho(A)$  for which  $A$  is an antichain of  $P$ , we can deduce that  $P$  possesses exactly  $d + 4$  antichains. Keeping in mind that  $\emptyset, \{1\}, \dots, \{d\}$  are antichains of  $P$ , we find that there is no antichain  $A$  in  $P$  with  $|A| \geq 3$ . Otherwise, the number of antichains of  $P$  is at least  $d + 5$ . It then follows that there are exactly three 2-element antichains in  $P$ . We need to consider the following four cases:

- (i) Let, say,  $\{1, 2\}, \{1, 3\}, \{1, 4\}$  be the 2-element antichains of  $P$ . Then the maximal chains of  $P$  are  $P \setminus \{1\}$  and  $P \setminus \{2, 3, 4\}$ .
- (ii) Let, say,  $\{1, 2\}, \{1, 3\}, \{2, 4\}$  be the 2-element antichains of  $P$ . Then the maximal chains of  $P$  are  $P \setminus \{1, 2\}, P \setminus \{1, 4\}$  and  $P \setminus \{2, 3\}$ .
- (iii) Let, say,  $\{1, 2\}, \{1, 3\}, \{4, 5\}$  be the 2-element antichains of  $P$ . Then the maximal chains of  $P$  are  $P \setminus \{1, 4\}, P \setminus \{1, 5\}, P \setminus \{2, 3, 4\}$  and  $P \setminus \{2, 3, 5\}$ .
- (iv) Let, say,  $\{1, 2\}, \{3, 4\}, \{5, 6\}$  be the 2-element antichains of  $P$ . It can be shown easily that  $P$  possesses exactly eight maximal chains.

Recall that the number of facets of  $\mathcal{C}(P)$  is equal to  $d + c(P)$ , it follows from the assumption that there are exactly 7 maximal chains in  $P$ , which is a contradiction. As a result, none of the chain polytopes  $\mathcal{C}(P)$  of a finite poset  $P$  on  $[d]$  with  $d + 4$  vertices can possess  $d + 7$  facets, as desired.

(2) Let  $P$  be a finite poset on  $[d]$  and suppose that the number of vertices of  $\mathcal{O}(P)$  is  $d + 4$  and the number of facets of  $\mathcal{O}(P)$  is  $d + 7$ . Since the number of vertices of  $\mathcal{O}(P)$  and that of  $\mathcal{C}(P)$  coincide, it follows from the proof of (a) that there is no antichain  $A$  in  $P$  with  $|A| \geq 3$  and that  $P$  includes exactly three 2-element antichains. On the other hand, it is known [9, Corollary 1.2] that the number of facets of  $\mathcal{O}(P)$  is less than or equal to that of  $\mathcal{C}(P)$ . Hence the number of maximal chains of  $P$  is at least 7. Thus, by using the argument in the proof of (a), we can assume that the antichains of  $P$  are  $\{1, 2\}$ ,  $\{3, 4\}$  and  $\{5, 6\}$ . Then, it is easy to prove that the number  $|E(\hat{P})|$  of edges in the Hasse diagram of  $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$  is at most  $d + 6$ . So we deduce that the number of facets of  $\mathcal{O}(P)$  is at most  $d + 6$ , a contradiction with the assumption.  $\square$

We remark that, by modifying the argument of the statement (1) in Lemma 3.6, we can prove directly that the order polytope of Example 3.5 cannot be unimodularly equivalent to any chain polytope.

**Example 3.7.** Let  $P$  be the finite poset shown in Figure 3. Let  $\ell$  be the edge partition with

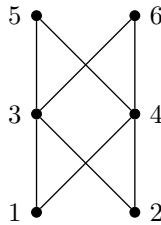


Figure 3: Poset  $P$  from Example 3.7.

$oE(P) = \{\{3, 5\}, \{3, 6\}\}$  and  $cE(P) = E(P) \setminus oE(P)$ . Then it is easy to verify that  $\mathcal{OC}_\ell(P)$  is an integral polytope with 10 vertices and 13 facets. (Since the number of facets of the order-chain polytope is small, we can compute this by hand. Of course, we can also compute this by using the software `polymake` [4].) So it follows from Lemma 3.6 that the integral order-chain polytope  $\mathcal{OC}_\ell(P)$  cannot be unimodularly equivalent to any order polytope or any chain polytope.

In fact, for any  $d > 6$ , let  $P_d$  be the poset shown in Figure 4 and let  $\ell$  be the edge partition with

$$oE(P_d) = \{\{3, 5\}, \{3, 6\}, \{5, 7\}, \{6, 7\}, \{7, 8\}, \dots, \{d - 1, d\}\}.$$

It is easy to see that the order-chain polytope  $\mathcal{OC}_\ell(P_d)$  has  $d + 4$  vertices and  $d + 7$  facets. Therefore  $\mathcal{OC}_\ell(P_d)$  cannot be unimodularly equivalent to any order polytope or any chain polytope.

Recall that Example 3.5 shows that there is an order polytope which is not unimodularly equivalent to any chain polytope. To conclude this section, we will prove that, for each  $d \geq 9$ , there exists a finite poset  $P$  on  $[d]$  for which the chain polytope  $\mathcal{C}(P)$  cannot be unimodularly equivalent to any order polytope.

Recall that, for a finite poset  $P$  on  $[d]$ , we have

$$f_{d-1}(\mathcal{O}(P)) = m_\star(P) + m^\star(P) + |E(P)|$$

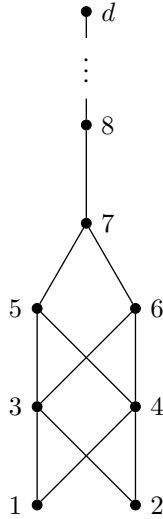


Figure 4: Poset  $P_d$  from Example 3.7.

and

$$f_{d-1}(\mathcal{C}(P)) = d + c(P),$$

To present our results, we firstly discuss upper bounds for  $f_{d-1}(\mathcal{O}(P))$  and  $f_{d-1}(\mathcal{C}(P))$ . By [9, Theorem 2.1], if  $d \leq 4$ , then  $\mathcal{O}(P)$  and  $\mathcal{C}(P)$  are unimodularly equivalent and  $f_{d-1}(\mathcal{O}(P)) = f_{d-1}(\mathcal{C}(P)) \leq 2d$ . Moreover, for each  $1 \leq d \leq 4$ , there exists a finite poset  $P$  on  $[d]$  with  $f_{d-1}(\mathcal{O}(P)) = f_{d-1}(\mathcal{C}(P)) = 2d$ .

**Lemma 3.8.** *Let  $d \geq 5$  and  $P$  be a finite poset on  $[d]$ . Then*

$$f_{d-1}(\mathcal{O}(P)) \leq \left\lfloor \frac{d+1}{2} \right\rfloor \left( d - \left\lfloor \frac{d+1}{2} \right\rfloor \right) + d \tag{3.3}$$

and

$$f_{d-1}(\mathcal{C}(P)) \leq \begin{cases} 3^k + d, & d = 3k \\ 4 \cdot 3^{k-1} + d, & d = 3k + 1 \\ 2 \cdot 3^k + d, & d = 3k + 2. \end{cases} \tag{3.4}$$

Furthermore, both upper bounds for  $f_{d-1}(\mathcal{O}(P))$  and  $f_{d-1}(\mathcal{C}(P))$  are tight.

*Proof. (Order polytope)* Let  $d = 4$ . Since the right-hand side of (3.3) is equal to  $2d (= 8)$ , the inequality (3.3) also holds for  $d = 4$ . Let  $d \geq 5$  and  $P$  be a finite poset on  $[d]$ . We will prove (3.3) by induction on  $d$ . Suppose that 1 is a minimal element of  $P$  and let  $a$  be the number of elements in  $P$  which cover 1.

If  $a = 0$ , then  $\mathcal{O}(P) = \mathcal{O}(P \setminus \{1\}) \times [0, 1]$  and so

$$\begin{aligned} f_{d-1}(\mathcal{O}(P)) &= f_{d-2}(\mathcal{O}(P \setminus \{1\})) + 2 \\ &\leq \left\lfloor \frac{d}{2} \right\rfloor \left( d - 1 - \left\lfloor \frac{d}{2} \right\rfloor \right) + d - 1 + 2 \\ &\leq \left\lfloor \frac{d+1}{2} \right\rfloor \left( d - \left\lfloor \frac{d+1}{2} \right\rfloor \right) + d. \end{aligned}$$

If  $1 \leq a \leq \lfloor d/2 \rfloor$ , then from the facts that  $|E(P \setminus \{1\})| = |E(P)| - a$ ,  $m_*(P \setminus \{1\}) \geq m_*(P) - 1$  and  $m^*(P \setminus \{1\}) = m^*(P)$ , we have

$$\begin{aligned} f_{d-1}(\mathcal{O}(P)) &= m^*(P) + m_*(P) + |E(P)| \\ &\leq m^*(P \setminus \{1\}) + m_*(P \setminus \{1\}) + 1 + |E(P \setminus \{1\})| + a \\ &\leq \left\lfloor \frac{d}{2} \right\rfloor \left( d - 1 - \left\lfloor \frac{d}{2} \right\rfloor \right) + (d - 1) + \left\lfloor \frac{d}{2} \right\rfloor + 1 \\ &\leq \left\lfloor \frac{d+1}{2} \right\rfloor \left( d - \left\lfloor \frac{d+1}{2} \right\rfloor \right) + d. \end{aligned}$$

Now we consider the case  $\lfloor d/2 \rfloor + 1 \leq a \leq d - 1$ . Let, say, 2 be an element of  $P$  which covers 1. Since the set of the elements of  $P$  which cover 1 is an antichain of  $P$ , it follows that  $|E(P \setminus \{2\})| \geq |E(P)| - (d - a)$ ,  $m_*(P \setminus \{2\}) \geq m_*(P)$  and  $m^*(P \setminus \{2\}) \geq m^*(P) - 1$ . Hence

$$\begin{aligned} f_{d-1}(\mathcal{O}(P)) &= m^*(P) + m_*(P) + |E(P)| \\ &\leq m^*(P \setminus \{2\}) + 1 + m_*(P \setminus \{2\}) + |E(P \setminus \{2\})| + (d - a) \\ &\leq \left\lfloor \frac{d}{2} \right\rfloor \left( d - 1 - \left\lfloor \frac{d}{2} \right\rfloor \right) + (d - 1) + \left( d - \left\lfloor \frac{d}{2} \right\rfloor - 1 \right) + 1 \\ &\leq \left\lfloor \frac{d+1}{2} \right\rfloor \left( d - \left\lfloor \frac{d+1}{2} \right\rfloor \right) + d. \end{aligned}$$

Therefore, the inequality (3.3) holds. We proceed to show that this upper bound for  $f_{d-1}(\mathcal{O}(P))$  is tight. In fact, let  $P$  be the finite poset  $P$  on  $[d]$  with

$$E(P) = \left\{ \{i, j\} \in [d] \times [d] : 1 \leq i \leq \left\lfloor \frac{d+1}{2} \right\rfloor, \left\lfloor \frac{d+1}{2} \right\rfloor + 1 \leq j \leq d \right\}.$$

Clearly, we have

$$f_{d-1}(\mathcal{O}(P)) = \left\lfloor \frac{d+1}{2} \right\rfloor \left( d - \left\lfloor \frac{d+1}{2} \right\rfloor \right) + d.$$

**(Chain polytope)** Let  $d \geq 5$ . Let  $P_1$  be a finite poset on  $[d]$  and  $M_1$  the set of minimal elements of  $P_1$ . If  $P_1$  is an antichain, then  $f_{d-1}(\mathcal{C}(P_1)) = 2d$ . Suppose that  $P_1$  is not an antichain. Let  $P_2 = P_1 \setminus M_1$  and  $M_2$  be the set of minimal elements of  $P_2$ . In general, if  $P_i$  is not an antichain and  $M_i$  is the set of minimal element of  $P_i$ , then we set  $P_{i+1} = P_i \setminus M_i$ . By continuing this construction, we can get an integer  $r \geq 1$  such that each of the  $P_1, \dots, P_{r-1}$  is not an antichain and that  $P_r$  is an antichain. Let  $P$  be the finite poset

on  $[d]$  such that  $i_1 \prec i_2 \prec \dots \prec i_r$  if  $i_j \in M_j$  for  $1 \leq j \leq r$ . One has  $c(P_1) \leq c(P) = |M_1| \cdots |M_r|$ . For any integer  $d \geq 5$ , let

$$M(d) = \max \left\{ \prod_{i=1}^r m_i : 1 \leq r \leq d, m_1 + m_2 + \dots + m_r = d, m_i \in \mathbb{N}^+ \right\}.$$

Then the desired inequalities (3.4) follows immediately from the following claim:

$$M(d) = \begin{cases} 3^k, & d = 3k \\ 4 \cdot 3^{k-1}, & d = 3k + 1 \\ 2 \cdot 3^k, & d = 3k + 2. \end{cases} \tag{3.5}$$

So it suffices to prove this claim. Since for any integer  $m \geq 4$ ,

$$m \leq \left\lfloor \frac{m+1}{2} \right\rfloor \left( m - \left\lfloor \frac{m+1}{2} \right\rfloor \right),$$

we can assume that, to maximize the product  $\prod_{i=1}^r m_i$ , all parts  $m_i \leq 3$ . We can also assume without loss of generality that there are at most two  $m_i$ s that are equal to 2, since  $2^3 < 3^2$ . Then the claim (3.5) follows immediately.

Finally, for each  $d \geq 5$ , the existence of a finite poset  $P$  on  $[d]$  for which the equality holds in (3.4) follows easily from the above argument. □

**Remark 3.9.** The special case  $d = 1976$  of claim (3.5) is exactly the problem 4 in the International Mathematical Olympiad (IMO) in 1976, where the maximum value of a product of positive integers summing up to 1976 is asked for. The answer is  $2 \cdot 3^{658}$  since  $1976 = 3 \cdot 658 + 2$ .

A routine computation shows that, for each  $1 \leq d \leq 8$ , the right-hand side of (3.3) coincides with that of (3.4) and that, for each  $d \geq 9$ , the right-hand side of (3.3) is strictly less than that of (3.4). Hence

**Corollary 3.10.** *For each  $d \geq 9$ , there exists a finite poset  $P$  on  $[d]$  for which the chain polytope  $\mathcal{C}(P)$  cannot be unimodularly equivalent to any order polytope.*

### 4 Volumes of $\mathcal{OC}_\ell(P)$

Given a poset  $P$  on  $[d]$ , Corollary 4.2 in [16] shows that the volumes of  $\mathcal{O}(P)$  and  $\mathcal{C}(P)$  are given by

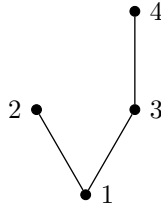
$$V(\mathcal{O}(P)) = V(\mathcal{C}(P)) = \frac{e(P)}{d!},$$

where  $e(P)$  is the number of linear extensions of  $P$ . (Recall that a linear extension of  $P$  is a permutation  $\pi = \pi_1 \pi_2 \cdots \pi_d$  of  $[d]$  such that  $\pi^{-1}(i) < \pi^{-1}(j)$  if  $i \prec j$  in  $P$ .)

For order-chain polytopes, different edge partitions usually give rise to polytopes with



different volumes. For example, let  $P$  be the poset as follows:



It is easy to see that

$$V(\mathcal{O}(P)) = V(\mathcal{C}(P)) = \frac{3}{4!}.$$

Let

$$\ell = (\{1, 2\}, \{\{1, 3\}, \{3, 4\}\}), \quad \ell' = (\{\{1, 2\}, \{1, 3\}\}, \{3, 4\}),$$

then we have

$$V(\mathcal{OC}_\ell(P)) = \frac{1}{4!} \quad \text{and} \quad V(\mathcal{OC}_{\ell'}(P)) = \frac{5}{4!}.$$

Hence one has the following inequality:

$$V(\mathcal{OC}_\ell(P)) < V(\mathcal{O}(P)) = V(\mathcal{C}(P)) < V(\mathcal{OC}_{\ell'}(P)).$$

It should be noted that, for an arbitrary poset  $P$ , we can not always find edge partitions such that this inequality holds. For example, if  $P$  is a chain, then there is no edge partition  $\ell$  such that  $V(\mathcal{OC}_\ell(P)) < V(\mathcal{O}(P)) = V(\mathcal{C}(P))$ . Then a natural question is to ask which edge partition  $\ell$  gives rise to an order-chain polytope with maximum volume. It seems very difficult to solve this problem in general case. In this section, we consider the special case when  $P$  is a chain  $P$  on  $[d]$ . We transform it to a problem of maximizing descent statistics over certain family of subsets. For references on this topic, we refer the reader to [3] and [14].

Let  $P$  be a chain on  $[d]$ . By the proof of Theorem 3.3, for an edge partition  $\ell$  of  $P$ , the order-chain polytope  $\mathcal{OC}_\ell(P)$  is unimodularly equivalent to a chain polytope  $\mathcal{C}(P_1)$ , where  $P_1$  is a zigzag poset such that all maximal chains, except the first one (containing 1) and the last one (containing  $d$ ), consist of at least three elements. So we have

$$V(\mathcal{OC}_\ell(P)) = V(\mathcal{C}(P_1)) = \frac{e(P_1)}{d!}.$$

Conversely, for such a zigzag poset  $P_1$ , it is easy to find an edge partition  $\ell$  of  $P$  such that  $\mathcal{OC}_\ell(P)$  is unimodularly equivalent to  $\mathcal{C}(P_1)$ . Denote by  $\mathcal{Z}(d)$  the set of such zigzag posets  $P_1$  on  $[d]$ . Thus, to compute the maximum volume over all order-chain polytopes of the chain  $P$ , it suffices to compute the maximum number of linear extensions for all zigzag posets  $P_1 \in \mathcal{Z}(d)$ . Next we shall represent this problem as a problem of maximizing descent statistic over a certain class of subsets. To this end, we recall some notions and basic facts. Given a permutation  $\pi = \pi_1\pi_2 \cdots \pi_d$ , let  $\text{Des}(\pi)$  denote its descent set  $\{i \in [d - 1] : \pi_i > \pi_{i+1}\}$ . For  $S \subseteq [d - 1]$ , define the descent statistic  $\beta(S)$  to be the number of permutations of  $[d]$  with descent set  $S$ . Note that there is an obvious bijection between zigzag posets on  $[d]$  and subsets of  $[d - 1]$  given by

$$S: P \mapsto \{j \in [d - 1] : j \succ j + 1\}.$$

Moreover, a permutation  $\pi = \pi_1\pi_2 \cdots \pi_d$  of  $[d]$  is a linear extension of  $P$  if and only if  $\text{Des}(\pi^{-1}) = S(P)$ . Let  $\mathcal{F}(d) = S(\mathcal{Z}(d))$ . Then we can transform the problem of maximizing volume of order-chain polytopes of a  $d$ -chain to the problem of maximizing the descent statistic  $\beta(S)$ , where  $S$  ranges over  $\mathcal{F}(d)$ .

Observe that  $\beta(S) = \beta(\bar{S})$ , where  $\bar{S} = [d - 1] \setminus S$ . Following [3], we will encode both  $S$  and  $\bar{S}$  by a list  $L = (l_1, l_2, \dots, l_k)$  of positive integers such that  $l_1 + l_2 + \dots + l_k = d - 1$ . Given  $S \subseteq [d - 1]$ , a run of  $S$  is a set  $R \subseteq [d - 1]$  of consecutive integers of maximal cardinality such that  $R \subseteq S$  or  $R \subseteq \bar{S}$ . For example, if  $d = 10$ , then the set  $S = \{1, 2, 5, 8, 9\}$  has 5 runs:  $\{1, 2\}$ ,  $\{3, 4\}$ ,  $\{5\}$ ,  $\{6, 7\}$ ,  $\{8, 9\}$ . Suppose that  $S$  has  $k$  runs  $R_1, R_2, \dots, R_k$  with  $|R_i| = l_i$ , let  $L(S) = (l_1, l_2, \dots, l_k)$ .

**Lemma 4.1.** *Suppose that  $S \subseteq [d - 1]$  and  $L(S) = (l_1, l_2, \dots, l_k)$ . Then  $S \in \mathcal{F}(d)$  if and only if  $l_i \geq 2$  for all  $2 \leq i \leq k - 1$ .*

*Proof.* The lemma follows immediately from the fact that  $\mathcal{Z}(d)$  consists of zigzag posets  $P$  such that all maximal chains in  $P$ , except the first one (containing 1) and the last one (containing  $d$ ), contain at least three elements. □

Denote by  $F_d$  the  $d$ th Fibonacci number. By Lemma 4.1, it is easy to see that  $|\mathcal{F}(d)| = 2F_d$  for  $d \geq 2$ . Based on computer evidences, we conjectured the following results about maximizing descent statistic over  $\mathcal{F}(d)$ , which in fact<sup>1</sup> is a special case of Theorem 6.1 in [3].

**Proposition 4.2.** *Suppose that  $d \geq 2$  and  $S \subseteq [d - 1]$ .*

(1) *If  $d = 2m$  and*

$$L(S) = (1, \underbrace{2, 2, \dots, 2}_{m-1}) \quad \text{or} \quad L(S) = (\underbrace{2, 2, \dots, 2}_{m-1}, 1),$$

*then  $\beta(T) \leq \beta(S)$  for any  $T \in \mathcal{F}(d)$ .*

(2) *If  $d = 2m + 1$  and*

$$L(S) = (1, \underbrace{2, 2, \dots, 2}_{m-1}, 1),$$

*then  $\beta(T) \leq \beta(S)$  for any  $T \in \mathcal{F}(d)$ .*

Equivalently, by the proof of Theorem 3.3, we have

**Proposition 4.3.** *Let  $P$  be a chain on  $[d]$ . Then the alternating edge partition  $\ell = (oE(P), cE(P))$  with*

$$oE(P) = \begin{cases} \{\{1, 2\}, \{3, 4\}, \dots, \{d - 1, d\}\}, & \text{if } d \text{ is even} \\ \{\{1, 2\}, \{3, 4\}, \dots, \{d - 2, d - 1\}\}, & \text{otherwise} \end{cases}$$

*gives rise to an order-chain polytope  $OC_\ell(P)$  with maximum volume.*

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